Geometry of the conjugacy problem

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1. Conjugacy length function
   - Bounds length of short conjugators.
2. Permutation conjugacy length function
   - Inspired by fast solutions to the conjugacy problem in hyerbolic and relatively hyperbolic groups (Bridson–Howie, Epstein–Holt, Bumagin).

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Lemma

Γ *finitely generated with solvable WP,* $|\cdot|$ *word length. Then:*

Conjugacy problem is solvable $\iff$ CLF_Γ *is recursive.*

Example: free groups

F free group, finite generating set X.

u, v reduced words on $X \cup X^{-1}$.

e.g. $u = aabbbaba^{-1}$
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The conjugator will be a product of subwords of u and v. Hence

$$\mathrm{CLF}_F(x) \leq x.$$

$g = bababa^{-1}$

$v = gug^{-1}$

State of the art

Known results include:

Class of groups	$\mathrm{CLF}(x)$	
Hyperbolic groups	linear	Lysenok
$\mathrm{CAT}(0)$ & biautomatic groups	$\preceq \exp(x)$	Bridson–Haefliger
RAAGs & special subgroups	linear	Crisp–Godelle–Wiest
Mapping class groups	linear	Masur–Minsky; Behrstock–Druțu; J. Tao.
2-Step Nilpotent	quadratic	Ji–Ogle–Ramsey
$\pi_1(M)$, M prime 3–manifold	$\preceq x^2$	Behrstock–Druțu, S
Free solvable groups	$\preceq x^3$	S

Plus:

wreath products (S),
group extensions (S),
relatively hyperbolic groups (Ji–Ogle–Ramsey, Z. O'Conner, Bumagin).

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$\exists$ cyclic permutations u', v' of u, v and $g \in G$ such that

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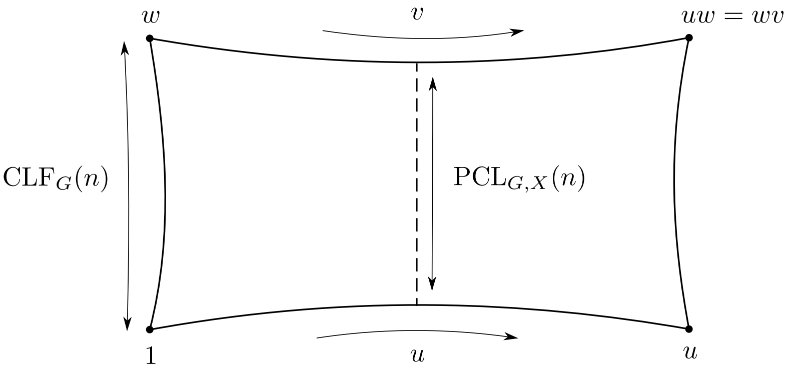
(ii) $|g| \leq \mathrm{PCL}_{G,X}(n)$.

e.g. For a free group $\mathrm{PCL} = 0$.

Relationship to CLF:

$$\mathrm{PCL}_{G,X}(n) \leq \mathrm{CLF}_G(n) \leq \mathrm{PCL}_{G,X}(n) + n.$$



Sublinear PCL

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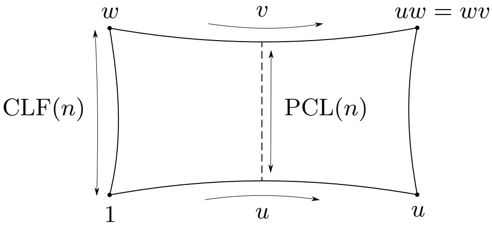

If $\mathrm{PCL}_{G,X}(n) \leq K$ for all n, then conjugacy problem is almost as fast as word problem: (on input geodesic words).

Apply the word problem n^2 times, on words of length $n + 2K$, where n is the sum of the length of the input words.

Relatively hyperbolic groups

Theorem (Antolín–S '15)

Let G be hyperbolic relative to a finite collection of subgroups $\{H_\omega\}_{\omega\in\Omega}$. There exists a finite generating set X such that $\langle X \cap H_\omega \rangle = H_\omega$ and

$$\mathrm{PCL}_{G,X}(n) \preceq \max_{\omega\in\Omega}\left\{ \mathrm{PCL}_{H_\omega, X\cap H_\omega}(n)\right\}.$$

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In particular, hyperbolic groups and groups that are hyperbolic relative to abelian groups will all have PCL bounded by a constant.

Consequences of constant PCL

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1. **Potentially fast algorithm** to solve the conjugacy problem.
2. Exponential conjugacy growth rate controlled by exponential growth rate.
3. (Ciobanu-Hermiller-Holt-Rees)
   ConjGeo(G, X) is a regular language whenever either
   - Geo(G, X) has a biautomatic structure,
   - (G, X) has falsification by fellow traveller property.

G hyperbolic. Take u, v geodesic words, conjugate in G. Cyclic permutations $u' = u_2u_1$, $v' = v_2v_1$ and $v'w = wu'$ with $|w|$ minimal. Let w_i be prefix of w.

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Claim: If $4\delta < i < |w| - 4\delta$ then $d(w_i, u'w_i) < 8\delta$.
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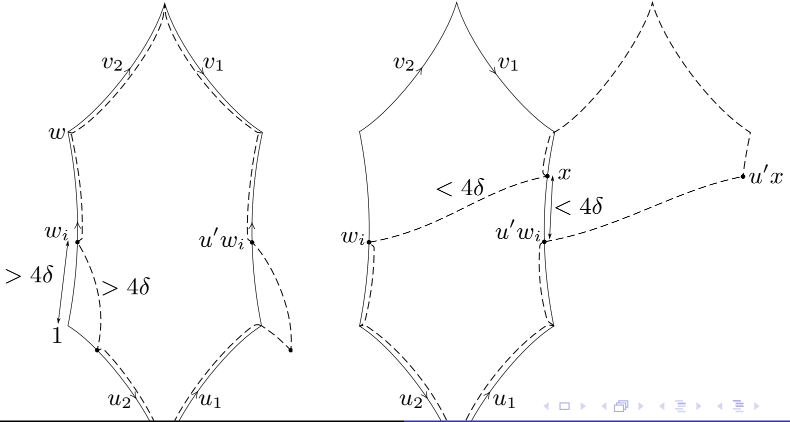

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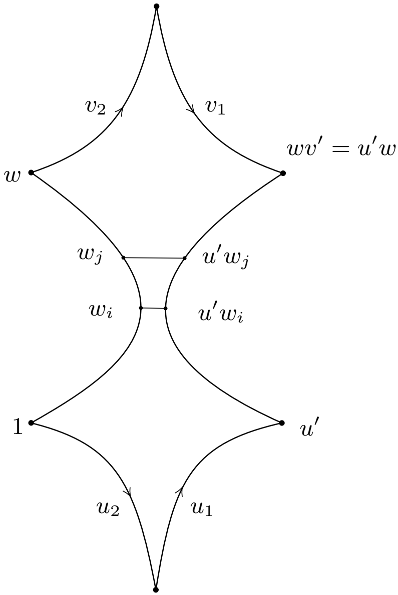

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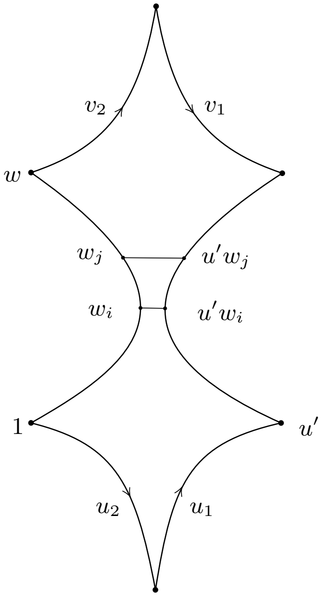

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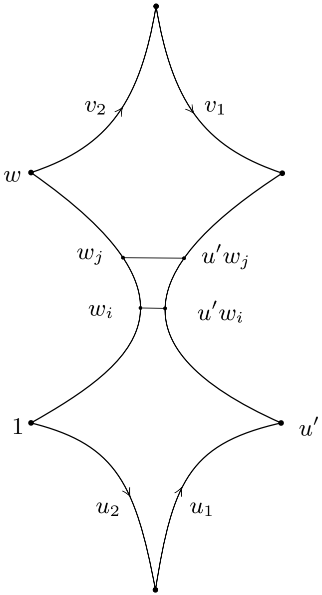

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So $w_i^{-1}v'w_i$ are distinct.

$\implies |w| \leq 8\delta + B_X(8\delta)$.

$\implies \mathrm{PCL}_{G,X}(n) \leq 8\delta + B_X(8\delta)$.

Yago says:

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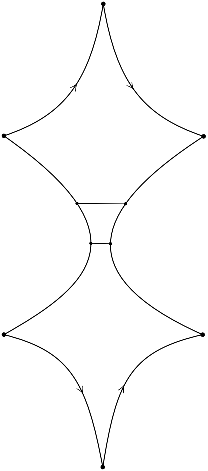

Thank you for your attention!