Question 1.

(a) What is a vector field?

(b) Give a few examples that have real, or physical, meanings. For each, think about whether the divergence or curl represent anything particular.

(a) A vector field assigns to each point in its domain D in $\mathbb{R}^n$ a vector in $\mathbb{R}^n$.

(b) Examples can be things like fluid flow (the fluid can be things like water, blood etc or gases, or certain particles in gases, like oxygen flow), or force field (gravitational field, magnetic field etc).

Another example can be developed from the following: Each person has a velocity vector at each point in time. Only one person can occupy a point in space, so this assigns to each point in space a vector, given either by $\mathbf{0}$ if no person occupies that point, or by the velocity vector of the person occupying that point. (In practice we would have to do some type of averaging process over space and a period of time). Points with positive divergence would correspond to maternity wards in a hospital, for example, where new people are born. Points with negative divergence would be, for example, a hospice.

Question 2.

(a) What is a conservative vector field? Give at least two characterizations.

(b) How can we determine whether $\mathbf{F}$ is conservative?

(a) A conservative vector field is a vector field $\mathbf{F}$ such that:

- it has a potential function, i.e. a scalar-valued function f with the same domain as $\mathbf{F}$ such that $\mathbf{F} = \nabla f$,
- $\mathbf{F}$ is path independent:
  - $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of $\mathcal{C}$;
  - $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve $\mathcal{C}$.

(b) The following facts apply to a three-dimensional vector field $\mathbf{F}$. Replace $\text{curl}(\mathbf{F})$ with $\text{curl}_z(\mathbf{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ for $\mathbf{F} = \langle F_1, F_2 \rangle$.

Fact 1. $\text{curl}(\nabla f) = \mathbf{0}$.

Fact (Theorem) 2. If the domain of $\mathbf{F}$ is simply connected and $\text{curl}(\mathbf{F}) = \mathbf{0}$ then $\mathbf{F}$ is conservative.

Fact 1 can be used to show a vector field is *not* conservative, while Fact 2 can be used to show a vector field is conservative, but only on a simply connected domain—remember the vortex field example from 17.3.

Question 3.

(a) How do you evaluate $\int_{\mathcal{C}} f(x, y, z)\, ds$?

(b) How do you evaluate $\int_{\mathcal{C}} \mathbf{F}(x, y, z) \cdot d\mathbf{r}$?

(c) In each case, what is $\mathcal{C}$?

(d) Can you think of any tricks that might help evaluate (a) or (b) in special cases?

A general strategy for (a) and (b) (compare with that for surface integrals below):

1. Parametrize the curve by $\mathbf{r}(t)$ for $t \in [a, b]$.
2. Determine the tangent to the curve $\mathbf{r}'(t)$.
3. (For (b)) check the given orientation of the curve agrees with the orientation on the curve given by the parametrization.
4. Use one of the following:

   (a) $\displaystyle\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t)) \left|\left|\mathbf{r}'(t)\right|\right|\, dt$

   (b) $\displaystyle\int_{\mathcal{C}} \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_a^b f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$

(c) The curve $\mathcal{C}$ should be sufficiently "nice" so that we can perform the integration in step 4 above. Typically we ask that the curve is *piecewise smooth*: there is a parametrization $\mathbf{r}(t)$ so that $\mathbf{r}'(t)$ is a continuous function except possibly at finitely many points. (This will make the curve look smooth, hence the name, except it may have a few corners or sharp turns, where $\mathbf{r}'(t)$ fails to be continuous, or is not defined).

(d) Typical tricks may be to use symmetry, often to either simplify the expression, or to show it is zero; or if $f(x, y, z)$ is a constant then you have a constant multiple of the length of the curve, which maybe can be calculated geometrically.

If $\mathbf{F}$ is conservative, you could use the Fundamental Theorem for Line Integrals. If $\mathcal{C}$ is closed, you could use Green's Theorem/Stokes' Theorem to integrate $\mathbf{F}$ over $\mathcal{C}$.

Question 4.

Think of a few examples where $\int_{\mathcal{C}} f(x, y, z)\, ds$ or $\int_{\mathcal{C}} \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ have a real or physical meaning.

The most common example will be to calculate work done against/by a force field.

Another situation is the flux across a curve: $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n}\, ds$. This may be applicable if we have a flow on a surface, for example.

Question 5.

(a) How do you evaluate $\iint_S f(x, y, z)\, dS$?

(b) How do you evaluate $\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$?

(c) In each case, what is S?

(d) Can you think of any tricks that might help evaluate (a) or (b) in special cases?

A general strategy for (a) and (b) (compare with that for curve integrals above):

1. Parametrize the surface by $G(u, v)$ for (u, v) in D, the domain of parametrization.
2. Determine the normal to the surface $\mathbf{N}(u, v) = \frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v}$.
3. (For (b)) check the given orientation of the surface agrees with the orientation on the surface given by the parametrization (do the normals point in the same direction?).
4. Use one of the following:

(a) $\displaystyle\iint_S f(x, y, z)\, dS = \iint_D f(G(u, v))\, ||\mathbf{N}(u, v)||\ du\, dv$

(b) $\displaystyle\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S} = \iint_D \mathbf{F}(G(u, v)) \cdot \mathbf{N}(u, v)\, du\, dv$

(c) As in Question 3, the surface should be sufficiently "nice" so that we can perform the integration in step 4. Typically we ask for the surface to be *smooth*, meaning that there is a parametrization $G(u, v)$ such that the normals $\mathbf{N}(u, v)$ obtained in step 2 above are always non-zero (i.e. every point is regular). We do allow for surfaces with corners (cubes, tetrahedron etc) by breaking it into finitely many smooth pieces.

(d) Tricks may include using area if $f(x, y, z)$ or $\mathbf{F} \cdot \mathbf{n}$ is constant (where $\mathbf{n}$ is the unit normal), or using symmetry.

We may be able to apply Stokes' Theorem if $\mathbf{F} = \mathrm{curl}(\mathbf{A})$ for some vector potential function $\mathbf{A}$ (see Question 9 (b)). If S is a closed surface then we can use the divergence theorem to integrate $\mathbf{F}$ over S.

Question 6.

Think of a few examples where $\displaystyle\iint_S f(x, y, z)\, dS$ or $\displaystyle\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$ have a real or physical meaning.

Some ideas:

∘ If $\mathbf{F}$ represents a fluid flow then its surface integral represents the total rate of flow across S. For example, we could look at the flow of oxygen in/out of a human brain, or blood through the heart.

∘ If $\mathbf{F}$ represents the movement of a population (like that described in Question 1), then the flux of that vector field would represent net migration, e.g. draw a bubble around Boston and we will get net migration in/out of Boston.

∘ If f represents pressure in the ocean, then integrating f over the exterior surface of a submarine would give the total pressure exerted on the submarine by the ocean.

Question 7.

State each of the following Theorems:

Don't let yourself get bogged down by the hypotheses on the functions/curves/surfaces in the following statements. Focus on the equation itself (but you should be aware there are restrictions).

(a) The fundamental theorem of calculus

Suppose $F(x)$ has a continuous first-order derivative on $[a, b]$. Then

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(b) The fundamental theorem for line integrals

Suppose f is a differentiable function whose gradient ∇f is continuous, and $\mathcal{C}$ is a piecewise smooth oriented curve from P to Q, then:

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(Q) - f(P)$$

(c) Green's Theorem

Suppose $\mathbf{F} = \langle F_1, F_2 \rangle$, where F_1 and F_2 have continuous first-order derivatives, and D is a region in $\mathbb{R}^2$. Then:

$$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

(Recall that the boundary ∂D is oriented so that as we walk around it we keep D on our left. When D is simply connected, this is the same as travelling anticlockwise.)

(d) Stokes' Theorem

Suppose $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$, where F_1, F_2 and F_3 have continuous first-order derivatives, and S is a piecewise smooth oriented surface in $\mathbb{R}^3$. Then:

$$\iint_S \operatorname{curl}(\mathbf{F})\,dA = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

(Recall that the boundary ∂S is oriented so that as we walk around it, on the side of S determined by the normal vectors) we keep S on our left.)

(e) The divergence theorem

Suppose $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$, where F_1, F_2 and F_3 have continuous first-order derivatives, and W is a solid region in $\mathbb{R}^3$ whose boundary is piecewise smooth. Then:

$$\iiint_W \operatorname{div}(\mathbf{F})\,dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$$

(Recall that the boundary ∂W is oriented with normals pointing outwards from W.)

Question 8.

Can you describe a common theme that links the 5 theorems in the previous question?

In each case we have a function (F, f, or $\mathbf{F}$), and an operation on this function (differentiation, gradient, curl_z, curl or div).

One side of each equation involves integrating the function produced by this operation over some appropriate region. The other side involves integrating the original function over the boundary of that region.

(In (a) and (b) the integral on the right hand side is a "discrete integral", which is just a sum over finitely many points.)

Question 9.

(a) A question asks you to find $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$. If $\text{curl}(\mathbf{F}) \neq \mathbf{0}$, can we use the fundamental theorem for line integrals?

(b) A question asks you to find $\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$. If $\text{div}(\mathbf{F}) \neq \mathbf{0}$, can we use Stokes' Theorem? If S is a closed surface, can we us the divergences theorem?

(a) We can apply the fundamental theorem for line integrals only to conservative vector fields. Recall $\text{curl}(\nabla f) = \mathbf{0}$, so if $\text{curl}(\mathbf{F}) \neq 0$ then $\mathbf{F}$ is not conservative.

(b) Similarly, Stokes' Theorem applies to a surface integral of the curl of a vector field, so we would need $\mathbf{F} = \text{curl}(\mathbf{A})$ for some vector potential function $\mathbf{A}$. Since $\text{div}(\text{curl}\,\mathbf{A}) = \mathbf{0}$ for every vector field $\mathbf{A}$, we can't use Stokes' Theorem.

When S is closed, we may however use the divergence theorem (assuming S and $\mathbf{F}$ satisfy the hypotheses).