4130 HOMEWORK 3

Due Thursday February 18

(1) Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences of rational numbers. Prove that $\{x_n\} \sim \{y_n\}$ if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $|x_m - y_n| < \varepsilon$.

Let $(*)$ be the condition:

for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $|x_m - y_n| < \varepsilon$.

We need to prove that $(*)$ holds if and only if:

for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - y_n| < \varepsilon$.

Clearly, $(*)$ implies the second condition. So we just need to check that if $\{x_n\} \sim \{y_n\}$ then $(*)$ holds. So suppose $\{x_n\} \sim \{y_n\}$. Then let $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that if $n > N$, then $|x_n - y_n| < \varepsilon/2$. Since $\{y_n\}$ is a Cauchy sequence, we may also choose $N_2 \in \mathbb{N}$ such that if $m, n > N_2$ then $|y_m - y_n| < \varepsilon/2$. Now suppose $m, n > \max\{N_1, N_2\}$. Then

$$|x_m - y_n| \leq |x_m - y_m| + |y_m - y_n| < \varepsilon$$

as required.

(2) (a) Using the formula for the partial sums of a geometric series, or otherwise, check that the sequence of rational numbers whose n^{th} term is

$$x_n = \sum_{i=1}^{n} \frac{e_i}{10^i}$$

is a Cauchy sequence, for any integers e_i with $0 \leq e_i \leq 9$. (Remark: the real number defined by this Cauchy sequence is denoted $0.e_1e_2e_3\ldots$)

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $\frac{1}{10^N} < \varepsilon$.

[Note: Why is there an N such that $1/10^N < \varepsilon$? It is tempting to take $N > \log_{10}(1/\varepsilon)$. This is not a good idea, because we haven't yet defined the logarithm function. Remember, all we know about $\mathbb{R}$ is that it is a complete ordered field. Instead, we can show by induction on N that $1/10^N < 1/N$ for all $N \in \mathbb{N}$.

Then, using the archimedean property, we can take $1/N < \varepsilon$ and then we are guaranteed to have $1/10^N < \varepsilon$ as well.
How do we show that $1/10^N < 1/N$ for all N? We need to show that $N < 10^N$. To see this, we use induction on N. The base step is $1 < 10$. Now suppose we have $N-1 < 10^{N-1}$. Then $N = N-1+1 < 10^{N-1}+1 < 10^{N-1}.2 < 10^{N-1}.10 = 10^N$, which proves the induction step.]
Now suppose $m, n > N$. Assume $n > m$. The sequence $\{x_n\}$ is increasing, so we have

$$|x_m - x_n| = x_n - x_m = \sum_{i=m+1}^{n} \frac{e_i}{10^i}$$

Since $e_i \leq 9$, we have

$$|x_m - x_n| \leq 9 \sum_{i=m+1}^{n} \frac{1}{10^i} = \frac{9}{10^{m+1}} \sum_{i=0}^{n-m-1} \frac{1}{10^i}$$

The sum on the right hand side is $(1-(1/10)^{n-m})/(1-(1/10))$ and so the right hand side simplifies to:

$$\frac{1}{10^m}(1-(\frac{1}{10})^{n-m}) = \frac{1}{10^m} - \frac{1}{10^n} < \frac{1}{10^m} < \frac{1}{10^N} < \varepsilon.$$

This shows that the sequence is Cauchy.

(b) Show that $0.999\ldots = 1$.

The question is asking us whether the Cauchy sequence $\{x_n\}$ whose n^{th} term is

$$x_n = \sum_{i=1}^{n} \frac{9}{10^i}$$

is equivalent to the Cauchy sequence of all 1's. To show this, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $1/10^N < \varepsilon$. Suppose $n > N$. Then

$$|\sum_{i=1}^{n} \frac{9}{10^i} - 1| = |9(\frac{1-\frac{1}{10^{n+1}}}{1-\frac{1}{10}} - 1) - 1| = |10 - \frac{1}{10^n} - 9 - 1| = \frac{1}{10^n} < \frac{1}{10^N} < \varepsilon.$$

Therefore, the two Cauchy sequences are equivalent.

(3) Section 2.2.4 #5.

We wish to show that between any two distinct real numbers x and y, there are infinitely many rationals. Without loss of generality, assume $x < y$. Then $y - x > 0$.

By the archimedean property, there exists $A \in \mathbb{N}$ with $y - x > 1/A$. Therefore, $x < x + 1/A < y$. Now consider the set

$$S = \{n \in \mathbb{N} : \frac{n}{A} \geq x\}.$$

By the well-ordering principle, S has a least element n. This n must satisfy

$$\frac{n-1}{A} < x \leq \frac{n}{A}$$

or else $n - 1$ would be in S, contradicting that n is the least element. Therefore, we have $\frac{n}{A} < x + \frac{1}{A} < y$ and so $x \leq \frac{n}{A} < y$. Now use the archimedean property again to choose a $B \in \mathbb{N}$ with $y - \frac{n}{A} > \frac{1}{B}$. Then $x \leq \frac{n}{A} < \frac{n}{A} + \frac{1}{B} < y$ and so the set

$$\{\frac{n}{A} + \frac{1}{kB} : k \in \mathbb{N}\}$$

is an infinite set of rationals with lie between x and y.

(4) Using the triangle inequality, show that for any $a, b \in \mathbb{R}$, we have

$$||a| - |b|| \leq |a - b|.$$

Using this, show that if $\{x_n\}$ is a sequence of real numbers which converges to L, then $\{|x_n|\}$ converges to $|L|$.

To show the inequality, let $a, b \in \mathbb{R}$. Then

$$|a| = |a - b + b| \leq |a - b| + |b|$$

by the triangle inequality. Therefore,

$$|a| - |b| \leq |a - b|.$$

Because $|a - b|$ is always positive, we must have $||a| - |b|| \leq |a - b|$ as required.

To show the statement about convergence, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that if $n > N$ then $|x_n - L| < \varepsilon$. Then for $n > N$, we have

$$||x_n| - |L|| \leq |x_n - L| < \varepsilon$$

and so $\{|x_n|\} \to |L|$, as required.