MATH 413 HONORS INTRODUCTION TO ANALYSIS I
PRELIM 1.
PRACTICE

(Note: attempt all questions. You have 70 minutes. Good luck!)

(1) **(9 marks)** Let $X = (0, 1) \cup (2, 3) \subset \mathbb{R}$. State whether the following statements about X are true or false and give a brief reason in each case.

(a) $3 \in \mathbb{R}$ is a cluster point (a.k.a. limit-point) of X.

(b) X is a closed set.

(c) The set $f^{-1}(X)$ is open, where $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^5 + 2x^3 - 9x + 1$.

(2) **(25 marks)** Let $\{x_n\}$ be a sequence of real numbers.

(a) **(3 marks)** Define what it means for $\{x_n\}$ to converge to a limit $L \in \mathbb{R}$.

(b) **(10 marks)** Show that if $\{x_n\}$ converges to the limits $L \in \mathbb{R}$ and to $M \in \mathbb{R}$ then $L = M$.

(c) **(6 marks)** Let $a < b$. Prove the following theorem using any method you wish:
Theorem: If $\{x_n\}$ is a monotonically increasing sequence of points in $(a, b]$, then $\{x_n\}$ converges to a point of $(a, b]$.

(d) **(6 marks)** Show that the converse of the theorem in part (c) is false.

(3) **(16 marks)** Here is an unfinished proof of the following theorem:

Theorem: If $\{x_n\}$ and $\{y_n\}$ are bounded sequences then

$$\liminf\{x_n + y_n\} \geq \liminf\{x_n\} + \liminf\{y_n\}.$$

Proof: Let $n \in \mathbb{N}$. For each $t > n$, we have $x_t \geq \inf_{k>n}\{x_k\}$ and $y_t \geq \inf_{k>n}\{y_k\}$. Therefore, $x_t + y_t \geq \inf_{k>n}\{x_k\} + \inf_{k>n}\{y_k\}$. Therefore, the number $r = \inf_{k>n}\{x_k\} + \inf_{k>n}\{y_k\}$ is a lower bound for the set $\{x_t + y_t : t > n\}$...

(a) **(9 marks)** Finish the proof of the theorem.

(b) **(7 marks)** Give an example of bounded sequences $\{x_n\}$ and $\{y_n\}$ such that $\liminf\{x_n + y_n\} \neq \liminf\{x_n\} + \liminf\{y_n\}$. **[END OF PAPER]**