Question 1.

Let $f : D \to \mathbb{R}$ be a scalar-value function defined on a domain D in $\mathbb{R}^3$; $\mathbf{F}$ be a vector field defined on $\mathbb{R}^3$; S be a smooth surface in $\mathbb{R}^3$.

For each of the following expressions, state whether they represent *scalars*, *vectors* or *nonsense*.

(a) $\nabla \times \mathbf{F}$

(b) $\operatorname{div} \mathbf{F} \operatorname{curl} \mathbf{F}$

(c) $\operatorname{div}(\operatorname{curl} \mathbf{F})$

(d) $\operatorname{curl}(\nabla f)$

(e) $\nabla(\operatorname{div}(\nabla f))$

Question 2.

For each of the following statements, determine if they are true or false, and give a justification.

(a) If (a, b) is a critical point of a function $f(x, y)$, then the directional derivatives $D_{\mathbf{u}} f(a, b)$ of f at (a, b) are all zero.

(b) Since the direction of maximum slope of $f(x, y)$ at a point (c, d) is given by $\nabla f(c, d)$, it follows that $\nabla f(c, d)$ is positive.

(c) Let C be a smooth, oriented curve lying in the x, y–plane (so its z–coordinate is zero), and let $\mathbf{F}(x, y, z) = \langle ze^x, xy\sin(z), x + y\cos(z)\rangle$. Then $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Question 3.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2y - xy^2 + xy$.

(a) Verify that $(-\frac{1}{3}, \frac{1}{3})$ is a critical point of f, and find the full list.

(b) Use the second derivative test to identify whether f has a local maximum, local minimum or saddle point at $(-\frac{1}{3}, \frac{1}{3})$.

Question 4.

Let $f(x, y, z) = 5x - 8y - 5z$. Using Lagrange multipliers, find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $x^2 + 4y^2 + z^2 = 33$, and write down the locations where they are obtained.

Question 5.

Use polar coordinates to evaluate $\displaystyle\iint_R e^{-x^2-y^2}\, dA$ where $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4,\ 0 \leq y \leq x\}$.

Question 6.

Evaluate $\displaystyle\int_C x - y + 2z\, ds$, where C is given by $\mathbf{r}(t) = \langle 1,\ 3\cos(t),\ 3\sin(t)\rangle$ for $t \in [0, 2\pi]$.

Question 7.

A force field is given by $\mathbf{F}(x, y, z) = \langle 2xy,\ x^2 + z\cos(y),\ \sin(y) + 2z\rangle$. Show that $\mathbf{F}$ is conservative and use this to find the work done by the force in moving an object along a curve from $(0, 0, 1)$ to $(-2, \pi/2, 4)$.