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   - Bounds length of short conjugators.

2. **Permutation conjugacy length function**
   - Inspired by fast solutions to the conjugacy problem in hyperbolic and relatively hyperbolic groups (Bridson–Howie, Epstein–Holt, Bumagin).

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Geometry of the conjugacy problem
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Conjugacy Length Function

$G$ group with length function $|\cdot| : G \rightarrow [0, \infty)$
(e.g. word length if finitely generated).

Definition (Conjugacy length function)

CLF$_G : [0, \infty) \rightarrow [0, \infty)$ minimal function satisfying:

For $x \geq 0$, $u, v \in G$ such that $|u| + |v| \leq x$, then $u$ is conjugate to $v$ $\iff$ $\exists g \in G$ such that (i) $gug^{-1} = v$ and (ii) $|g| \leq \text{CLF}_G(x)$.

Lemma

If $\Gamma$ finitely generated with solvable WP, $|\cdot|$ word length. Then:

Conjugacy problem is solvable $\iff$ CLF$_\Gamma$ is recursive.

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Example: free groups

$F$ free group, finite generating set $X$.

$u, v$ reduced words on $X \cup X^{-1}$.

e.g. $u = aabbbaba^{-1}$
$v = babababba^{-1}b^{-1}$
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(i) Cyclically reduce \( u, v \) to \( u', v' \),

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(i) Cyclically reduce $u, v$ to $u', v'$,

(ii) Cyclically conjugate $u'$ to $v'$.

The conjugator will be a product of subwords of $u$ and $v$. Hence

$$\text{CLF}_F(x) \leq x.$$
### State of the art

**Known results include:**

<table>
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<tr>
<th>Class of groups</th>
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<tr>
<td>Hyperbolic groups</td>
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<tr>
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<td>$\leq \exp(x)$</td>
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<td>$\pi_1(M)$, $M$ prime 3–manifold</td>
<td>$\leq x^2$</td>
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<td>Free solvable groups</td>
<td>$\leq x^3$</td>
<td>S</td>
</tr>
</tbody>
</table>

**Plus:**

- wreath products (S),
- group extensions (S),
- relatively hyperbolic groups (Ji–Ogle–Ramsey, Z. O’Conner, Bumagin).
$G$ group, $X$ (finite) generating set, $|\cdot|$ word length.

**Definition (Permutation conjugacy length function)**

$PCL_{G,X} : \mathbb{N} \to \mathbb{N}$ minimal function satisfying:

(i) For geodesic words $u, v$ on $X$ such that $|u| + |v| \leq n$, then $u, v$ represent conjugate elements of $G$ iff there exist cyclic permutations $u', v'$ of $u, v$ and $g \in G$ such that

$$gu'g^{-1} = v' \quad \text{and} \quad |g| \leq PCL_{G,X}(n).$$

(ii) For a free group $PCL = 0$. 

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\end{align*}
\]

e.g. For a free group \( \text{PCL} = 0 \).
Relationship to CLF:

\[ PCL_{G,X}(n) \leq CLF_G(n) \leq PCL_{G,X}(n) + n. \]
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\[ \text{PCL}_{G,X}(n) \leq \text{CLF}_G(n) \leq \text{PCL}_{G,X}(n) + n. \]

If \( \text{PCL}_{G,X}(n) \leq K \) for all \( n \), then conjugacy problem is almost as fast as word problem: (on input geodesic words).

Apply the word problem \( n^2 \) times, on words of length \( n + 2K \), where \( n \) is the sum of the length of the input words.
Theorem (Antolín–S ’15)

Let $G$ be hyperbolic relative to a finite collection of subgroups \{${H_\omega}\}_{\omega \in \Omega}$. There exists a finite generating set $X$ such that $\langle X \cap H_\omega \rangle = H_\omega$ and

$$\text{PCL}_{G,X}(n) \leq \max_{\omega \in \Omega} \left\{ \text{PCL}_{H_\omega,X \cap H_\omega}(n) \right\}.$$
Relatively hyperbolic groups

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In particular, hyperbolic groups and groups that are hyperbolic relative to abelian groups will all have PCL bounded by a constant.
Suppose \( \text{PCL}_{G,X}(n) \leq K \).

1. **Potentially fast algorithm** to solve the conjugacy problem.
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1. **Potentially fast algorithm** to solve the conjugacy problem.

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3. (Ciobanu-Hermiller-Holt-Rees) $\text{ConjGeo}(G, X)$ is a regular language whenever either
   
   - $\text{Geo}(G, X)$ has a biautomatic structure,
   - $(G, X)$ has falsification by fellow traveller property.
G hyperbolic. Take $u, v$ geodesic words, conjugate in $G$. Cyclic permutations $u' = u_2u_1$, $v' = v_2v_1$ and $v'w = wu'$ with $|w|$ minimal. Let $w_i$ be prefix of $w$. 

Claim: If $4\delta < i < |w| - 4\delta$ then $d(w_i, u'w_i) < 8\delta$. Use that geodesic hexagons are $4\delta$–thin.
$PCL_{G,X}(n) < K$ for hyperbolic groups

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If $w_{j-1}v_{i+1}w_j = w_{i-1}v_{i+1}w_j$ then cut middle chunk out of diagram:

obtains shorter conjugator.

So $w_{j-1}v_{i+1}w_j$ are distinct.

$\Rightarrow |w| \leq 8\delta + B_{X}(8\delta)$.

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$$\implies \text{PCL}_{G,X}(n) \leq 8\delta + B_X(8\delta).$$
Yago says:
Yago says: “Please don’t feed the hexagons!!”

Thank you for your attention!