Random walks with local memory on $\mathbb{Z}^2$

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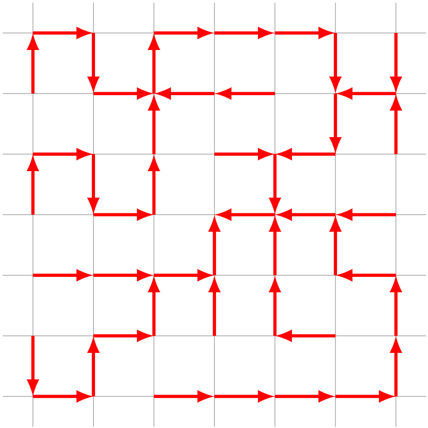
Joint work with Lila Greco, Lionel Levine, Boyao Li



Simple random walk on $\mathbb{Z}^2$

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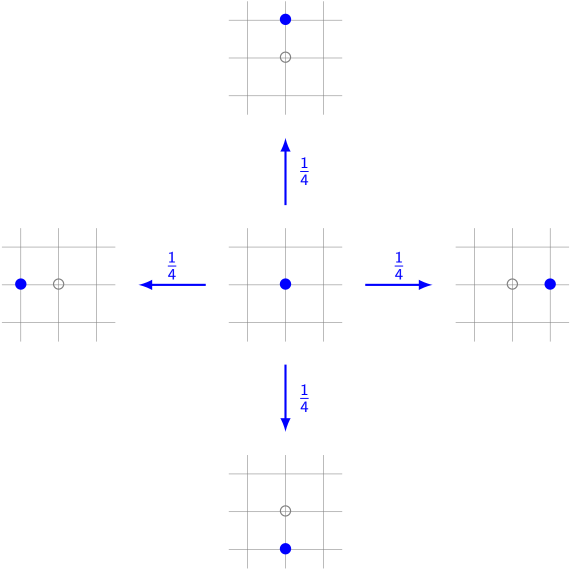


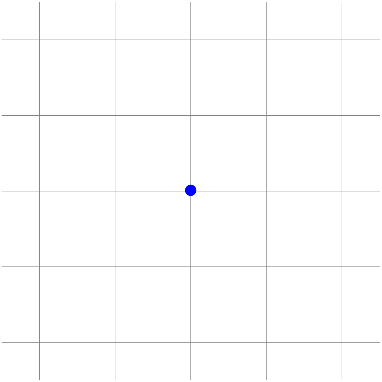
Simple random walk on $\mathbb{Z}^2$

- Visits every site infinitely often? Yes!
- Scaling limit? The standard 2-D Brownian motion, i.e.

$$\frac{1}{\sqrt{n}} \underbrace{X_{[nt]}}_{\text{Location of the walker at time } [nt]} \overset{n\to\infty}{\Longrightarrow} \frac{1}{\sqrt{2}} \big(\underbrace{B_1(t), B_2(t)}_{\text{Independent standard Brownian motions}}\big) \qquad t \geq 0.$$

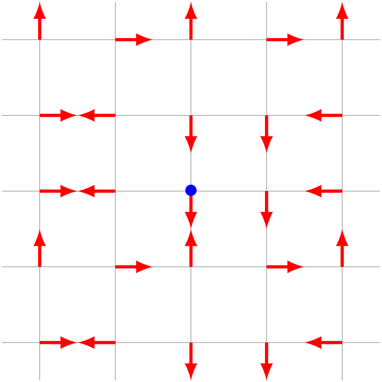
Rotor walk on $\mathbb{Z}^2$

Rotor walk on $\mathbb{Z}^2$



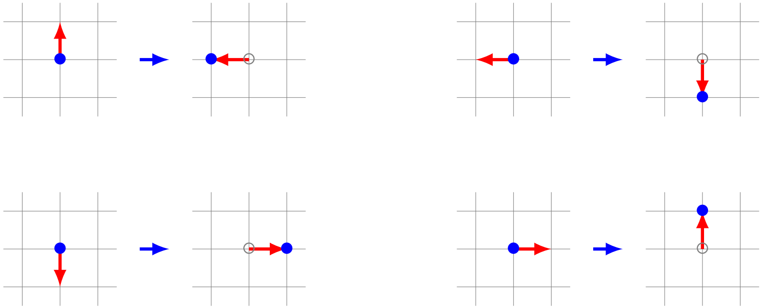
Rotor walk on $\mathbb{Z}^2$

Put a signpost at each site.



Rotor walk on $\mathbb{Z}^2$

Turn the signpost 90° counterclockwise, then follow the signpost.



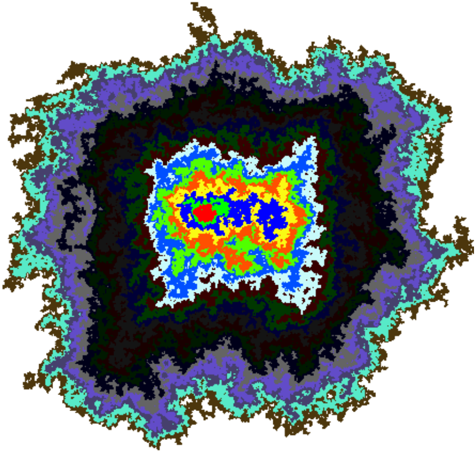
The signpost says:
"This is which way you went last time you were here",
(assuming you ever were!)

Conjectures for a rotor walk on $\mathbb{Z}^2$

If the initial signposts are distributed independently and uniformly among the four directions, then

- (PDDK '96) Visits every site infinitely often?
- (Kapri-Dhar '09) Scaling limit? The asymptotic shape of $\{X_1, \ldots, X_n\}$ is a disc (!)

The set of sites visited by a rotor walk in $\mathbb{Z}^2$



$\{X_1, \ldots, X_n\}$ after 20 returns to the origin (by Laura Florescu).

More randomness please!

Well studied

Many open problems

Random

Deterministic

More randomness please!

Well studied

Let's study this!!!

Many open problems

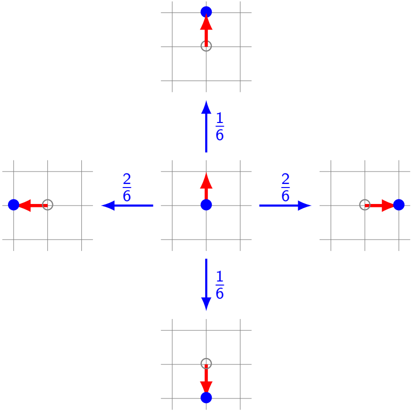
Random ⟷ Deterministic

Random walk with local memory (RWLM) on $\mathbb{Z}^2$

Random walk with local memory (RWLM) on $\mathbb{Z}^2$

Turn the signpost in a random manner, then follow the signpost.

Example:

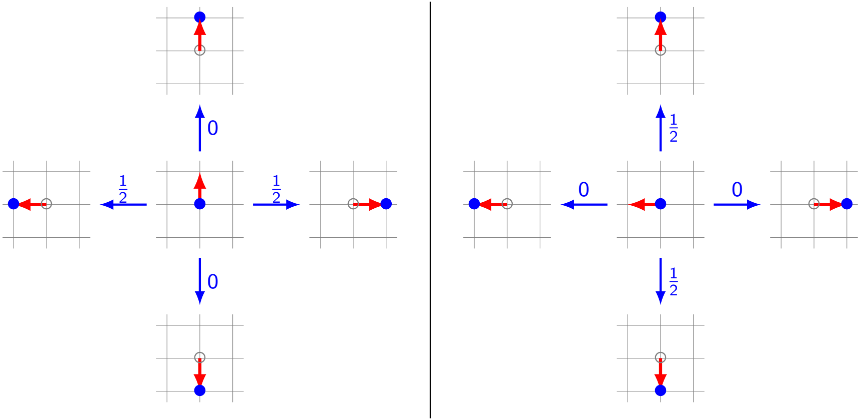


(Compare with turning the signpost 90° counterclockwise for rotor walk.)

Random walk with local memory (RWLM) on $\mathbb{Z}^2$

The distribution of the next signpoint depends on the current signpost.

Example:



(Compare with the simple random walk which ignores the signpost.)

Scaling limit for a RWLM on $\mathbb{Z}$

(Huss, Levine, Sava-Huss 16+) The scaling limit for a RWLM on $\mathbb{Z}$ is a perturbed Brownian motion $(Y(t))_{t\geq 0}$, i.e.

$$Y(t) = \underbrace{B(t)}_{\substack{\text{Standard}\\ \text{Brownian}\\ \text{motion}}} + \underbrace{a \sup_{0\leq s\leq t} Y(s)}_{\substack{\text{Perturbation when}\\ \text{hitting maximum}}} + \underbrace{b \inf_{0\leq s\leq t} Y(s)}_{\substack{\text{Perturbation when}\\ \text{hitting minimum}}}, \qquad t \geq 0,$$

Scaling limit for a RWLM on $\mathbb{Z}^2$

- Conjecture: the scaling limit for a RWLM on $\mathbb{Z}^2$ is a "2-D perturbed Brownian motion (?)".
- Problem: Not clear how to define "2-D perturbed Brownian motion (?)".
- Weaker conjecture: the scaling limit for a Martingale RWLM on $\mathbb{Z}^2$ is a 2-D Brownian motion.

Scaling limit for a Martingale RWLM on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '17+)

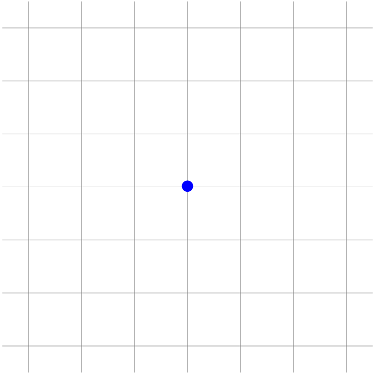
Suppose that a RWLM on $\mathbb{Z}^2$ is a non-degenerate Martingale, and with initial signposts picked randomly. Then with probability 1 the RWLM scales to a 2-D Brownian motion, i.e.

$$\frac{1}{\sqrt{n}} \underbrace{X_{[nt]}}_{\substack{\text{Location of the} \\ \text{walker at time } [nt]}} \overset{n\to\infty}{\Longrightarrow} \underbrace{(a_1 B_1(t), a_2 B_2(t))}_{\substack{\text{Independent} \\ \text{Brownian motions}}} \qquad t \geq 0.$$

WARNING: The initial signposts are picked according to the free spanning forest distribution, NOT the i.i.d. uniform distribution.

Free spanning forest (FSF) distribution

The following construction converges to a distribution on the spanning trees of $\mathbb{Z}^2$ (Pemantle '91):



From the perspective of the walker, the free spanning forest is a stationary distribution for RWLM.

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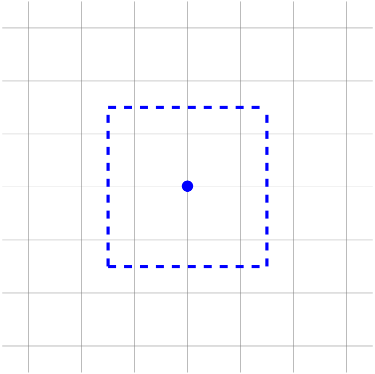

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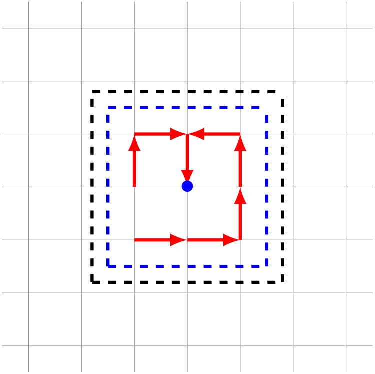

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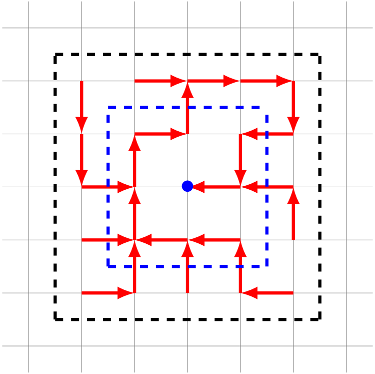

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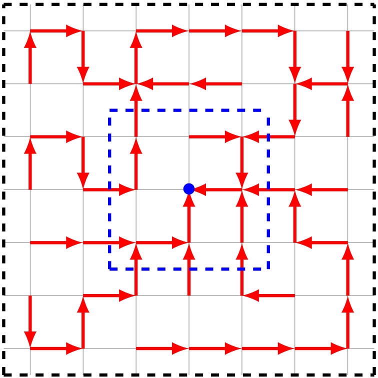

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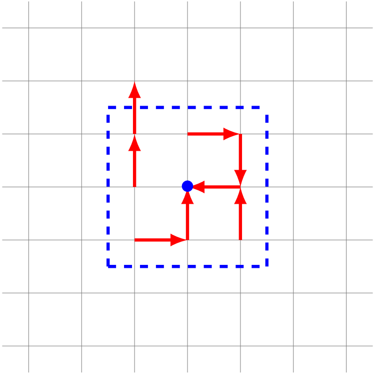

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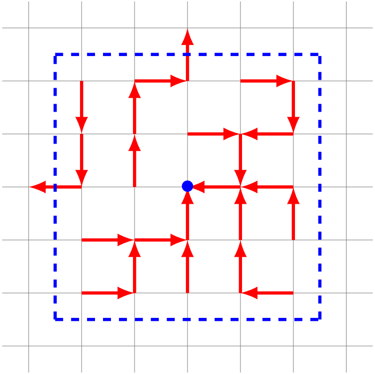

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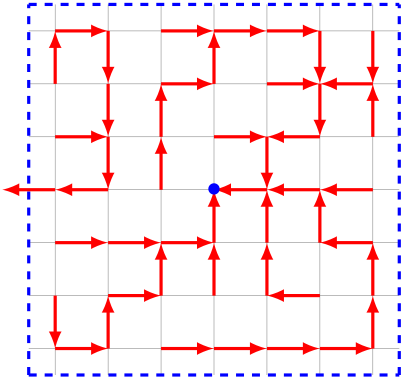

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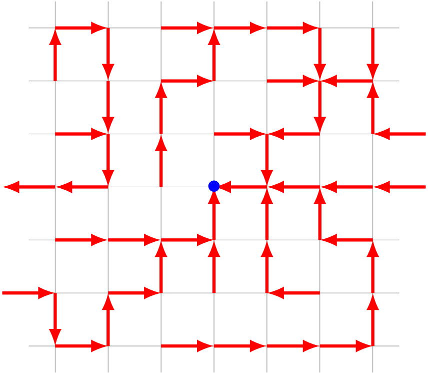

From the perspective of the walker, the free spanning forest is a stationary distribution for RWLM.

Scaling limit for a Martingale RWLM on $\mathbb{Z}^2$

Theorem (C., Greco, Levine, Li '17+)

For a non-degenerate Martingale RWLM on $\mathbb{Z}^2$ with initial signposts picked according to the free spanning forest distribution, the scaling limit of the walk is a 2-D Brownian motion.

What is next?

- Scaling limit for non-Martingale RWLMs?
  - Need to define the "2-D perturbed Brownian motion (?)".
- Scaling limit for Martingale RWLMs on $\mathbb{Z}^d$?
  - Yes for $d \in \{3, 4\}$.
  - Open for $d \geq 5$.
- Rotor walks with initial signposts picked according to FSF?
  - Does NOT visit every site infinitely often (Florescu, Levine, Peres '16).
  - Scaling limit?

THANK YOU!