

4130 HOMEWORK 6

Due Thursday April 1

(1) Let $A \subset \mathbb{R}$. A point $x \in A$ is called *isolated* if it is not a cluster point of A .

(a) Can an open set have an isolated point? Can a closed set have one?

An open set U cannot have an isolated point because if $x \in U$ and $\delta > 0$ then $(x - \delta, x + \delta)$ contains an interval and hence contains infinitely many points of U . On the other hand, for any x , $\{x\}$ is a closed set which does have an isolated point, namely x itself.

(b) Give an example of a countable set with no isolated points.

The set \mathbb{Q} is countable and has no isolated points because if $q \in \mathbb{Q}$ and $\delta > 0$, then $(q - \delta, q + \delta)$ contains infinitely many rational numbers, and so q is a cluster point of \mathbb{Q} .

(2) Section 3.3.1 # 8.

If $A \subset \mathbb{R}$ is compact, then A is bounded, so $\sup(A)$ and $\inf(A)$ exist. For each $n \in \mathbb{N}$, there exists a point $a_n \in A$ with $\sup(A) \geq a_n > \sup(A) - \frac{1}{n}$. The sequence $\{a_n\}$ converges in \mathbb{R} to $\sup(A)$. Since A is compact, we have that

$$\lim_{n \rightarrow \infty} a_n \in A$$

and thus $\sup(A) \in A$. Similarly, $\inf(A) \in A$.

For the counterexample, take $[-1, 0) \cup (0, 1]$. This is not closed, hence noncompact, but it contains its sup and inf.

(3) Section 4.2.4 # 3. (Recall that an *interval* is, by definition, a subset I of \mathbb{R} such that for all $x, y \in I$ and all $z \in \mathbb{R}$ with $x < z < y$, we have $z \in I$.)

We are asked to show that the continuous image of an interval is an interval. Let I be an interval. Let $f : I \rightarrow \mathbb{R}$ be a continuous function. Let $X = f(I)$. We need to show that X is an interval. Let $x, y \in X$. Let $z \in \mathbb{R}$ be a number such that $x < z < y$. We need to show that $z \in X$.

Since $x, y \in X$, we have $x = f(a)$ and $y = f(b)$ for some $a, b \in I$. There are three cases: either $a < b$ or $b < a$ or $a = b$.

If $a < b$, then $[a, b] \subset I$. The function f restricted to $[a, b]$ is continuous, and therefore by the intermediate value theorem, it takes on the value z , since $f(a) < z < f(b)$. Thus, there exists $t \in [a, b] \subset I$ with $f(t) = z$ and so $z \in f(I) = X$.

If $a > b$ then $[b, a] \subset I$ and we can apply the Intermediate Value Theorem in the same way to conclude that $z \in X$.

If $a = b$ then $x = y$, which is a contradiction since we assumed $x < y$.

Thus, in all possible cases, we have $z \in X$. Thus, X is an interval, as required.

For the example part, take $f(x) = x$ and let I be any open interval.

(4) In this question, we will show that every positive real number has an n^{th} root.

(a) Let $x \in (0, \infty)$ and $n \in \mathbb{N}$. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha^n < x < \beta^n$.

Let $0 < \alpha < \min\{x, 1\}$ and let $\beta > \max\{x, 1\}$. Then $\alpha^n < \alpha$ since $\alpha < 1$ and $\beta^n > \beta$ since $\beta > 1$. Thus,

$$\alpha^n < \alpha < x < \beta < \beta^n.$$

(b) Show that there exists $y \in \mathbb{R}$ with $x = y^n$.

Define $f : (0, \infty) \rightarrow (0, \infty)$ by $f(x) = x^n$. Then f is continuous and $f(\alpha) = \alpha^n < x < \beta^n = f(\beta)$. By the Intermediate Value Theorem, f must assume the value x . Thus, there exists y with $y^n = x$.

(c) For $x \in [0, \infty)$, show that there exists a *unique* $y \in [0, \infty)$ with $x = y^2$. We denote this y by \sqrt{x} .

By the previous part, such a y exists. Suppose y_1 and y_2 are both positive and satisfy $y_1^2 = y_2^2 = x$. Then $y_1^2 - y_2^2 = 0 = (y_1 - y_2)(y_1 + y_2)$. Since $y_1 + y_2 > 0$, we have $y_1 - y_2 = 0$ and so $y_1 = y_2$.

(d) Define $f : [0, \infty) \rightarrow \mathbb{R}$ by $f(x) = \sqrt{x}$. Show that f is a continuous function.

Let $x \in [0, \infty)$. We must show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|\sqrt{x} - \sqrt{y}| < \varepsilon$.

For all y , we have

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - \sqrt{y}| |\sqrt{x} + \sqrt{y}| / |\sqrt{x} + \sqrt{y}| = |x - y| / |\sqrt{x} + \sqrt{y}|.$$

This is $\leq |x - y|/\sqrt{x}$ since $y \geq 0$. Thus, given $\varepsilon > 0$, we may choose $\delta < \sqrt{x}\varepsilon$. Then if $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < |x - y|/\sqrt{x} < \varepsilon$, as required.

(5) Two monasteries, A and B , are joined by exactly one path AB which is 20 miles long. One morning, Brother Albert (a monk) sets out from monastery A at 9 am, arriving at monastery B at 9 pm. The next day, he sets out from monastery B at 9 am, arriving at monastery A at 9 pm. On both journeys, he may have stopped to rest, or even walked backwards for some of the time.

(a) Prove that there is a point x on the path AB such that Brother Albert was at x at exactly the same time on both days. (Hint: let $f_i(t)$ be the distance of Brother Albert from A at time t on day i , $i = 1, 2$. Apply the Intermediate Value Theorem to a suitable combination of f_1 and f_2 .) Using the hint, consider the function $g = f_1 - f_2$. This is continuous and $g(9am) = f_1(9am) - f_2(9am) = 0 - 20 = -20$, while $g(9pm) = 20$. Thus, by the Intermediate Value Theorem, there exists a time t between $9am$ and $9pm$ with $g(t) = 0$ and so $f_1(t) = f_2(t)$. The desired point is the point on the path AB which at distance exactly $f_1(t)$ from A .

(b) Another monk, Brother Gilbert, has been dabbling in forbidden knowledge. Once per day, by snapping his fingers, he can instantaneously teleport himself to any point within a 3 ft. radius of his current location. Suppose Brother Gilbert makes the same journey as Brother Albert. Does the conclusion from part (a) still hold?

No. Brother Gilbert's motion is no longer necessarily continuous, so we cannot apply the IVT. As an exercise, you can try to construct an example to show that it is indeed possible that there might be no such point.

(6) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let $a > 0$. We say that f is *periodic with period a* if

$$f(x + a) = f(x)$$

for all $x \in \mathbb{R}$.

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is periodic with period a and define

$$g(x) = f(1/x)$$

for $x > 0$.

(a) Show that for all $x > 0$, we have

$$f([x, x+a]) = g\left(\left[\frac{1}{x+a}, \frac{1}{x}\right]\right).$$

Let $u \in [x, x+a]$. Then $1/u \in [\frac{1}{x+a}, \frac{1}{x}]$, and $f(u) = g(1/u) \in g([\frac{1}{x+a}, \frac{1}{x}])$. Thus,

$$f([x, x+a]) \subset g\left(\left[\frac{1}{x+a}, \frac{1}{x}\right]\right).$$

Now let $v \in [\frac{1}{x+a}, \frac{1}{x}]$. Then $1/v \in [x, x+a]$ and so $g(v) = f(1/v) \in f([x, x+a])$.

Thus,

$$f([x, x+a]) \supset g\left(\left[\frac{1}{x+a}, \frac{1}{x}\right]\right).$$

This proves the result.

(b) Suppose f is not constant. Show that g is not uniformly continuous on $(0, \infty)$.

Suppose f is not constant. Let $r \in \mathbb{R}$. Then there exist $b_0, b_1 \in [r, r+a]$ with $f(b_0) \neq f(b_1)$. Indeed, if not, then f is constant on $[r, r+a]$ and therefore constant on all of \mathbb{R} , by periodicity.

Choose and fix $r > 0$. Let $n \in \mathbb{N}$.

By periodicity of f , we have $f([r+na, r+na+a]) = f([r, r+a])$.

By (a), there exist $c_0, c_1 \in [\frac{1}{r+na+a}, \frac{1}{r+na}]$ with $g(c_0) = f(b_0)$ and $g(c_1) = f(b_1)$.

In particular, $|c_0 - c_1| \leq \frac{1}{r+na}$. Let $\varepsilon = |g(c_0) - g(c_1)| > 0$. Given $\delta > 0$, choose $n > (1/a)(1/\delta - r)$. Then $|c_0 - c_1| < \delta$, but $|g(c_0) - g(c_1)| \geq \varepsilon$.

Thus, $\exists \varepsilon > 0 \forall \delta > 0 \exists c_0, c_1 \in (0, \infty)$ with $|c_0 - c_1| < \delta$ but $|g(c_0) - g(c_1)| \geq \varepsilon$.

This is precisely the negation of the definition of uniform continuity. So g is not uniformly continuous.

[Remark: In particular, taking $f(x) = \sin(x)$ and $a = 2\pi$, we see that $\sin(1/x)$ is not uniformly continuous.]