4130 HOMEWORK 4

Due Tuesday March 2

(1) Let $\mathbb{N}^{\mathbb{N}}$ denote the set of all sequences of natural numbers. That is,

$$\mathbb{N}^{\mathbb{N}} = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{N}\}.$$

Show that $|\mathbb{N}^{\mathbb{N}}| = |\mathcal{P}(\mathbb{N})|$.

We use the Schröder-Bernstein Theorem. First, there is an injection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{N}^{\mathbb{N}}$, because we may regard a subset of $\mathbb{N}$ as a sequence of zeroes and ones, or equivalently as a sequence of 1's and 2's, and this gives the desired injection. The hard part is showing that there is an injection $\mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$. To see this, note that a sequence of natural numbers is the same thing as a function $\mathbb{N} \to \mathbb{N}$. But by definition, a function is a special kind of subset of $\mathbb{N} \times \mathbb{N}$. In this way, we get an injection $\mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N} \times \mathbb{N})$. But it was shown in class that $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ have the same cardinality, and hence so do their power sets. In this way, we get the desired injection $\mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$.

Working through the above proof, we can explicitly write down an injection if we like. An example is:

$$(a_1, a_2, \ldots) \mapsto \{2 \cdot 3^{a_1}, 2^2 \cdot 3^{a_2}, \ldots\} \subset \mathbb{N}.$$

(2) Let $\{x_n\}$ be a Cauchy sequence of rational numbers. Regarding $\{x_n\}$ as a sequence of real numbers, show that $\{x_n\}$ converges to the real number x defined as the equivalence class of the sequence $\{x_n\}$.

The hardest part of this is working out what to prove in the first place.

Let $\varepsilon > 0$ be a real number. Choose $A \in \mathbb{N}$ with $3/2A < \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence of rational numbers, there exists $N \in \mathbb{N}$ such that if $m, n > N$ then $|x_n - x_m| < 1/A$. In other words, if $m, n > N$ then $-1/A < x_m - x_n < 1/A$.

Now fix some $m > N$ and consider the sequence of rational numbers $\{y_n\}$ where

$$y_n = x_m - x_n$$

for $n \geq 1$. We have chosen N so that if $n > N$ then $-1/A < y_n < 1/A$. This implies that for $n > N$, we have

$$y_n + 3/2A > 1/2A$$

and

$$3/2A - y_n > 1/2A.$$

The first inequality is the statement that the real number defined as the equivalence class of the Cauchy sequence $\{y_n + 3/2A\}$ is positive. (Ex: why is this sequence Cauchy?) If we let $y = [\{y_n\}]$, then we have an inequality in the *real* numbers

$$y + 3/2A > 0.$$

Similarly, the other inequality gives

$$3/2A - y > 0.$$

Putting these together, we have, for $n > N$,

$$-3/2A < y < 3/2A.$$

But $y = [\{x_m - x_n\}_{n\geq 1}] = [\{x_m, x_m, \ldots\}] - [\{x_1, x_2, \ldots\}]$ which is the real number $x_m - x$, by definition of how the rationals are embedded in the reals. Thus, we have

$$-3/2A < x_m - x < 3/2A$$

and hence

$$|x_m - x| < 3/2A < \varepsilon.$$

This holds for any given $m > N$. So we have shown that for all $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that if $m > N$ then $|x_m - x| < \varepsilon$, as required.

(3) Section 2.2.4 # 4.

Let $x \in \mathbb{R}$ be the equivalence class of the Cauchy sequence $\{x_n\}$ where $x_i \in \mathbb{Q}$. We construct a sequence q_n of rational numbers which is increasing and converges to x. To start with, take $q_1 \in \mathbb{Q}$ with $x > q_1 > x - 1$. Now suppose we have constructed $q_1, \ldots, q_{n-1}$ with $q_{n-1} < x$. Let q_n be a rational number with $x > q_n > \max\{q_{n-1}, x - 1/n\}$. Then the sequence $\{q_n\}$ is increasing and $q_n < x$ for all n. Also,

$q_n > x - 1/n$, so $|x - q_n| = x - q_n < 1/n$, and thus $\{q_n\} \to x$. In particular, $\{q_n\}$ is Cauchy.

We now have an increasing Cauchy sequence of rational numbers $\{q_n\}$, and it remains to show that this sequence is equivalent to $\{x_n\}$. For this, let $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|x - q_n| < \varepsilon/2$. By the previous problem, we can choose $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|x - x_n| < \varepsilon/2$. Now if $n > \max\{N_1, N_2\}$ then $|x_n - q_n| \leq |x_n - x| + |x - q_n| < \varepsilon$, which shows that the sequences $\{x_n\}$ and $\{q_n\}$ are equivalent sequences of rationals.

(4) Show that every subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound. (Hint: see p. 75 of the textbook.)

Let S be a subset of $\mathbb{R}$ which is bounded below. Let

$$-S = \{-x : x \in S\}.$$

Then $-S$ is bounded above, so it has a least upper bound $s := \sup(-S)$. We claim that $-s$ is the greatest lower bound of S. First, $-s$ is a lower bound because if $x \in S$ then $-x \leq -s$ and so $x \geq s$. Also, if b is any other lower bound for $-S$, then $-b$ is an upper bound for S, and so $-b \geq s$, whence $b \leq -s$ and so $-s$ is the greatest lower bound, as required.

(5) Find, if they exist, the supremum (least upper bound) and infimum (greatest lower bound) of the following subsets of $\mathbb{R}$.

(a) $\{1, 2, 3\}$.

Sup $= 3$, inf $= 1$.

(b) $(0, 1) \cup \{2\} \cup [3, 4) = \{x \in \mathbb{R} : 0 < x < 1 \text{ or } x = 2 \text{ or } 3 \leq x < 4\}$.

Sup $= 4$, inf $= 0$.

(c) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$.

The set is $\{0, 1/2, 2/3, \ldots\}$. The supremum is 1 and the infimum is the smallest element, which is 0.

(d) $\mathbb{Q}$.

This set is not bounded above or below, and so it does not have a finite sup or inf.

(6) Prove Theorem 2.3.2 in the textbook.

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers.

(a) Suppose $\{x_k\} \to x$ and $\{y_k\} \to y$. We must show that $\{x_k + y_k\} \to x + y$. Let $\varepsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that if $k > N_1$ then $|x_k - x| < \varepsilon/2$. There also exists N_2 such that if $k > N_2$ then $|y_k - y| < \varepsilon/2$. Now suppose $k > \max\{N_1, N_2\}$. Then

$$|(x_k + y_k) - (x + y)| \leq |x_k - x| + |y_k - y| < \varepsilon.$$

Next, we need to show that $\{x_k y_k\} \to xy$. To do this, we first show that a convergent sequence of real numbers is bounded. We have

$$|x_k| \leq |x_k - x| + |x|.$$

Taking $\varepsilon = 1$, we know that there exists N such that if $k > N$ then $|x_k - x| < 1$. Therefore, if $k > N$ then $|x_k| < 1 + x$. It follows that for all k, $|x_k| \leq B$ where

$$B = \max\{|x_1|, |x_2|, \ldots, |x_N|, 1 + |x|\}.$$

Thus, every convergent sequence of real numbers is bounded.

Now we show that $\{x_k y_k\} \to xy$. Let $\varepsilon > 0$ and choose B such that $|x_k| \leq B$ for all k and $|y| \leq B$. Choose N_1 such that if $k > N_1$ then $|x_k - x| < \varepsilon/2B$. Choose N_2 such that if $n > N_2$ then $|y_k - y| < \varepsilon/2B$. Then if $N > \max\{N_1, N_2\}$ then

$$|x_k y_k - xy| = |x_k(y_k - y) + y(x_k - x)| \leq |x_k||y_k - y| + |y||x_k - x| < \varepsilon.$$

Next, suppose $y \neq 0$. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|y_n - y| < \varepsilon$. So $y_n \in (y - \varepsilon, y + \varepsilon)$. If $y > 0$ then $y - \varepsilon > 0$ for some $\varepsilon > 0$. If $y < 0$ then $y + \varepsilon < 0$ for some $\varepsilon > 0$. In either case, we have $y_n \neq 0$ for sufficiently large n, as desired.

We show that $\{1/y_k\} \to 1/y$. As stated in the book, we may as well assume that $y_k \neq 0$ for all k; otherwise we can neglect the terms with $y_k = 0$. By convergence of $\{y_n\}$, there exists $N \in \mathbb{N}$ such that if $n > N$ then

$$|y_n - y| < |y|/2.$$

Then if $n > N$, we have

$$|y_n| \geq |y| - |y - y_n| > |y| - |y|/2 = |y|/2.$$

Therefore,

$$|\frac{1}{y_n} - \frac{1}{y}| = \frac{|y_n - y|}{|y_n||y|} \leq \frac{2}{|y|^2}|y_n - y|,$$

for $n > N$. Let $\varepsilon > 0$. Choose N_2 such that if $n > N_2$ then $|y_n - y| < \varepsilon|y|^2/2$. Then for $n > \max\{N, N_2\}$, we get

$$|\frac{1}{y_n} - \frac{1}{y}| < \varepsilon$$

as required.

Now the statement that $\{x_k/y_k\} \to x/y$ follows from writing $x_k/y_k = x_k \cdot (1/y_k)$.

(b) Suppose there is $m \in \mathbb{N}$ such that $x_k \geq y_k$ for $k > m$. Suppose for a contradiction that $x < y$. Then let $0 < \varepsilon < (y-x)/2$. There exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|x_n - x| < \varepsilon$ and there exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|y_n - y| < \varepsilon$. Therefore, if $n > \max\{N_1, N_2, m\}$ then

$$x_n < x + \varepsilon < x + (y - x)/2 = (x + y)/2 = y - (y - x)/2 < y - \varepsilon < y_n$$

which contradicts $x_n \geq y_n$.