4130 HOMEWORK 5

Due Tuesday March 9

(1) A subset I of $\mathbb{R}$ is called an *interval* if for all $x, y \in I$ and all $z \in \mathbb{R}$, if $x < z < y$ then $z \in I$.

Show that if I is a bounded interval, then $(\inf I, \sup I) \subset I$. Using this, show that I must be one of the following four intervals:

$$(\inf I, \sup I) \qquad [\inf I, \sup I) \qquad (\inf I, \sup I] \qquad [\inf I, \sup I].$$

Let I be a bounded interval. Then $\sup I$ and $\inf I$ exist. Suppose $\inf I < z < \sup I$. Then there is some $x \in I$ with $\inf I \leq x < z$. Indeed, if there was no such x, then z would be a lower bound for I, but z is greater than the greatest lower bound $\inf I$. Similarly, there is some $y \in I$ with $z < y \leq \sup I$. Therefore, $x < z < y$ and so $z \in I$ by definition of an interval.

Therefore, $(\inf I, \sup I) \subset I$. Now suppose $w < \inf I$. Then $w \notin I$ because $\inf I$ is a lower bound for I. Similarly, if $w > \sup I$ then $w \notin I$. Therefore, we have $I \subset [\inf I, \sup I]$. Altogether, we have

$$(\inf I, \sup I) \subset I \subset [\inf I, \sup I]$$

which leaves only the four given possibilities.

(2) For each of the following $S \subset \mathbb{R} \cup \{\pm\infty\}$, state whether there is a sequence whose set of limit-point is S. If there is, find one. If not, give a reason.

(a) $S = \{0\}$.

Take the sequence $(0, 0, 0, \ldots)$. This converges to 0, so 0 is its unique limit point.

(b) $S = \{\infty, -\infty\}$.

Take the sequence $\{(-2)^n\}$. This has no convergent subsequence, and it has a subsequence $\{2^{2n}\}$ which diverges to ∞ and a subsequence $\{-2 \cdot 2^{2n}\}$ which diverges to $-\infty$.

(c) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

This cannot be the set of limit points of any sequence, because it does not contain its infimum. (This is Section 3.2.3 # 9.)

(d) $S = \mathbb{N}$.

This also cannot be the set of limit points of any sequence, because it does not contain its "supremum" ∞. Putting it another way, if a sequence $\{x_n\}$ has every $n \in \mathbb{N}$ as a limit point, then we must have $\limsup_n x_n = \infty$.

(e) $S = \mathbb{N} \cup \{\infty\}$.

Define a sequence as follows:

$$(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots).$$

In general, we take the first n natural numbers and then the first $n+1$ and then the first $n+2$, and so on.

This sequence has a constant subsequence with value n, for every natural number n. It does not have any other finite limit points (a sequence of natural numbers cannot converge unless it is eventually constant, by the Cauchy property), but it also has a subsequence which diverges to ∞.

(3) Let $\mathcal{U}$ be the following collection of subsets of $\mathbb{R}$.

$$\mathcal{U} = \{(q, r) : q, r \in \mathbb{Q}, q < r\}.$$

(a) Show that $\mathcal{U}$ is countable.

We can define an injection $\mathcal{U} \to \mathbb{Q} \times \mathbb{Q}$ by sending the interval (q, r) to the ordered pair (q, r) (confusing notation!) So $\mathcal{U}$ is countable because $|\mathbb{Q} \times \mathbb{Q}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

(b) Show that every open set can be expressed as a union of intervals from $\mathcal{U}$.

Let U be an open set. Given $x \in U$, there exists $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \subset U$. By the density of the rationals, there exists a rational q with $x - \varepsilon < q < x$ and there exists a rational r with $x < r < x + \varepsilon$. Thus, $x \in (q, r) \in \mathcal{U}$, and $(q, r) \subset (x - \varepsilon, x + \varepsilon) \subset U$. Therefore,

$$U = \bigcup_{\substack{(q,r) \in \mathcal{U} \\ (q,r) \subset U}} (q, r).$$

(c) Let $\mathcal{X}$ denote the set of all open subsets of $\mathbb{R}$. Show that $|\mathcal{X}| = |\mathbb{R}|$. (Hint: recall that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.)

We can define an injection $\mathbb{R} \to \mathcal{X}$ by sending $x \in \mathbb{R}$ to the interval $(x-1, x+1)$. We need to define an injection $\mathcal{X} \to \mathbb{R}$. Using the hint, we need to define an injection $\mathcal{X} \to \mathcal{P}(\mathbb{N})$. Since $|\mathbb{N}| = |\mathcal{U}|$, we have $|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathcal{U})|$. We therefore want to define an injection $f : \mathcal{X} \to \mathcal{P}(\mathcal{U})$. For $U \in \mathcal{X}$, we define

$$f(U) = \{(q, r) \in \mathcal{U} : (q, r) \subset U\}.$$

To show that f is injective, observe that if $f(U) = f(V)$ then by the previous part, we have

$$U = \bigcup_{\substack{(q,r)\in\mathcal{U} \\ (q,r)\subset U}} (q, r) = \bigcup_{(q,r)\in f(U)} (q, r) = \bigcup_{(q,r)\in f(V)} (q, r) = V.$$

(d) Show that the set of all *closed* subsets of $\mathbb{R}$ also has cardinality $|\mathbb{R}|$.

There is a bijection between the set of all closed subsets of $\mathbb{R}$ and the set of all open subsets of $\mathbb{R}$ defined by sending a closed set C to the open set $\mathbb{R} \setminus C$. Since a set C is closed if and only if $\mathbb{R} \setminus C$ is open, this is clearly a bijection. Therefore, the set of all closed sets has the same cardinality as the set of all open sets.

[Remark: Let $\mathcal{Y}$ denote the set of all subsets of $\mathbb{R}$ that are either open or closed. One can now show that $|\mathcal{Y}| < |\mathcal{P}(\mathbb{R}) \setminus \mathcal{Y}|$. In other words, loosely speaking, "most" subsets of $\mathbb{R}$ are neither open nor closed.]