Question 1.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$. Which of the following expressions are meaningful? For each meaningful expression, state whether it corresponds to a vector or a scalar.

i. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

ii. $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$

iii. $(\mathbf{u} \times \mathbf{v})\mathbf{w}$

v. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

vi. $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

vii. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

vii. $\big((\mathbf{u} \times \mathbf{v}) \times \mathbf{w}\big) \times \mathbf{u}$

viii. $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$

Question 2.

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in $\mathbb{R}^3$.

(a) Show that the vector $\mathbf{v} - \dfrac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||^2}\mathbf{u}$ is perpendicular to $\mathbf{u}$.

(b) Show that $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\mathbf{u} \cdot \mathbf{v}$.

Question 3.

(a) Let Σ_1 be the plane containing the points $P = (1, 3, 2)$, $Q = (3, 9, 6)$, $R = (0, 3, 1)$. Give an equation for the plane in the form $ax + by + cz = k$ where a, b, c, k are constants to be found.

(b) Let Σ_2 be the plane given by $3x - 2y + z = 0$. Find the equation of the line of intersection of Σ_1 and Σ_2.

Question 4.

Express the ball of radius 1 centered at $(0, 0, 1)$, given by $x^2 + y^2 + (z - 1)^2 \leq 1$, using spherical coordinates.

Question 5.

For each of the following, either determine the value of the limit or show it does not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3 + x^3}{xy^2}$

Question 6.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = 2x^2 - 6x + y^2 + 2y + 7$.

(a) Compute the gradient vector ∇f of f.

(b) Consider the point $P = (4, -1)$

i. What is the directional derivative of f at P in direction $\langle 1, 1\rangle$?

ii. Give a *unit* vector in the direction that f is *decreasing* at the fastest rate at P.

Question 7.

Find an equation for the tangent plane to the surface $x^2 + 2y^2 - z^2 = 5$ at $P = (2, 1, 1)$.

Question 8.

Consider the curve $\mathbf{r}(t) = \left\langle 2t^3, \dfrac{3}{2}t^2, -t^3 \right\rangle$ for $t \geq 0$. Show that the arc length function of $\mathbf{r}(t)$ from $\mathbf{0}$ for $t \geq 0$, is given by $s(t) = \dfrac{1}{5}(5t^2 + 1)^{3/2} - \dfrac{1}{5}$.