Surfaces review questions

In the following questions, when the surface involved is a quadric surface, identify the type of surface by looking at the horizontal and vertical traces, and sketch it.

Question 1.

Parametrize the following surfaces

(a) $x^2 + 2y^2 + 3z^2 = 1$ for $y \le 0$,

(b) $4x^2 - 4y^2 - z^2 = 4$ for $0 \le x \le 2$,

(c) the torus obtained by rotating the circle in the xz–plane given by $(x-a)^2 + z^2 = R^2$, for $R < a$, about the z–axis.

Hint: think about what your two parameters should represent geometrically.

Question 2.

Let S be the part of $z = x^2 + y^2$ that lies under the plane $z = 4$. Evaluate $\iint_S z\,dS$.

Question 3.

Let S be the same surface as in Question 2, and let $\mathbf{F} = \langle x, xz, xy \rangle$.

(a) Calculate $\text{curl}(\mathbf{F})$ and $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$, where we take the orientation on S given by upward pointing normal vectors.

(b) Verify Stokes' Theorem holds.

Question 4.

Let S be the portion of the surface $z^2 = 3x^2 + 3y^2$ between the planes $z = 1$ and $z = 3$. Evaluate $\iint_S x^2 z^2\,dS$.

Question 5.

Let S be the same surface as in Question 4, oriented with upward pointing normals. Use Stokes' Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \text{curl}(\mathbf{A})$, where $\mathbf{A} = \langle 0, xy, xyz \rangle$.

Steps in evaluating surface integrals:

<u>Step 1</u>: Parametrize S by $G(u,v)$ for $(u,v) \in D$.

<u>Step 2</u>: Find the normal $\underline{N}(u,v)$.

[<u>Step 3</u>: for $\iint_S \underline{F} \cdot d\underline{S}$, check if the orientations agree.]

<u>Step 4</u>: $\iint_S f(x,y,z)\,dS = \iint_D f(G(u,v))\,\|\underline{N}(u,v)\|\,dA$

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_D \underline{F}(G(u,v)) \cdot \underline{N}(u,v)\,dA$$

$\boxed{\text{If they do } \underline{\text{not}} \text{ agree, then multiply your answer by } -1}$

Q1 (a) $\quad x^2 + 2y^2 + 3z^2 = 1 \quad$ for $\quad y \leq 0$.

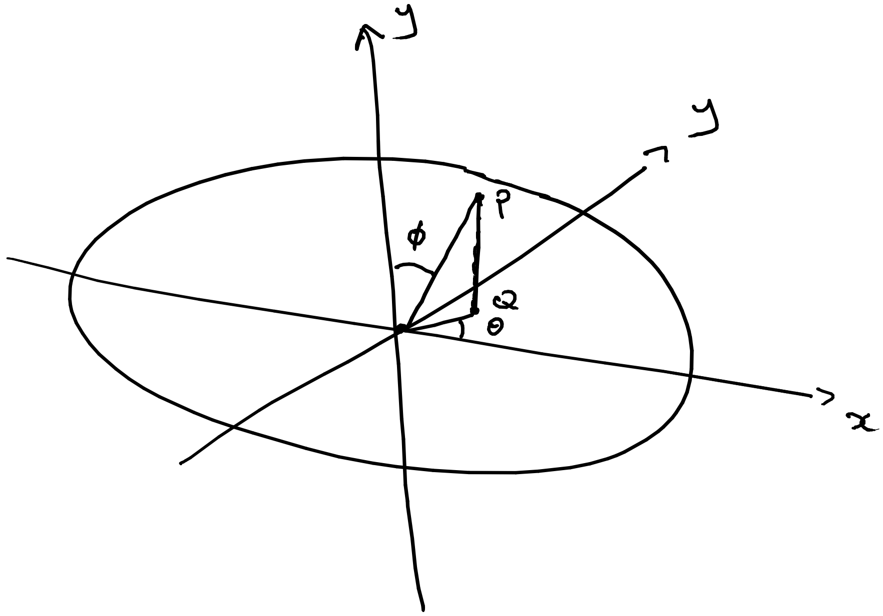
This is an ellipsoid: look at any trace in a plane $x = k$, $y = k$ or $z = k$, for suitable values of k, and we see the trace is an ellipse.

eg. $x = k$: $\quad k^2 + 2y^2 + 3z^2 = 1$

$$\Leftrightarrow \quad 2y^2 + 3z^2 = 1 - k^2$$

This is an ellipse provided $k^2 \leq 1$.

We will use the parameters θ and ϕ from spherical coordinates to map out the ellipsoid:



P = point on ellipsoid

Q = projection of P to xy-plane

We will essentially stretch / compress the coordinate axes to turn the unit sphere into the ellipsoid:

The x-axis should be unchanged.

The y-axis should be stretched by a factor of $\frac{1}{4}$.

The z-axis should be stretched by a factor of $\frac{1}{9}$.

Then take

$$x = \sin\phi \cos\theta$$

$$y = \frac{1}{4}\sin\phi \sin\theta$$

$$z = \frac{1}{9}\cos\phi$$

(we can quickly verify there satisfy $x^2 + 2y^2 + 3z^2 = 1$).

Our domain should be :

$$\phi \in [0, \pi], \quad \theta \in [\pi, 2\pi]$$

(since $y \leq 0$).

(b) $4x^2 - 4y^2 - z^2 = 4$, $0 \leq x \leq 2$.

This is a (two-sheeted) hyperboloid :

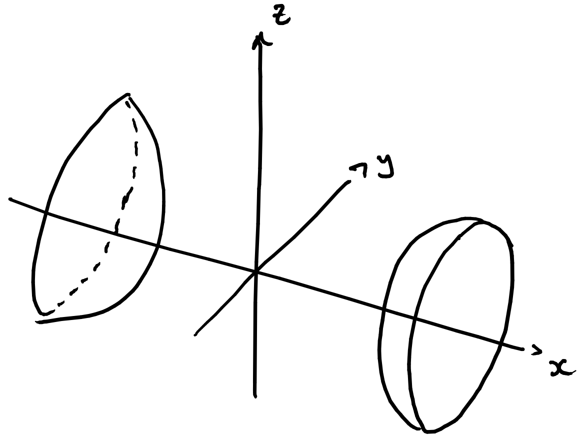
Trace :

$z = k \longrightarrow 4x^2 - 4y^2 = 4 + k^2$ - hyperbola

$y = k \longrightarrow 4x^2 - z^2 = 4 + 4k^2$ - hyperbola

$x = k \longrightarrow 4y^2 + z^2 = 4k^2 - 4$

↳ ellipse, provided $k^2 \geq 1$.



The constraint $0 \leq x \leq 2$ tells us we should consider only one sheet, so that the surface can be viewed as a graph with $x = f(y,z)$, over a suitable domain.

We now determine $f(y,z)$. Start with the given equation for the hyperboloid and rearrange to make x the subject:

$$4x^2 - 4y^2 - z^2 = 4 \Rightarrow x^2 = y^2 + \frac{1}{4}z^2 + 1$$

$$\Rightarrow x = \sqrt{y^2 + \frac{1}{4}z^2 + 1}$$

We take the positive square root since we know $x \in [0,2]$.

Thus we may parametrize the surface like a graph:

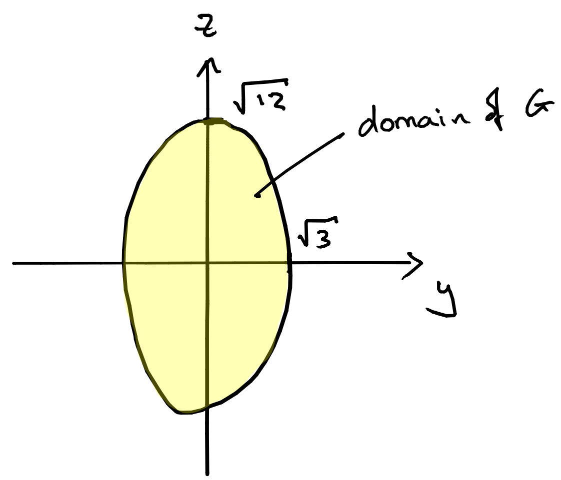
$$G(y,z) = \left(\sqrt{y^2+\frac{1}{4}z^2+1}, y, z\right).$$

The domain will be the projection of the surface to the yz-plane. This will be given by points (y,z) such that

$$y^2 + \frac{1}{4}z^2 + 1 = x^2 \in [0,4]$$

$$\Rightarrow y^2 + \frac{1}{4}z^2 \in [-1,3]$$

$$\Rightarrow \underline{y^2 + \frac{1}{4}z^2 \leq 3}$$



The domain is this elliptic region.

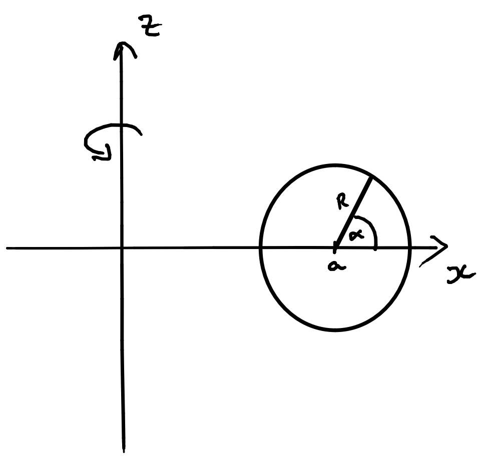
(c) The torus. (This question is hard!)

We need two parameters, and we use the geometry of the torus to describe them.

The torus is obtained by rotating a circle in the xz-plane about the z-axis.

- A parameter α will map out this circle,
- A parameter θ will describe the rotation.

The circle we rotate is:



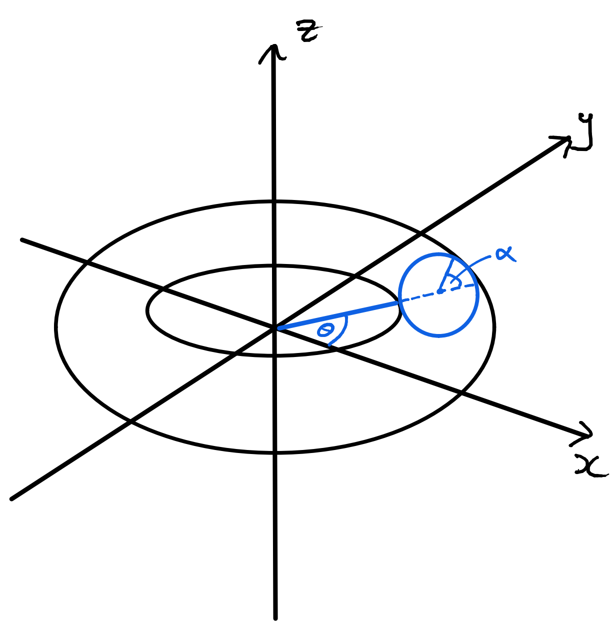
Let α be the usual angular parameter to describe the circle. So:

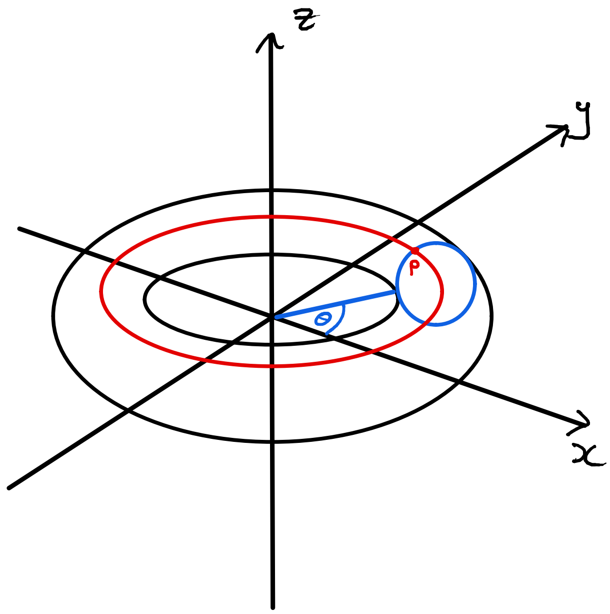
$$\left.\begin{aligned} x &= a + R\cos\alpha \\ z &= R\sin\alpha \end{aligned}\right\} \to \text{This is valid in the } xz\text{-plane (ie } \theta = 0)$$

Rotating this circle gives the torus. Fixing θ we get a circle that should be described by α.

Let r be the distance to the z-axis (as for cylindrical coords).

Then $\begin{cases} r = a + R\cos\alpha \\ z = R\sin\alpha \end{cases}$





Fixing a value of α determines a point on the circle in the xz-plane. When rotated we get a circle with center on the z-axis (the red circle).

This circle has radius

$$r = a + R\cos\alpha$$

and lies in the plane

$$z = R\sin\alpha$$

So it can be parametrized by

$$x = (a + R\cos\alpha)\cos\theta$$
$$y = (a + R\cos\alpha)\sin\theta$$

This gives the parametrization:

$$G(\alpha, \theta) = \big((a+R\cos\alpha)\cos\theta, (a+R\cos\alpha)\sin\theta, R\sin\alpha \big)$$

with domain $\alpha \in [0, 2\pi]$

$\theta \in [0, 2\pi]$.

Q2. $\iint_S z \, dS$, S is the portion of $z = x^2+y^2$ under the plane $z=4$.

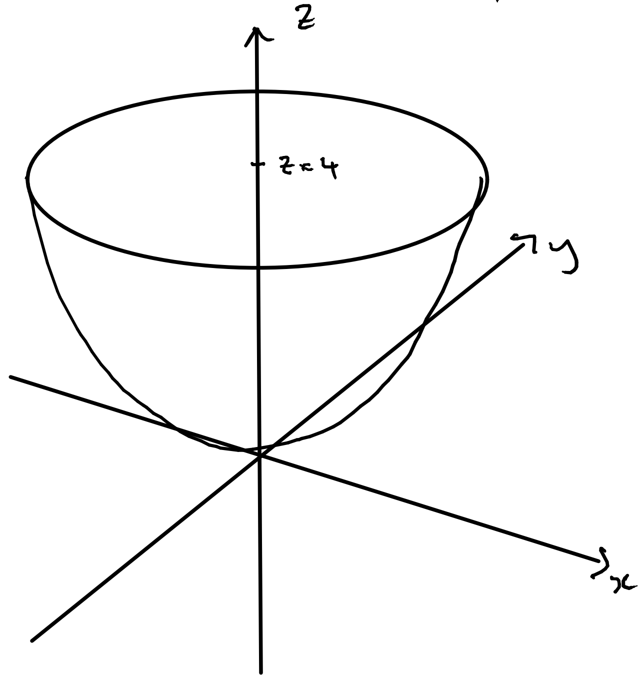
First we identify the type of surface:

Traces: $x=k \to z = k^2+y^2 \to$ parabola

$y=k \to z = x^2+k^2 \to$ parabola

$z=k \to k = x^2+y^2 \to$ circles

Thus S is a piece of a paraboloid.



Step 1: Parametrize S.

We could parametrize it as a graph:

$$G(x,y) = (x, y, x^2+y^2)$$

but since the domain is a disk we will use cylindrical coordinates (otherwise we will swap to polar coordinates when integrating).

So our parametrization comes from rewriting $z = x^2+y^2$ as $z = r^2$. Then:

$$G(r,\theta) = (r\cos\theta, r\sin\theta, r^2),$$

for $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Step 2: Determine the normal $\underline{N}(r,\theta)$.

$$\frac{\partial G}{\partial r} = (\cos\theta, \sin\theta, 2r)$$

$$\frac{\partial G}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$\Rightarrow \underline{N}(r,\theta) = \langle \cos\theta, \sin\theta, 2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$= \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$$

Step 4: $\|\underline{N}(r,\theta)\| = \sqrt{4r^4 + r^2}$

$$\iint_S z\,dS = \int_0^{2\pi}\int_0^2 (r^2)\sqrt{4r^4+r^2}\,dr\,d\theta$$

Remember, this trick only works when the domain is rectangular (ie the limits are constants)

$$= \int_0^{2\pi} d\theta \int_0^2 r^3 (4r^2+1)^{1/2}\,dr$$

$u = 4r^2+1$

$\frac{du}{dr} = 8r$

$r^2 = \frac{1}{4}(u-1)$

$$= 2\pi \int_1^{17} \frac{1}{32}(u-1)u^{1/2}\,du$$

$$= 2\pi \left[\frac{1}{32}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\right]_1^{17}$$

$$= \frac{\pi}{8}\left(17^{3/2}\left(\frac{1}{5}17 - \frac{1}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right)$$

$$= \frac{\pi}{8}\left(17^{3/2}\left(\frac{46}{15}\right) + \frac{2}{15}\right)$$

Q3.

(a) $\underline{F} = \langle x, xz, xy \rangle \Rightarrow \text{curl}(\underline{F}) = \langle 0, -y, z \rangle$.

Step 1 (from Q2) $\underline{G}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$,

for $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Step 2 (from Q2) $\underline{N}(r,\theta) = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$

Step 3: (In the original question sheet no orientation was given - sorry!)

Look at the z-component of $\underline{N}$. It is r, and we have $r \geq 0$ for all points in the domain D of G. So we have an upward pointing normals.

Step 4:

$$\iint_S \text{curl}(\underline{F}) \cdot d\underline{S} = \int_0^{2\pi}\int_0^2 \langle 0, -r\sin\theta, r^2 \rangle \cdot \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\sin^2\theta + r^3 \, dr\, d\theta$$

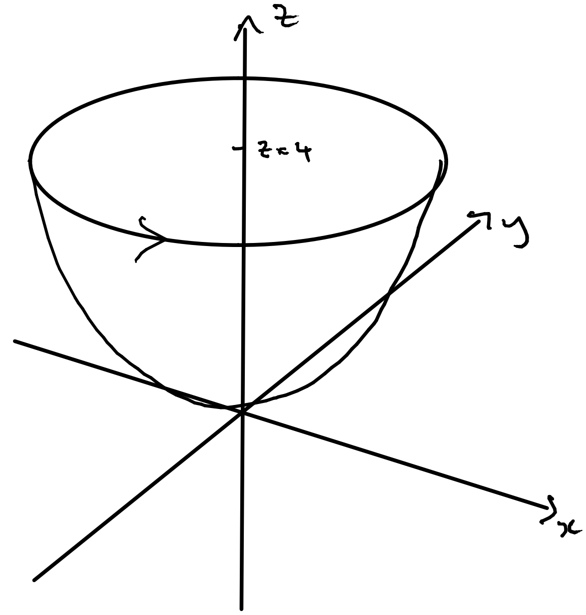
$$= \int_0^{2\pi} 4\sin^2\theta + 4 \, d\theta$$

$$= \int_0^{2\pi} 4(1-\cos 2\theta) + 4 \, d\theta$$

$$= \left[4\theta - \frac{2}{3}\sin 2\theta + 4\theta \right]_0^{2\pi} = \underline{16\pi}$$

(b) We need to verify Stokes' Theorem:

$$\int_{\partial S} \underline{F} \cdot d\underline{r} = \iint_S \text{curl}(\underline{F}) \cdot d\underline{S} = 16\pi$$



We orient ∂S as pictured:

Step 1: ∂S is the circle $4 = x^2 + y^2$ in the plane $z = 4$.

We can parametrize it:

$$\underline{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle$$

for $t \in [0, 2\pi]$.

Step 2: $\underline{r}'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

Step 3: At $t = 0$, $\underline{r}(t) = \langle 2, 0, 4 \rangle$, $\underline{r}'(t) = \langle 0, 2, 0 \rangle$, and ∂S travels in the positive y-direction. So our orientations do agree.

Step 4:

$$\int_{\partial S} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \langle 2\cos t, 8\cos t, 4\cos t \sin t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -4\sin t\cos t + 16\cos^2 t \, dt$$

$2\cos^2 t - 1 = \cos 2t$
$\Rightarrow 2\cos^2 t = \cos 2t + 1$

$$= \left[-2\sin^2 t + 8\left(\frac{1}{2}\sin 2t + t \right) \right]_0^{2\pi}$$

$$= \underline{16\pi}$$

Q4: $\iint_S x^2 z^2 \, dS$.

$S: \; z^2 = 3x^2 + 3y^2 \;, \quad 1 \le z \le 3$.

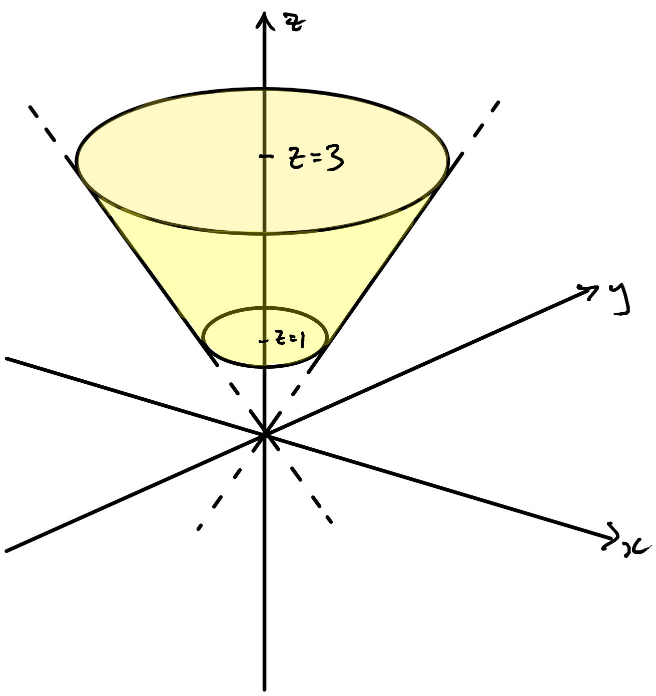
Classify the quadric surface:

Traces: $x = k: \quad z^2 = 3k^2 + 3y^2 \;\to$ hyperbola, except $x = 0$ when we get the lines $z = \pm\sqrt{3}\,y$

$y = k \;\to$ similar to $x = k$.

$z = k: \quad 3x^2 + 3y^2 = k^2 \;\to$ circles.

S is a portion of a circular cone.

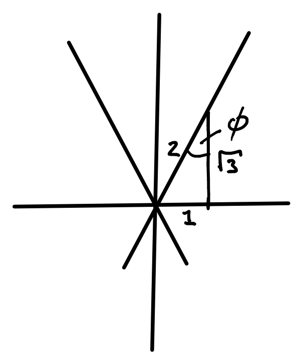


Step 1: Parametrize S.

Since we are dealing with a cone, we can use spherical coordinates with ϕ fixed.
[There are alternatives, eg as a graph / cylindrical]
To find the appropriate value of ϕ, consider the trace in $y = 0$. It is a pair of lines of slope $\pm\sqrt{3}$:

So $\tan\phi = \frac{1}{\sqrt{3}}$

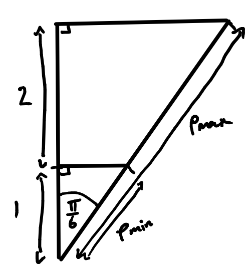
$\Rightarrow \phi = \pi/6$.



Our parametrization is therefore:

$$\underline{r}(\rho, \theta) = \left(\frac{\rho}{2}\cos\theta \;,\; \frac{\rho}{2}\sin\theta \;,\; \frac{\sqrt{3}}{2}\rho \right)$$

The domain will have $\theta \in [0, 2\pi]$. For the values of ρ we consider the triangles:



$\rho_{min} = \frac{\sqrt{3}}{2}$

$\rho_{max} = \frac{3\sqrt{3}}{2}$

$\Rightarrow \rho \in \left[\frac{\sqrt{3}}{2} \,,\, \frac{3\sqrt{3}}{2} \right]$.

Step 2 : $\frac{\partial G}{\partial \rho} = \left(\frac{1}{2}\cos\theta \,,\ \frac{1}{2}\sin\theta \,,\ \frac{\sqrt{3}}{2} \right)$

$$\frac{\partial G}{\partial \theta} = \left(-\frac{\rho}{2}\sin\theta \,,\ \frac{\rho}{2}\cos\theta \,,\ 0 \right)$$

$$\Rightarrow \quad \underline{N}(\rho,\theta) = \left\langle -\frac{\sqrt{3}}{4}\rho\cos\theta \,,\ -\frac{\sqrt{3}}{4}\rho\sin\theta \,,\ \frac{\rho}{4} \right\rangle$$

Step 4 : $\|\underline{N}(\rho,\theta)\| = \sqrt{\frac{3}{16}\rho^2 + \frac{\rho^2}{16}} = \frac{\rho}{4}$

$$\iint_S x^2 z^2 \, dS = \int_0^{2\pi} \int_{\frac{\sqrt{3}}{2}}^{\frac{3\sqrt{3}}{2}} \left(\frac{\rho^2}{4}\cos^2\theta \right) \left(\frac{3}{4}\rho^2 \right) \left(\frac{\rho}{4} \right) d\rho \, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_{\frac{\sqrt{3}}{2}}^{\frac{3\sqrt{3}}{2}} \frac{\rho^5}{2^6} \, d\rho$$

$$= \left[\frac{1}{4}\sin 2\theta + \frac{\theta}{2} \right]_0^{2\pi} \left[\frac{\rho^6}{(3)(2^7)} \right]_{\frac{\sqrt{3}}{2}}^{\frac{3\sqrt{3}}{2}}$$

$$= \frac{\pi}{(3)(2^7)} \left(\frac{3^6 \cdot 3^3}{2^6} - \frac{3^3}{2^6} \right) = \underline{\frac{3^2 \pi}{2^{13}} \left(3^6 - 1 \right)}$$

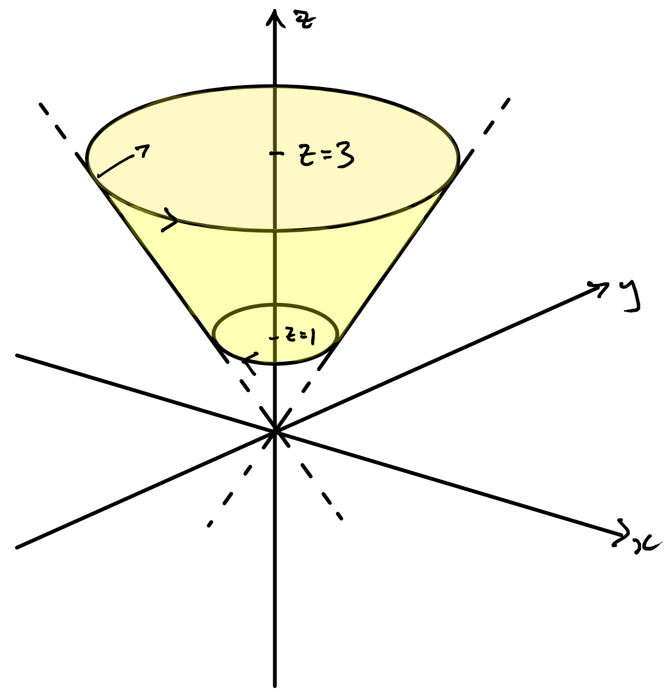
Q5. $\underline{F} = \text{curl}(\underline{A})$, $\underline{A} = \langle 0, xy, xyz\rangle$

We want to find $\iint_S \underline{F} \cdot d\underline{S}$.

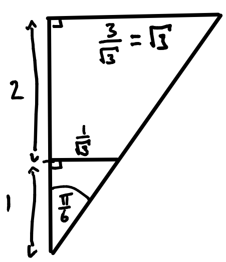
Stokes' Theorem tells us:

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_S \text{curl}(\underline{A}) \cdot d\underline{S} = \int_{\partial S} \underline{A} \cdot d\underline{r}$$

Note ∂S has two components. With upward pointing normals on S, the orientation on ∂S is as shown.



Step 1:



- At $z=3$, the boundary is a circumference of radius $\sqrt{3}$
- At $z=1$ it has radius $\frac{1}{\sqrt{3}}$.

$\underline{r}_1(t) = \langle \frac{1}{\sqrt{3}}\cos t, \frac{1}{\sqrt{3}}\sin t, 1\rangle$

$\underline{r}_2(t) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, 3\rangle$ (*) both for $t \in [0, 2\pi]$.

Step 2:

$\underline{r}_1'(t) = \langle -\frac{1}{\sqrt{3}}\sin t, \frac{1}{\sqrt{3}}\cos t, 0\rangle$

$\underline{r}_2'(t) = \langle -\sqrt{3}\sin t, \sqrt{3}\cos t, 0\rangle$

Step 3: $\underline{r}_1$ travels anticlockwise $\rightarrow$ orientations agree.

$\underline{r}_2$ travels anticlockwise $\rightarrow$ orientations <u>disagree</u>.

(*) The eagle-eyed among you may have foreseen the orientation issue, and may choose to change $\underline{r}_2$ to $\langle \sqrt{3}\sin t, \sqrt{3}\cos t, 3\rangle$

Since the orientations on $\underline{r}_2$ disagreed.

Step 4:

$$\int_{\partial S} \underline{A} \cdot d\underline{S} = \int_0^{2\pi} \underline{A}(\underline{r}_1(t)) \cdot \underline{r}_1'(t)\,dt - \int_0^{2\pi} \underline{A}(\underline{r}_2(t)) \cdot \underline{r}_2'(t)\,dt$$

$$= \int_0^{2\pi} \left\langle 0, \tfrac{1}{3}\cos t \sin t, \tfrac{1}{3}\cos t \sin t \right\rangle \cdot \left\langle -\tfrac{1}{\sqrt{3}}\sin t, \tfrac{1}{\sqrt{3}}\cos t, 0 \right\rangle dt$$

$$- \int_0^{2\pi} \langle 0, 3\cos t \sin t, 9\cos t \sin t \rangle \cdot \langle -\sqrt{3}\sin t, \sqrt{3}\cos t, 0 \rangle\,dt$$

$$= \int_0^{2\pi} \frac{1}{3\sqrt{3}}\cos^2 t \sin t\,dt - \int_0^{2\pi} 3\sqrt{3}\cos^2 t \sin t\,dt$$

$$= \left(\frac{1}{3\sqrt{3}} - 3\sqrt{3}\right) \int_0^{2\pi} \cos^2 t \sin t\,dt$$

$$= \left(\frac{1}{3\sqrt{3}} - 3\sqrt{3}\right) \left[\frac{-1}{3}\cos^3 t \right]_0^{2\pi}$$

$$= 0.$$

((Can you see how to use the symmetry of $\underline{A}$ and ∂S to avoid doing all this calculation?))