A geometric approach to the conjugacy problem for semisimple Lie groups

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Lemma

Γ *finitely generated with solvable WP,* $|\cdot|$ *word length. Then:*

Conjugacy problem is solvable $\iff$ CLF_Γ *is recursive.*

Example: free groups

F free group, finite generating set X.

u, v reduced words on $X \cup X^{-1}$.

e.g. $u = aabbbaba^{-1}$
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The conjugator will be a product of subwords of u and v. Hence

$$\mathrm{CLF}_F(x) \leq x.$$

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$g = bababa^{-1}$

$v = gug^{-1}$

State of the art

Known results include:

Class of groups	$\mathrm{CLF}(x)$	
Hyperbolic groups	linear	Bridson–Haefliger
$\mathrm{CAT}(0)$ and biautomatic groups	$\preceq \exp(x)$	Bridson–Haefliger
RAAGs & special subgroups	linear	Crisp–Godelle–Wiest
2-Step Nilpotent	quadratic	Ji–Ogle–Ramsey
$\pi_1(M)$ where M prime 3–manifold	$\preceq x^2$	Behrstock–Druțu, S
Free solvable groups	$\preceq x^3$	S

Plus:

wreath products (S),
group extensions (S),
relatively hyperbolic groups (Ji–Ogle–Ramsey, Z. O'Conner, Bumagin).

State of the art, continued

Mapping class groups

S connected, oriented surface of genus g and p punctures.

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Question: What about for arithmetic groups? Or $\mathrm{Out}(F_n)$?

Semisimple Lie groups

G real semisimple Lie group, finite centre and no compact factors.

d_G left-invariant Riemannian metric.

$X = G/K$ associated symmetric space.

$\Gamma < G$ non-uniform lattice.

e.g. $\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$ and $X = \mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}(n)$.

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Jordan decomposition:

Each $g \in G$ has unique decomposition as

$$g = su$$

where:

- s is semisimple (translates along an axis in X);
- u is unipotent (fixes a point in the boundary of X),

and s, u commute.

Complete Jordan decomposition:

Each $g \in G$ has unique decomposition as

$$g = kau$$

where:

- k is elliptic
- a is real hyperbolic
- u is unipotent

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Slope

Let $a \in G$ be real hyperbolic. The *slope* of a tells you the location of translated geodesics in Weyl chambers. (It lies in $\partial X/G$).

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Theorem (S '14)

Fix slope ξ. Then there exists $d_\xi, \ell_\xi > 0$ such that for $a, b \in G$ real hyperbolic of slope ξ and such that $|a|, |b| > d_\xi$

Note: $|a| = d_G(1, g)$

Conjugacy of real hyperbolic elements

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a *is conjugate to* $b \iff \exists\, g \in G$ *such that (i)* $ga = bg$ *and*
(ii) $|g| \leq \ell_\xi(|a| + |b|)$.

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Consequence for lattices

Assume G is higher rank and $\Gamma < G$ is an irreducible lattice.

Corollary

Fix a slope ξ. Then there exists $\ell_\xi > 0$ such that $a, b \in \Gamma$, real hyperbolic of slope ξ, are conjugate if and only if there is a conjugator $g \in G$ such that

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If $Z_\Gamma(a)$ is virtually $\mathbb{Z}$, then g can be "pushed" to a conjugator γ in Γ, retaining the linear bound on its length.

Idea of proof

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$$\mathrm{Min}(a) := \left\{ x \in X \mid d(x, ax) = \inf_{y \in X} d(y, ay) \right\}$$

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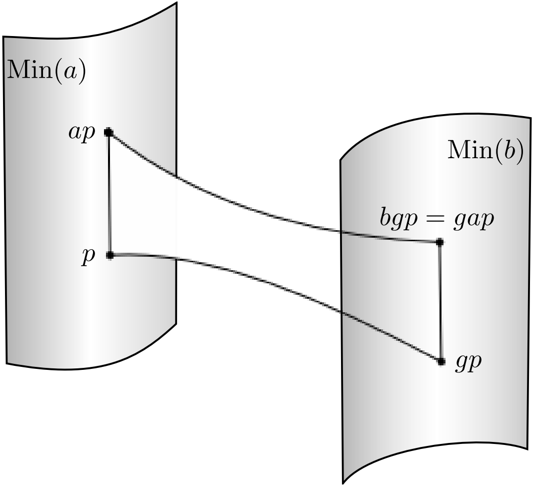
Lemma

- *If $ga = bg$ then $g\,\mathrm{Min}(a) = \mathrm{Min}(b)$;*
- *if $g\,\mathrm{Min}(a) = \mathrm{Min}(b)$ then $\exists\, k \in G$ fixing a point in $\mathrm{Min}(a)$ such that*

$$(gk)a = b(gk).$$

Idea of proof, continued



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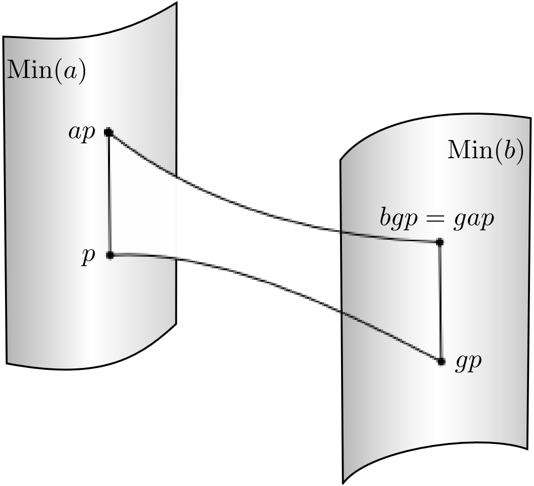

Minimal distance between the flats is important — corresponds to length of shortest conjugator.

Thank you for your attention!