4130 HOMEWORK 7

Due Tuesday April 13

(1) Let $D \subset \mathbb{R}$. Let $f, g : D \to \mathbb{R}$ and let a be a cluster point of D. Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Show that $\lim_{x\to a} f(x)g(x) = LM$.

First, we show that $f(x)$ and $g(x)$ are bounded near a. Since $\lim_{x\to a} f(x) = L$, there exists $\delta > 0$ such that if $x \in D$ and $|x-a| < \delta$ then $|f(x) - L| < 1$, whence $|f(x)| \le |f(x) - L| + |L| = 1 + |L|$. Similarly, there exists δ' such that if $|x - a| < \delta'$ then $|g(x)| \le 1+|M|$. Replacing δ by the minimum of δ and δ', we may assume that $|x-a| < \delta$ implies $|f(x)| \le 1 + |L|, |g(x)| \le 1 + |M|$.

Now, there is $\delta_1 > 0$ such that $|x - a| < \delta_1$ implies $|f(x) - L| < \varepsilon/2(1 + |M|)$. Similarly, there is $\delta_2 > 0$ such that $|x - a| < \delta_2$ implies $|g(x) - M| < \varepsilon/2(1 + |L|)$. Let $\delta_3 = \min\{\delta_1, \delta_2, \delta\}$. Then $|x - a| < \delta_3$ implies:

$$|f(x)g(x) - LM| \le |f(x)||g(x) - M| + |M||f(x) - L| < \varepsilon.$$

(2) Section 5.2.4 Exercise 3.

The converse of the Mean Value Theorem is false. For example, let $f(x) = x^3$ on $(-1, 1)$. Then f is strictly increasing and so if $x_1 < x_2$ then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

but $f'(0) = 0$, so there cannot be x_1, x_2 with the Newton quotient equal to $f'(0)$. We could construct a similar example for any strictly increasing/decreasing function with a critical point which is not a local max/min.

(3) Section 5.2.4 Exercise 4.

An example of such a function is $h(x) = |x|$, which can be written as

$$h(x) = \begin{cases} -x, & x \le 0 \\ x, & x \ge 0. \end{cases}$$

We already know that this is not differentiable at $x = 0$ but it is of the required form.

For the one-sided derivatives, we make the following definition. Suppose f is defined on a half-open interval $(a, b]$. Define

$$f'_-(b) = \lim_{x \to b^-} \frac{f(b) - f(x)}{b - x}.$$

Similarly, if g is defined on $[b, c)$ define

$$g'_+(b) = \lim_{x \to b^+} \frac{g(x) - g(b)}{x - b}.$$

We wish to to prove the following theorem:

Theorem: If f is differentiable on (a, b) and g is differentiable on (b, c) then the function h defined by

$$h(x) = \begin{cases} f(x) & a < x < b \\ g(x) & b < x < c \end{cases}$$

is differentiable at b if and only if $f'_-(b)$ and $g'_+(b)$ exist and are equal.

Proof: h is differentiable at b if and only if the limit

$$\lim_{x \to b} \frac{h(x) - h(b)}{x - b}$$

exists. But this limit exists if and only if both of the one-sided limits

$$\lim_{x \to b^-} \frac{h(x) - h(b)}{x - b}$$

and

$$\lim_{x \to b^+} \frac{h(x) - h(b)}{x - b}$$

exist and are equal. By definition of h, these one-sided limits are precisely $f'_-(b)$ and $g'_+(b)$, so we are done.

(4) Let U be an open subset of $\mathbb{R}$ and suppose $f : U \to U$ is a C^1 bijection and that $f'(x) \neq 0$ for all $x \in U$.

(a) Use the Inverse Function Theorem to show that the inverse function $f^{-1} : U \to U$ is also C^1.

Let $x \in U$. We need to show that f^{-1} is C^1 at x. By hypothesis, f is C^1 at $f^{-1}(x)$. Furthermore, $f'(f^{-1}(x)) \neq 0$. Therefore, there exists a neighborhood (a, b) of $f^{-1}(x)$ such that $f : (a, b) \to (c, d)$ is a bijection, for some (c, d). By the Inverse Function Theorem, the inverse function $g : (c, d) \to (a, b)$ is C^1.

But if $x \in (c,d)$ then $f(g(x)) = x = f(f^{-1}(x))$, so g coincides with f^{-1} on (c,d). Therefore, f^{-1} is C^1 at each point of (c,d). In particular, f^{-1} is C^1 at $x = f(f^{-1}(x)) \in (c,d)$.

(b) Give an example to show that f^{-1} may not be C^1 if $f'(x) = 0$ for some $x \in U$.

We can take $f(x) = x^3$ (again). Take $U = \mathbb{R}$. Then $f^{-1} = x^{1/3}$ is not differentiable at $x = 0$.

(5) Section 5.4.6 Exercise 22(b). (Hint: let $y = p_1x_1 + \cdots + p_nx_n$. For each $1 \leq i \leq n$, estimate $f(x_i)$ using the Taylor expansion of f about the point y, applying the Lagrange Remainder Theorem from page 188.)

Following the hint, for each i, we have

$$f(x_i) = f(y) + f'(y)(x_i - y) + \frac{1}{2}f''(u_i)(x_i - y)^2$$

for some u_i between y and x_i.

We sum these up for i between 1 and n to get

$$\sum_{i=1}^{n} p_i f(x_i) = (\sum p_i) f(y) + \sum_{i=1}^{n} p_i f'(y)(x_i - y) + \sum_{i=1}^{n} \frac{p_i}{2} f''(u_i)(x_i - y)^2.$$

The third term on the RHS is nonnegative by hypothesis. The second term is

$$\sum_{i=1}^{n} p_i f'(y)(x_i - y) = f'(y)(\sum_{i=1}^{n} p_i x_i - (\sum_{i=1}^{n} p_i)y) = f'(y)(y - y) = 0.$$

We therefore get

$$\sum_{i=1}^{n} p_i f(x_i) \geq f(y)$$

as required.