4130 HOMEWORK 8

Due Tuesday May 3

(1) Let $f_n : A \to \mathbb{R}$ be functions which converge uniformly on A to a function f. Let x_0 be a cluster point of A. Suppose $\lim_{x\to x_0} f_n(x)$ exists for all n. Let $L_n = \lim_{x\to x_0} f_n(x)$.

(a) Show that the sequence $\{L_n\}$ converges.

We show that $\{L_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$.

Since the sequence $\{f_n\}$ converges uniformly, it is uniformly Cauchy, and so there exists $N \in \mathbb{N}$ such that if $m, n > N$ then $|f_n(x) - f_m(x)| < \varepsilon/3$ for all $x \in A$.

Let $m, n > N$.

There exists $\delta_1 > 0$ such that if $|x - x_0| < \delta_1$, we have $|f_n(x) - L_n| < \varepsilon/3$.

There exists $\delta_2 > 0$ such that if $|x - x_0| < \delta_2$, we have $|f_m(x) - L_m| < \varepsilon/3$.

By the triangle inequality:

$$|L_n - L_m| \le |L_n - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(x) - L_m|$$

for all $x \in A$. In particular, we can choose some x with $|x - x_0| < \delta_1, \delta_2$. Then we get

$$|L_n - L_m| < \varepsilon.$$

Therefore, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m, n > N$ then $|L_m - L_n| < \varepsilon$. So $\{L_n\}$ is a Cauchy sequence, and hence it converges.

(b) Show that $\lim_{x\to x_0} f(x)$ exists and equals $\lim_{n\to\infty} L_n$.

Let $\varepsilon > 0$. Let $L = \lim_{n\to\infty} L_n$, which exists by part (a).

There exists $N_1 \in \mathbb{N}$ such that if $n > N_1$ then $|f(x) - f_n(x)| < \varepsilon/3$ for all $x \in A$.

There exists $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|L - L_n| < \varepsilon/3$.

Let $n > N_1, N_2$.

There exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f_n(x) - L_n| < \varepsilon/3$.

Let $|x - x_0| < \delta$. By the triangle inequality:

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L| < \varepsilon.$$

Therefore, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$, and so $\lim_{x \to x_0} f(x) = L$.

(2) Section 7.3.4 Exercise 11.

Define $f_n(x)$ to be a function given by a "spike" of the following form:

$f_n(x)$ is zero if $0 \leq x \leq 1 - 1/n$.

$f_n(x)$ is a straight line on the interval $[1 - 1/n, 1 - 1/2n]$, which joins the point $(1 - 1/n, 0)$ to the point $(1 - 1/2n, h)$, where h is chosen so that the area under the graph of this straight line is $1/2$.

$f_n(x)$ is a straight line on the interval $[1-1/2n, 1]$, which joins the point $(1-1/2n, h)$ to the point $(1, 0)$.

You can write a formula for the function if you like. It is continuous and piecewise-linear and has the property that

$$\int_0^1 f_n(x)dx = 1.$$

The f_n converge pointwise to zero, because at $x = 1$, $f_n(x) = 0$ for all n, and if $x < 1$ then $x < 1 - 1/N$ for N sufficiently large, and so $f_n(x) = 0$ for $n > N$. Therefore, $\{f_n(x)\} \to 0$ for all x, yet the sequence $\{\int_0^1 f_n(x)dx\}$ converges to 1.

(3) Find the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} (n^2 + n + 1)x^n.$$

Find a pair of polynomials $p(x)$ and $q(x)$ such that $f(x) = \frac{p(x)}{q(x)}$ within its radius of convergence.

The coefficient of x^n is $a_n = n^2 + n + 1$. We have

$$n \leq a_n \leq 3n^2$$

for all $n \in \mathbb{N}$.

Therefore,

$$n^{1/n} \leq a_n^{1/n} \leq 3^{1/n}(n^{1/n})^2$$

for all $n \in \mathbb{N}$. We know from class that $\{3^{1/n}\}$ and $\{n^{1/n}\}$ tend to 1 as $n \to \infty$, and so by the Sandwich Principle, we have $\lim_{n\to\infty} |a_n|^{1/n} = 1$. It follows that $\limsup_n |a_n|^{1/n} = 1$ and so the radius of convergence is $R = 1/1 = 1$.

We can use term-by-term differentiation inside $(-1, 1)$. From the sum of a geometric series, we know that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

for $|x| < 1$. From this, we get

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2},$$

and so

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Differentiating again and simplifying, we get

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3},$$

and so

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}.$$

Adding together, we get

$$f(x) = \sum_{n=0}^{\infty} (n^2+n+1)x^n = \sum_{n=0}^{\infty} n^2 x^n + \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n = \frac{1+x^2}{(1-x)^3}.$$