Andrew Sale **Research Statement**

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My research is in the area of **geometric group theory**. The basic principal is to investigate the nature of infinite groups using *geometric* and *topological* methods. An example where geometric group theory has been vital is in Agol's famous proof of the Virtual Haken Conjecture [12], completing the plan envisaged by Thurston to understand 3–manifolds. Agol's proof, and the theory it uses that was developed by Haglund, Wise and others, uses much of the insight that Gromov brought to geometric group theory in the 1980's.

Introduction to geometric group theory. Historically, the area developed from *combinatorial group theory*, and there remains a strong combinatorial influence. For example, finite rank free groups F_n have long been of key interest, and from this came a need to understand the *automorphisms* of the free group, denoted $\mathrm{Aut}(F_n)$. Earlier methods for handling $\mathrm{Aut}(F_n)$, or its quotient group $\mathrm{Out}(F_n)$ of outer automorphisms, were very combinatorial. However recent geometric techniques, particularly the introduction of the "outer space" of the free group by Culler and Vogtmann in the 80's, have accelerated research progress in this area.

This outer space provides us with a geometric object on which the group $\mathrm{Out}(F_n)$ acts in a nice way. In general, finding a space for a group to act on can provide a powerful tool, as the group can often be better understood by looking at the geometry of the space, and vice versa.

Sometimes, it can be helpful enough just to endow your group with a metric. One way to do this is via the *Cayley graph*. Let G be a group and X a finite generating set. The vertices of the Cayley graph with respect to X are in bijection with G itself, and two edges g, h are connected if there exists $x \in X \cup X^{-1}$ such that $g = hx$. The graph metric on the Cayley graph then gives a *word metric* on G.

Another key example of the power of geometry in group theory comes from Gromov's *hyperbolic groups* [23]. Hyperbolic groups are a geometric generalization of free groups, with which they share many properties. A metric space is δ–hyperbolic for some $\delta > 0$ if geodesic triangles are δ–thin: any side of the triangle is contained in the δ–neighborhood of the other two sides (Figure 1). A group is δ–hyperbolic if, with respect to some finite generating set, its Cayley graph is δ–hyperbolic

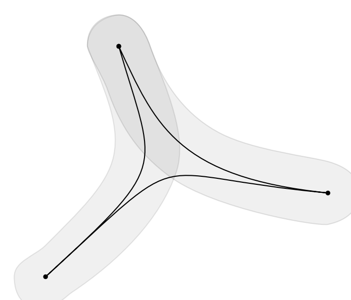

FIGURE 1. A δ–thin triangle.

Among the properties enjoyed by hyperbolic groups is that computations in the group are efficient. This can be expressed, for example, by the fact that the *word problem* in hyperbolic groups has a linear-time solution. The word problem is a classic algorithmic problem in group theory, originally posed by Dehn over a century ago. It asks, given a finite presentation $\langle X \mid R \rangle$ of a group, is there an algorithm that determines, on input a word w on $X \cup X^{-1}$, whether w represents the identity element in the group or not.

Dehn also posed the *conjugacy problem*, which generalizes the word problem, instead asking for an algorithm that determines when two input words represent conjugate elements in the group. Understanding solutions, or indeed solubility, of the word and conjugacy problems has been a constant theme for decades, and finding new geometric techniques has been a core motivation for geometric group theorists.

A brief introduction to my research. My research program has three main strands. After a brief introduction, I discuss each in more detail below.

1. Automorphism groups of graph products. The most common example of graph products, right-angled Artin groups (RAAGs), have become recognised as a very important class of groups thanks to their fundamental involvement in Agol's proof of the Virtual Haken Conjecture [12]. Examples of RAAGs include free groups F_n and free abelian groups $\mathbb{Z}^n$, and one way to view RAAGs is as a bridge in the space of finitely presented groups between these extreme examples. I am interested in their (outer) automorphism groups, which we can think of as spanning a range of groups with $\mathrm{Out}(F_n)$ and $\mathrm{GL}(n, \mathbb{Z})$ at the two extremes.

I have studied the question of how *vast* these groups are, and in work with V. Guirardel, gave a definitive answer for when they are vast and when they are not. Currently on-going, we are investigating when they are linear groups—whether they embed into $\mathrm{GL}(n, \mathbb{C})$ for some n.

A second important class of graph products are right-angled Coxeter groups (RACGs). These arise in situations involving reflections in vector spaces, and their algebraic, topological, and geometric properties have been much studied.

With T. Susse, we proved a striking dichotomy concerning their "vastness," and are working towards proving they satisfy the Tits Alternative—whether every subgroup either contains a copy of F_2, or has a finite-index solvable subgroup. The Tits Alternative is named after Jacques Tits, who proved the alternative holds for finitely generated linear groups This was a classic and fundamentally important result, as it provides a potential obstruction to linearity, and also spurred on the search for other classes of groups where it holds.

2. The geometry of the conjugacy problem. Interest in understanding geometric aspects of the conjugacy problem have grown recently, notably that of *conjugacy length*, which has found applications, such as in a version of the Bass conjecture (which is related to the Baum–Connes conjecture) [27], and to the study of formal languages [14].

Furthermore, geometric techniques often appear in algorithms solving the conjugacy problem. With proposed applications to cryptography, stemming from the initial paper of Anshel–Anshel–Goldfeld [13], interest in the *complexity* of such algorithms has grown significantly.

A common theme in my research has been to study the conjugacy length function (CLF), introduced in my thesis [10] (though the ideas had been floating around beforehand). This is a notion that geometrically quantifies the conjugacy problem. I have obtained bounds in many classes of groups, and in some used it as a tool to give fast algorithms [5, 6, 7, 9]. An upper bound on CLF gives a naive algorithm to solve the conjugacy problem, however this algorithm is not fast (it typically runs in exponential time). My recent work in this area is aimed at using geometry to give faster (polynomial time) algorithms, introducing the Permutation Conjugacy Length function with Y. Antolín and studying it in relatively hyperbolic groups [1].

3. Group extensions, particularly wreath products. When considering a group property, an important question is what permanence properties does it satisfy. A particularly important type of extension is a wreath product of two groups, as it is often involved in constructions of group embeddings or of new groups that provide counter-examples to conjectures. I have looked at several properties, including conjugacy length, and asked whether they hold under general group extensions, or specifically wreath products [7, 9].

Most recently, B. Hayes and I proved that the wreath product of two sofic groups is again sofic [3]. Sofic groups are of much interest as one is able to prove a wide array of conjectures in various areas of mathematics under the assumption of soficity, such as in ergodic theory, topological dynamics, and group rings (see for example [22, 28]).

1. Automorphism groups of graph products

The two main examples of graph products are right-angled Artin groups and right-angled Coxeter groups (RAAGs and RACGs). Both are defined by using a simplicial graph Γ, giving a RAAG, denoted A_Γ, and a RACG, denoted W_Γ. In both cases, the vertex set of Γ serves as a generating set for the group, with two vertices commuting as group elements when they are connected by an edge. RACGs require the extra relations that each vertex has order 2.

Thus, when Γ is a complete graph, all generators commute and $A_\Gamma = \mathbb{Z}^n$, while $W_\Gamma = (\mathbb{Z}_2)^n$, where n is the number of vertices of Γ. On the other hand, when Γ has no edges we get $A_\Gamma = F_n$, the free group, and W_Γ is the n–fold free product $\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$.

I am interested in automorphism groups of graph products, particularly their outer automorphism groups $\mathrm{Out}(A_\Gamma)$ and $\mathrm{Out}(W_\Gamma)$. This is obtained by taking the quotient of the group of automorphisms by the normal subgroup consisting of the inner automorphisms—those that are given by conjugating by an element in the group.

In the case of the RAAGs, the two extreme examples $\mathrm{Out}(\mathbb{Z}^n) \cong \mathrm{GL}(n,\mathbb{Z})$ and $\mathrm{Out}(F_n)$ are both hugely important classes of group. They share many properties, and analogies are often drawn between them. However there are keys areas where they have fundamental differences.

1.1. Vastness properties.

A group G is said to be *large* if it has a finite index subgroup H and an epimorphism $H \twoheadrightarrow F_2$. Being large has notable consequences, for example it says the group does not satisfy Kazhdan's Property (T). Another way to interpret being large is to say that every finitely generated group occurs as a quotient of some finite-index subgroup of G. Roughly, it says that G has got many quotients.

For groups $\mathrm{GL}(n,\mathbb{Z})$, with $n \geq 3$, there are classic results that tell us their finite index subgroups and their normal subgroups are heavily restrained. The implication being that they do not have many quotients, and in particular are not large.

When considering $\mathrm{Out}(F_n)$, for $n \geq 4$, whether or not they are large or satisfy Property (T) are both important open questions. When $n = 2$, $\mathrm{Out}(F_2)$ is isomorphic to $\mathrm{GL}(2,\mathbb{Z})$, which has a finite index free subgroup, hence is large. For $n = 3$, Grunewald and Lubotzky constructed a family of representations of finite-index subgroups of $\mathrm{Out}(F_n)$ with arithmetic images [24]. For $\mathrm{Out}(F_3)$ they get a representation with image $\mathrm{GL}(2,\mathbb{Z})$, implying $\mathrm{Out}(F_3)$ is large.

With V. Guirardel we looked at the question of largeness for $\mathrm{Out}(A_\Gamma)$, and identified many cases where it is, and many where it is not [2]. What we also did was to study properties that are slightly weaker than that of being large. For example, we say a group G has *all finite groups involved* (AFGI) if for every finite group F there is a finite-index subgroup H and an epimorphism $H \twoheadrightarrow F$. We called having AFGI, or the other properties we considered, *vastness properties*, with the ultimate vastness property is being large.

The representations of Grunewald and Lubotzky [24] imply that $\mathrm{Out}(F_n)$ always has AFGI, while the same classic results imply $\mathrm{GL}(n,\mathbb{Z})$ does not, for $n \geq 3$. Using a mix of topological, combinatorial, and group theoretic techniques, with Guriardel we obtained the following.

Theorem 1 (Guirardel–S. [2])**.** *There is a criterion on simplicial graphs Γ that determines whether $\mathrm{Out}(A_\Gamma)$ has AFGI. Furthermore, when it does not have AFGI, there is a short exact*

$$1 \to N \to \mathrm{Out}^0(A_\Gamma) \to \prod \mathrm{SL}(n_i,\mathbb{Z}) \to 1$$

where N is finitely generated nilpotent, $\mathrm{Out}^0(A_\Gamma)$ has finite index in $\mathrm{Out}(A_\Gamma)$, and the product in the quotient is finite and each n_i is at least 3.

The short exact sequence strongly restricts the structure of $\mathrm{Out}(A_\Gamma)$ when it does not have AFGI, giving us a dichotomy between those $\mathrm{Out}(A_\Gamma)$ that are vast, and those which are not.

We also apply methods from the proof of Theorem 1 to similar questions for *McCool groups*: subgroups of $\mathrm{Out}(F_n)$ consisting of automorphisms that preserve a given list of conjugacy classes. We show such a group is either large, surjects onto $\mathrm{Out}(F_n)$, surjects onto a mapping class group, or is virtually abelian. In particular we are able to conclude the following:

Theorem 2 (Guirardel–S. [2])**.** *Any McCool group of a free group either has AFGI, or is virtually abelian.*

I have studied a similar question of vastness for the RACG case with T. Susse. Here we also found a dichotomy occurring between those Γ for which $\mathrm{Out}(W_\Gamma)$ is vast, and those where it is not. Here though the dichotomy is more striking:

Theorem 3 (S.–Susse [11])**.** *There is a criterion on* Γ *that determines whether or not* $\mathrm{Out}(W_\Gamma)$ *is large. Furthermore, if it is not large it has a finite-index subgroup that is abelian.*

The two sides of the dichotomy are more extreme than the RAAG case. The reason for this is that to generate a finite-index subgroup of $\mathrm{Out}(A_\Gamma)$ we need two types of generator: partial conjugations and transvections. For $\mathrm{Out}(W_\Gamma)$, we only need partial conjugations. One way of interpreting Theorems 1 and 3 is that, in some sense, the transvections give the automorphism group more varied structures.

1.2. **Future work.**

1.2.1. *Linearity.* A linear group is any group that embeds into $\mathrm{GL}(n,\mathbb{C})$ for some $n \in \mathbb{N}$. It is shown by Formanek and Procesi [20] that $\mathrm{Out}(F_n)$ is not linear for $n \geq 4$. The question once again becomes, where is the boundary between linear and non-linear?

Aramayona and Martínez-Pérez [15] have used poison subgroups, a notion from [20], to show that for certain Γ, $\mathrm{Aut}(A_\Gamma)$ cannot be linear. We can modify their hypothesis on Γ to give a (slightly narrower) class of graphs for which $\mathrm{Out}(A_\Gamma)$ is not linear. (The modification is necessary to ensure that the conditions of the poison subgroup of [20] carry through to the outer automorphism group).

At the other end of the spectrum, using techniques similar to those of Grunewald and Lubotzky for automorphisms of free groups [24], with Guirardel we are working on constructing faithful representations for finite index subgroups of $\mathrm{Out}(A_\Gamma)$ whenever Γ does not contain a SIL.

This deals with many cases, but leaves a big gap in the middle where linearity is still open.

Question 1. For any graph Γ, can we either prove linearity or non-linearity of $\mathrm{Out}(A_\Gamma)$, or can we at least reduce it to the question of whether $\mathrm{Out}(F_3)$ is linear?

The plan is to construct embeddings of finite index subgroups of $\mathrm{Out}(A_\Gamma)$ into products of groups $\mathrm{Out}(A_\Lambda)$, where each Λ is smaller than Γ (in terms of the number of vertices). This we expect will allow us to reduce it to the question of linearity for $\mathrm{Out}(F_3)$, which remains open.

1.2.2. *Tits alternative.* Jacques Tits proved that every finitely generated linear group either contains a non-abelian free group or has a solvable subgroup of finite index [39]. This is a very important result, and has been used to prove non-linearity and generalized to other classes of groups. Nowadays, we say a group G satisfies the Tits Alternative if the same statement holds for every subgroup of G. Horbez proved that $\mathrm{Out}(A_\Gamma)$ satisfies the Tits Alternative [26].

With T. Susse, I am investigating the same question for RACGs:

Question 2. For any Γ, does $\mathrm{Out}(W_\Gamma)$ satisfy the (stronger) Tits alternative: each subgroup either contains a non-abelian free group or is virtually abelian?

We note that the result holds at both extreme examples, when Γ has no edges or when it is a complete graph. Also, thanks to Horbez, we need only consider those Γ that are connected.

We believe this stronger version of the Tits alternative should hold because the known examples of non-abelian solvable subgroups of $\mathrm{Out}(A_\Gamma)$ come about from transvections, which we no longer need to consider for a RACG.

Once the Tits alternative is dealt with, we have a host of further problems to attack. For example, which $\mathrm{Out}(W_\Gamma)$ are linear; which are themselves virtually a RACG (or quasi-isometric to, or commensurable to); when are they hyperbolic; when are they of type FP_∞?

1.2.3. *Arithmetic quotients and largeness.* Grunewald and Lubotzky's representations [24] provided a big step forward in understanding quotients of $\mathrm{Out}(F_n)$ and its finite index subgroups, and proving largeness of $\mathrm{Out}(F_3)$ was a very significant result. They used the homology of certain covers of the bouquet of n circles to produce representations of finite-index subgroups of $\mathrm{Out}(F_n)$ with arithmetic image.

Generalizing this to RAAGs means taking certain covers of the Salvetti complex for A_Γ, and exploiting its homology (which needs to be calculated). In [2] we have done this already when $A_\Gamma = \mathbb{Z}^a * \mathbb{Z}^b * \mathbb{Z}^c$ for some $a, b, c \geq 1$, when proving Theorem 1, and in particular largeness of $\mathrm{Out}(A_\Gamma)$ for many graphs Γ. We would like to know if we generalize these representations further, can we get largeness for more graphs:

Question 3. For which graphs Γ can we obtain a cover of the Salvetti complex which gives rise to representation whose image is commensurable to $\mathrm{GL}(2, \mathbb{Z})$?

A different question one may ask concerns the kernel of such representations: can every automorphism of a RAAG be seen to act on the integer homology of some finite cover of the Salvetti complex? In particular we can ask the following, inspired by a similar result of Koberda for mapping class groups [29].

Question 4. Fix a graph Γ. Is there a sequence of finite covers X_i of the Salvetti complex for A_Γ so that every automorphism $\varphi \in \mathrm{Aut}(A_\Gamma)$ acts non-trivially on $H_1(X_i, \mathbb{Z})$, for some i?

2. The geometry of the conjugacy problem and finding efficient algorithms

The conjugacy problem was introduced over 100 years ago by Dehn and it remains a fundamental algorithmic question in group theory. It asks whether, given a group G with finite generating set X, there is an algorithm that decides when two input words u, v on $X \cup X^{-1}$ represent conjugate elements in G, i.e. whether there is some $w \in G$ such that $uw = wv$.

The word length of an element $g \in G$, denoted $|g|$ is equal to the minimal length of a word on $X \cup X^{-1}$ that represents g. (This naturally defines a metric on G, which agrees with the metric on the Cayley graph of G with generating set X.) The *conjugacy length function* of G is the minimal function

$$\mathrm{CLF}_G : \mathbb{N} \to \mathbb{N}$$

such that for each $n \geq 0$, two elements $u, v \in G$ with $|u| + |v| \leq n$ are conjugate in G if and only if there is an element $g \in G$ with $gu = vg$ and $|g| \leq \mathrm{CLF}_G(n)$.

Note that changing the generating set will not change the asymptotic behaviour of the conjugacy length function. The above definition can naturally be extended to any group, replacing the word-length by a length function on G.

The conjugacy length function has been determined in a number of cases. Examples include:

- hyperbolic groups, where it is linear [30];
- mapping class groups, where it is linear [16, 32, 38];
- right-angled Artin groups, where it is linear (this follows from [37]);
- CAT(0)–groups and biautomatic groups, where it is at most exponential [17], though there is no case known where it is not linear;
- nilpotent groups, where it is at most polynomial [27, 31].
- solvable groups may have non-recursive conjugacy length function [25];

My contributions are summarized below.

- In relatively hyperbolic groups the CLF is linear, modulo the CLF of the parabolic subgroups, joint with Y. Antolín [1] (this extends a previous result [27]).
- There are partial results for semisimple Lie groups (using a length function obtained from the left-invariant Riemannian metric) and their lattices, particularly when focusing on either hyperbolic or unipotent elements where the CLF exhibits linear bounds (with some caveats) [6].
- In free solvable groups the CLF is at most cubic [9].
- In various metabelian groups, including Baumslag-Solitar groups and lamplighter groups, it is linear, while in others it is at most exponential [7, 5].
- The behaviour of the CLF under wreath products is well-understood [9].
- For general group extensions, the conjugacy problem is badly behaved and solubility may not pass even to/from index 2 subgroups [21, 18]. However, a bound on the conjugacy length function can be determined in general [7].

2.1. **Which functions can be conjugacy length functions?** The conjugacy length function quantifies the conjugacy problem in a geometric way, a notion shared in the relationship between the Dehn function and the word problem (another of Dehn's decision problems). Some of the major advances in geometric group theory during and since the 1990's involved understanding what functions can be Dehn functions of a group. The paper of Sapir, Birget and Rips [36] gives deep connections between Dehn functions and computational complexity, concluding that many functions can indeed be Dehn function. We ask the same question of conjugacy length functions.

Question 5. What functions can occur as conjugacy length functions of finitely presented groups?

This question is comparable the aforementioned result for Dehn functions, but is also closely related to the question of which functions can be distortion functions for subgroups of finitely presented groups. Olshanskii proved that any function satisfying basic conditions can be such a distortion function [34].

Looking at the current list of known conjugacy length function, we have only a short list. In abelian groups it is trivially zero. In many groups it is known to be linear. In some groups it is known to be quadratic (Heisenberg group). But here the list stops. Part of the problem is that in most cases only an upper bound is found. There are groups, such as CAT(0)–groups, for which we can only prove an exponential upper bound [17], and have no evidence that it should be anything more than linear.

In an on-going project with M. Bridson and T. Riley, we use carefully constructed group extensions to turn subgroup distortion functions into conjugacy length functions. In view of Olshanskii's result [34], this is very promising. However, conjugacy in groups has very subtle, delicate behaviour, and we need a good understanding of the input groups.

One class of groups we are using in this are a family of free-by-cyclic groups. Their conjugacy length function is currently unknown, but we conjecture that for this family it is linear. When we plug these groups into our construction we will get, for each integer n, a group with polynomial conjugacy length function of degree n. The complexity of algorithms solving the conjugacy problem for free-by-cyclic groups are currently not known.We ask the following:

Question 6. In the family of free-by-cyclic groups we consider, is there a polynomial-time algorithm solving the conjugacy problem?

Another class of groups we use are Baumslag–Solitar groups. These have linear conjugacy length function [7], but have a cyclic subgroup with exponential distortion. When plugged into our construction, we get the first group known to have exponential conjugacy length function.

2.2. **A more efficient method: the permutation conjugacy length function.** The naive algorithm solving the conjugacy problem that uses the conjugacy length function runs, typically, in at least exponential time (with respect to the length of the input words). Together with Y. Antolín, in [1] we introduced the notion of the permutation conjugacy length function (PCL). It uses ideas from efficient algorithms solving the conjugacy problem in hyperbolic and relatively hyperbolic groups.

The key difference between PCL and CLF is that for the PCL we are allowed to change the elements before looking for a short conjugator. We do this by taking the input as words u, v on a generating set, rather than elements. We are then allowed to consider all pairs of cyclic permutations of u, v, finding the length of the shortest conjugator among all such pairs. (A cyclic permutation of a word $w = x_1x_2 \dots x_m$ is a word of the form $x_{i+1} \dots x_mx_1 \dots x_i$.)

Regarding complexity of the conjugacy problem, the naïve algorithm associated to PCL, whereby each element in a ball of suitable radius is checked as to whether it is a conjugator for some pair of cyclic permutations of the input words, has the potential to run in up to cubic–time, relative to the length of the input words, provided there is a sufficiently fast solution to the word problem. In specific cases, the naïve algorithm can be tweaked to improve running-time, such as is the case for (relatively) hyperbolic groups.

A simple geometric argument tells us that CLF and PCL will agree unless CLF is at most linear. The first such example to consider would be a non-abelian free group F where CLF is linear, but PCL is zero. Antolín and I showed that this phenomenon extends to hyperbolic groups, and relatively hyperbolic groups, where PCL is bounded by a constant, but CLF is linear [1].

2.3. **Learning more about PCL and applying it.** The ability of PCL to give a fast algorithm solving the conjugacy problem is the core of its power. We want to learn more about PCL and how it behaves for different types of groups. Does its nice behaviour in the hyperbolic setting extend to other groups exhibiting negative curvature characteristics?

One class of groups I have been studying with Y. Antolín are RAAGs. Using the geometry of their associated CAT(0)–cube complex we can show that they have bounded PCL. Our plan is to use this geometry in a more general setting:

Question 7. Does the fundamental group of a special cube complex have bounded PCL?

We remark that such groups are already known to have fast (in fact, linear-time) solutions to the conjugacy problem [19].

The most prominent groups where complexity is unknown are perhaps *mapping class groups*. Given a closed, orientable surface S of finite type, the mapping class group $\mathrm{Mod}(S)$ is the group of homeomorphisms of S up to isotopy. It is a discrete, finitely generated group that captures, in a topological sense, the symmetries of S.

Mapping class groups are acylindrically hyperbolic, and in certain senses behave in a manner similar to relatively hyperbolic groups. I have been working with S. Dowdall on this problem, specifically considering the following:

Question 8. Do the ideas in [1] extend to mapping class groups? In particular, can we use PCL, restricted to pseudo-Anosov elements, to give a polynomial-time algorithm that determines whether or not two given pseudo-Ansov elements are conjugate?

We note that pseudo-Anosov elements are in a certain sense generic elements of $\mathrm{Mod}(S)$ [35], and we are further motivated to concentrate on them by the fact that the initial progress made on conjugacy length in $\mathrm{Mod}(S)$ was made for pseudo-Asnosov elements [32], before being later extended to all elements by Tao [38].

The main tools we use to study PCL for pseudo-Anosovs are the curve graph, a graph associated to the surface S that captures its topology in a discrete way, and hierarchies. The curve graph is known to be hyperbolic [33], however it is not locally finite. Masur–Minsky introduced the notion of hierarchies to help us understand the structure of the curve graph and its relationship to the mapping class group see [32], and their theory has since been further developed.

3. Group extensions, wreath products

3.1. **Soficity.** Sofic groups are of much interest as one is able to prove a wide array of conjectures in various areas of mathematics under the assumption of soficity, including in ergodic theory, topological dynamics, and group rings. However a crucial open problem is whether there exists a non-sofic group.

The notion of soficity simultaneously generalizes the concepts of amenability and residual finiteness. There are many classes of groups which are known to be sofic, but the number of results concerning permanence property are somewhat limited. Relatively straight-forward examples include closure under direct product and increasing unions, and the soficity of residually sofic groups. More substantial results generally require some amenability assumption. With B. Hayes, we showed that soficity is preserved by wreath products [3].

3.2. **The Magnus embedding.** The Magnus embedding takes a group given as a quotient of a free group F by the derived subgroup N' of a normal subgroup $N \triangleleft F$ and embeds it into a wreath product: $F/N' \hookrightarrow \mathbb{Z}^r \wr F/N$. The embedding is particularly powerful when studying free solvable groups, and I used it in particular to investigate the nature of the conjugacy length function for free solvable groups [9].

A crucial question I encountered concerned the geometric nature of the Magnus embedding. In particular I showed:

Theorem 4 (S. [8])**.** *For the natural choice of word metrics, the Magnus embedding is 2-bi-Lipschitz.*

3.3. **Width of groups.** In a different vein, with T. Riley, we investigated the width of wreath products of the form $G \wr \mathbb{Z}^r$ with respect to the subset consisting of palindromic words in a given generating set.

Given a finite generating set X for a group G, we say G has *finite palindromic width* with respect to X if there exists an integer k such that every $g \in G \wr \mathbb{Z}^r$ can be expressed as a product of at most k elements that can be realized as palindromic words on X.

Enticed by the lure of nice geometric techniques, we investigated this problem for $G \wr \mathbb{Z}^r$, and showed that when the palindromic width of G is finite with respect to a generating set X, then it is also finite for $G \wr \mathbb{Z}^r$ with respect to the natural generating set extending X [4]. We also looked at finitely generated metabelian groups, and finitely generated solvable groups satisfying max-n.

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