Necklaces and subset-sums: How can they be related?

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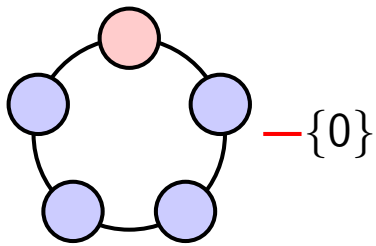
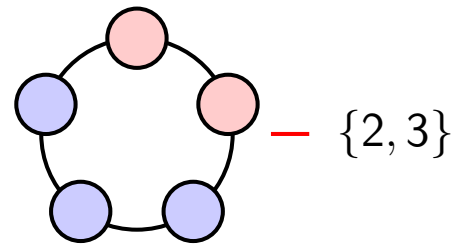
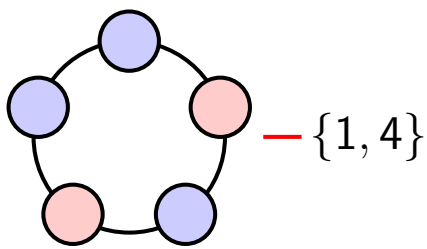
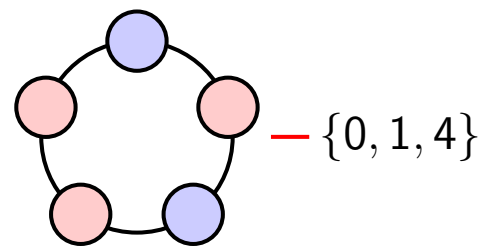
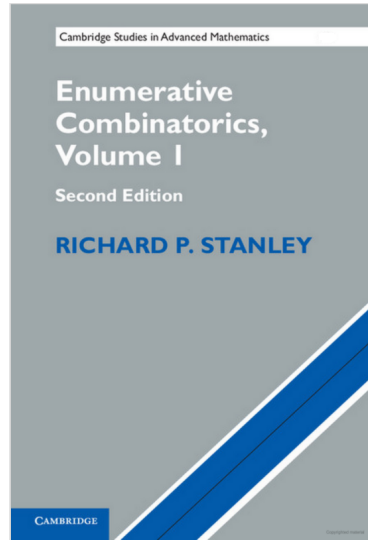










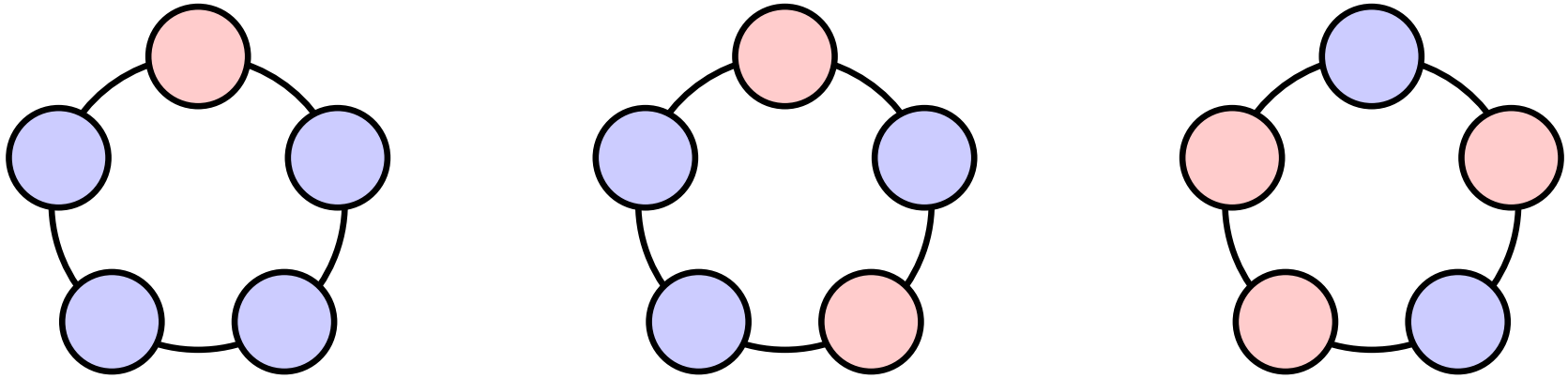
105. (a) [3–] Let $n \in \mathbb{P}$, and let $f(n)$ denote the number of subsets of $\mathbb{Z}/n\mathbb{Z}$ (the integers modulo n) whose elements sum to 0 in $\mathbb{Z}/n\mathbb{Z}$. For instance, $f(4) = 4$, corresponding to $\emptyset$, $\{0\}$, $\{1,3\}$, $\{0,1,3\}$. Show that

$$f(n) = \frac{1}{n} \sum_{\substack{d|n \\ d \text{ odd}}} \phi(d) 2^{n/d},$$

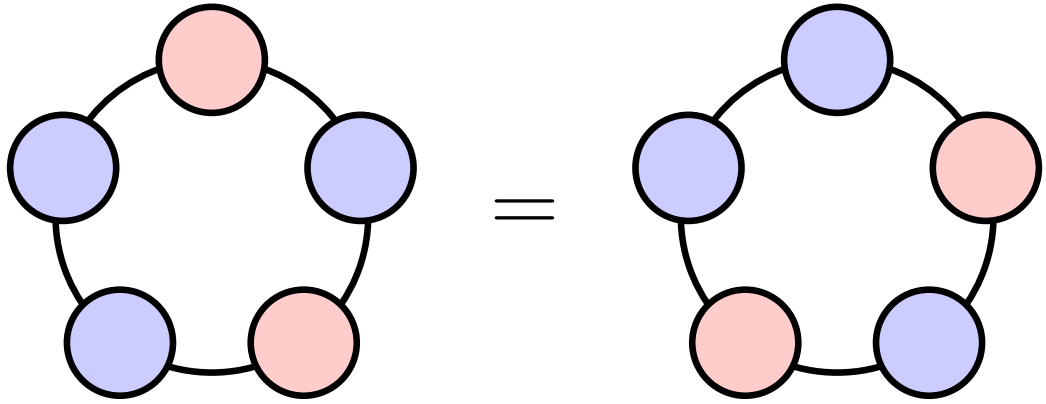
where ϕ denotes Euler's totient function.

(b) [5–] When n is odd, it can be shown using (a) (see Exercise 7.112) that $f(n)$ is equal to the number of necklaces (up to cyclic rotation) with n beads, each bead colored black or white. Give a combinatorial proof. (This is easy if n is prime.)

Necklaces with two colors



Necklaces are rotationally invariant.



Subsets of $\mathbb{Z}_n$ that sums to 0 (modulo n)

Let $n = 5$.

Example:

- $\{0, 1, 4\}$; $0 + 1 + 4 = 5 = 0 \mod 5$.
- $\{1, 2, 3, 4\}$; $1 + 2 + 3 + 4 = 10 = 0 \mod 5$.

Non-example:

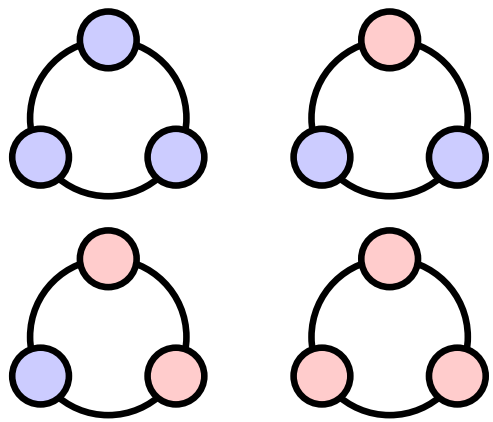
- $\{1, 3, 4\}$; $1 + 3 + 4 = 8 = 3 \mod 5$.
- $\{0, 1, 2, 3\}$; $0 + 1 + 2 + 3 = 6 = 1 \mod 5$.

The theorem

Theorem (EC1, 1.105(a))

If n is odd, then:

$$\begin{array}{c}\#\text{ of necklaces of length} \\ n \text{ with two colors}\end{array} = \begin{array}{c}\#\text{ of subsets of } \mathbb{Z}_n \\ \text{that sums to } 0.\end{array}$$



$\{\}, \{1,2\},$
$\{0\}, \{0,1,2\}$

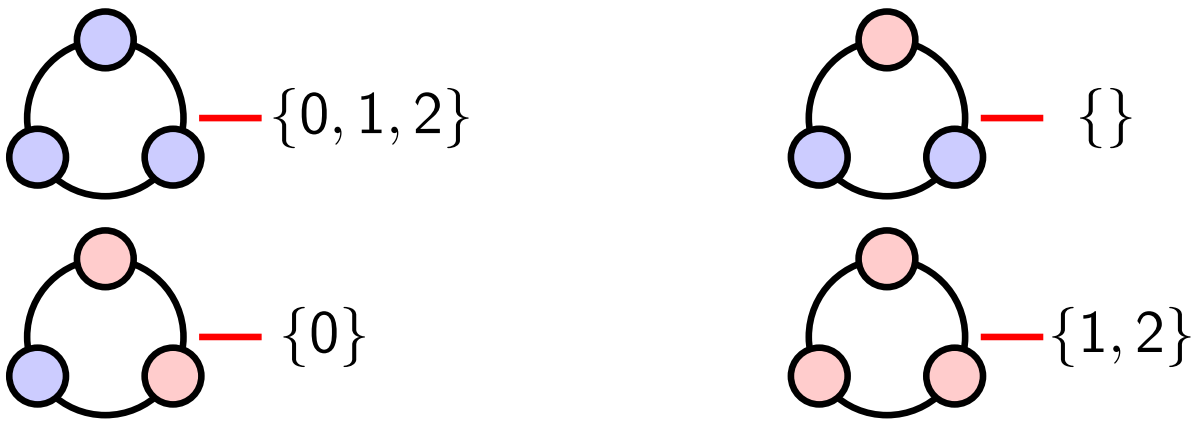
Proof in EC 1 uses orbit-counting theorem and generating function.
Stanley then asked for a combinatorial proof.

The theorem

Theorem (EC1, 1.105(a))

If n is odd, then:

$$\begin{array}{c}\#\text{ of necklaces of length}\\ n\text{ with two colors}\end{array} = \begin{array}{c}\#\text{ of subsets of } \mathbb{Z}_n\\ \text{that sums to } 0.\end{array}$$



We will give a combinatorial proof to this theorem.

Proof in EC1

Orbit-counting theorem: For a group G acting on a set X,

$$|\underbrace{X/G}_{\substack{\text{Orbit of}\\ \text{the action}}}| = \frac{1}{|G|}\sum_{g\in G} |\underbrace{X^g}_{\substack{\text{Elements fixed}\\ \text{by } g}}|.$$

For $G = \mathbb{Z}_n$ and $X =$ set of strings of length n with two colors,

$$\begin{array}{c}\#\text{ of necklaces of length}\\ n \text{ with two colors}\end{array} = \frac{1}{n}\sum_{d|n} 2^{n/d}\underbrace{\phi(d)}_{\substack{\text{Euler's}\\ \text{totient function}}}.$$

Proof in EC1 (ctd)

For any complex n-th root of unity $\zeta := e^{2\pi i/n}$,

$$(1+\zeta)(1+\zeta^2)\cdots(1+\zeta^n) = c_0 + c_1\zeta + \cdots + c_{n-1}\zeta^{n-1}.$$

By summing over all n-th roots of unity,

$$\sum_{\substack{d|n \\ d \text{ odd}}} 2^{n/d} \underbrace{\phi(d)}_{\substack{\text{Euler's} \\ \text{totient function}}} = n \left(\begin{array}{c} \# \text{ of subsets of } \mathbb{Z}_n \\ \text{that sums to } 0 \end{array} \right).$$

Proof in EC1 (ctd)

If n is odd, then:

$$\begin{array}{c}\#\text{ of necklaces of length} \\ n \text{ with two colors}\end{array} = \begin{array}{c}\#\text{ of subsets of } \mathbb{Z}_n \\ \text{that sums to 0,}\end{array}$$

and is equal to

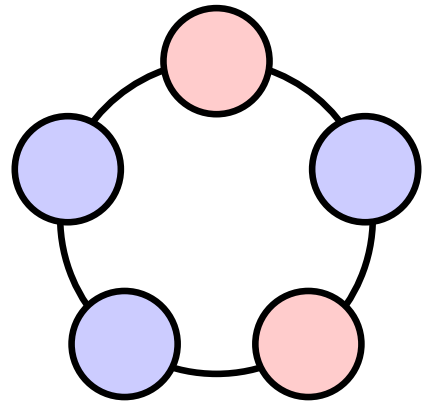
$$\frac{1}{n}\sum_{d|n} 2^{n/d}\underbrace{\phi(d)}_{\substack{\text{Euler's} \\ \text{totient function}}}.$$

Questions unanswered by this proof:

- How are they related?
- Why odd n?

View necklaces as polynomials

The necklace



can be viewed as:

$$\{1 + X^2, X + X^3, X^2 + X^4, X^3 + 1, X^4 + X\} \subset \mathbb{F}_2[X]/(X^5 - 1),$$

which is equal to:

$$\{X^k(1 + X^2) \mid 0 \leq k < 5\}.$$

Facts that we will use

For odd n, fix a primitive n-th root of unity ω over $\mathbb{F}_2$.
Write

$$C_i := \{s_i, 2s_i, \ldots, 2^{\ell_i - 1} s_i\} \subset \mathbb{Z}_n \qquad \text{(cyclotomic coset)}$$

$$P_i(X) := (X - \omega^{s_i})(X - \omega^{2s_i}) \ldots (X - \omega^{2^{\ell_i - 1} s_i}) \in \mathbb{F}_2[X].$$

Facts:

- $X^n - 1$ factors into irreducible polynomials $P_1(X) \cdots P_m(X)$;
- $(\mathbb{F}_2[X]/P_i(X))^\times$ is isomorphic to the cyclic group $\mathbb{Z}_{2^{\ell_i} - 1}$.

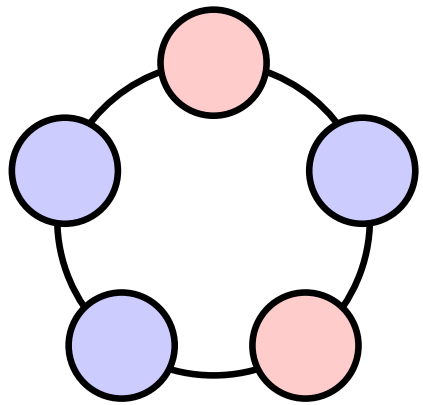
Example:

$$X^5 - 1 = (\underbrace{X + 1}_{P_1(X)})(\underbrace{1 + X + X^2 + X^3 + X^4}_{P_2(X)}),$$

$$(\mathbb{F}_2[X]/P_1(X))^\times \simeq \mathbb{Z}_1; \qquad (\mathbb{F}_2[X]/P_1(X))^\times \simeq \mathbb{Z}_{15}.$$

The bijection for $n = 5$

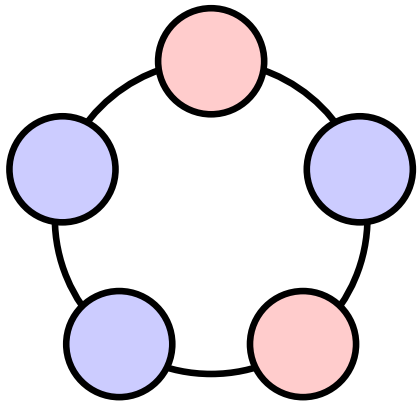
Necklaces divisible by $P_1(X)$ but not $P_2(X)$ $\leftrightarrow$ Nonempty subsets of $\{1, 2, 3, 4\}$ that sums to 0.



$\leftrightarrow$ $\{1, 4\}$.

$\{X^k(1 + X^2) \mid 0 \leq k < 5\}$

An example of the bijection



$$= \quad \{X^k(1+X^2) \mid 0 \le k < 5\}.$$

Take $1+X^3$ as the group generator of $(\mathbb{F}_2[X]/P_2(X))^\times$,

$$\{(1+X^3)^{9k}(1+X^3)^4 \mid 0 \le i < 5\} \mod P_2(X)$$

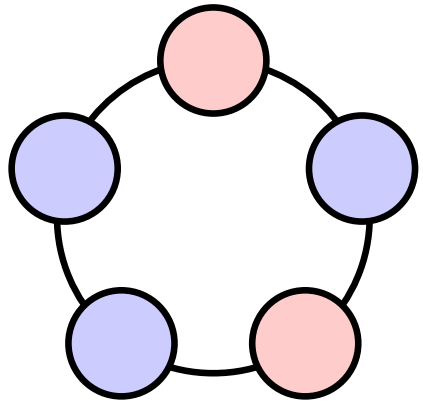
Viewing $(\mathbb{F}_2[X]/P_2(X))^\times$ as the group $\mathbb{Z}_{15}$,

$$\{9k+4 \mid 0 \le k < 5\} \subset \mathbb{Z}_{15}.$$

This gives us

$$\{4, 13, 7, 1, 10\}.$$

An example of the bijection (ctd)



$= \{4, 13, 7, 1, 10\}.$

Take the quotient and the remainder of division by 3:

$$\{\ 3 \cdot 1 + 1\ ,\ 3 \cdot 4 + 1\ ,\ 3 \cdot 2 + 1\ ,\ 3 \cdot 0 + 1\ ,\ 3 \cdot 3 + 1\ \}$$

Exchange the quotient with the remainder, then change 3 to 5:

$$\{\ 5 \cdot 1 + 1\ ,\ 5 \cdot 1 + 4\ ,\ 5 \cdot 1 + 2\ ,\ 5 \cdot 1 + 0\ ,\ 5 \cdot 1 + 3\ \}$$

This gives us:

$$\{6, 9, 7, 5, 8\}.$$

Take the unique subset of $\{1, 2, 4, 8\}$ that sums to 5 mod 15:

$$\{1, 4\}.$$

The bijection for necklaces coprime only to $P_i(X)$

Input: Necklaces coprime only to $P_i(X) = (X - \omega^{s_i}) \dots (X - \omega^{2^{\ell_i - 1} s_i})$.

Output: Nonempty subset of $\{s_i, \dots, 2^{\ell_i - 1} s_i\}$ that sums to 0 mod n.

Algorithm:

(1) View necklace as subset of $(\mathbb{F}_2[X]/P_i(X))^\times = \mathbb{Z}_{2^{\ell_i}-1}$;

(2) Take the quotient and remainder of division by $\frac{(2^{\ell_i}-1)\gcd(s_i,n)}{n}$;

(3) Exchange quotient with remainder;

(4) Change $\frac{(2^{\ell_i}-1)\gcd(s_i,n)}{n}$ to $\frac{n}{\gcd(s_i,n)}$;

(5) Take the unique number that is divisible by $\frac{n}{\gcd(s_i,n)}$;

(6) Output is the unique nonempty subset of $\{s_i, \dots, 2^{\ell_i - 1} s_i\}$ that sums to the number.

How about other necklaces?

Input: A necklace of length n with two colors.

Output: Subsets of $\mathbb{Z}_n$ that sums to 0 mod n.

Algorithm:

(1) View necklaces as elements of $\frac{\mathbb{F}_2[X]}{P_1(X)} \times \frac{\mathbb{F}_2[X]}{P_2(X)} \times \ldots \times \frac{\mathbb{F}_2[X]}{P_k(X)}$;

(2)-(5) Apply analogous steps to $\mathbb{Z}_{2^{\ell_1}-1} \times \mathbb{Z}_{2^{\ell_2}-1} \times \ldots \times \mathbb{Z}_{2^{\ell_k}-1}$;

(6) Output is a subset of $\mathbb{Z}_n$ viewed as $C_1 \cup C_2 \cup \ldots \cup C_k$.

Conclusion

Theorem

If n is odd, then:

$$\begin{array}{c}\#\text{ of necklaces of length} \\ n \text{ with two colors}\end{array} = \begin{array}{c}\#\text{ of subsets of } \mathbb{Z}_n \\ \text{that sums to } 0,\end{array}$$

and is equal to

$$\sum_{I \subseteq \{1,\ldots,m\}} \frac{\gcd(n, (s_i)_{i \in I})}{n} \prod_{i \in I} (2^{\ell_i} - 1).$$

This formula is different from (EC1)'s $\frac{1}{n} \sum_{d|n} 2^{n/d} \phi(d)$.

Conclusion

Theorem

If n is odd, then:

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and is equal to

$$\sum_{I \subseteq \{1,\ldots,m\}} \frac{\gcd(n, (s_i)_{i \in I})}{n} \prod_{i \in I} (2^{\ell_i} - 1).$$

Q: Why odd n?

- Reason 1: so $\mathbb{F}_2[X]/P_1(X), \ldots, \mathbb{F}_2[X]/P_m(X)$ are finite fields;
- Reason 2: so $C_1, \ldots, C_m$ form a partition of $\mathbb{Z}_n$.

Conclusion

Theorem

If n is odd, then:

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and is equal to

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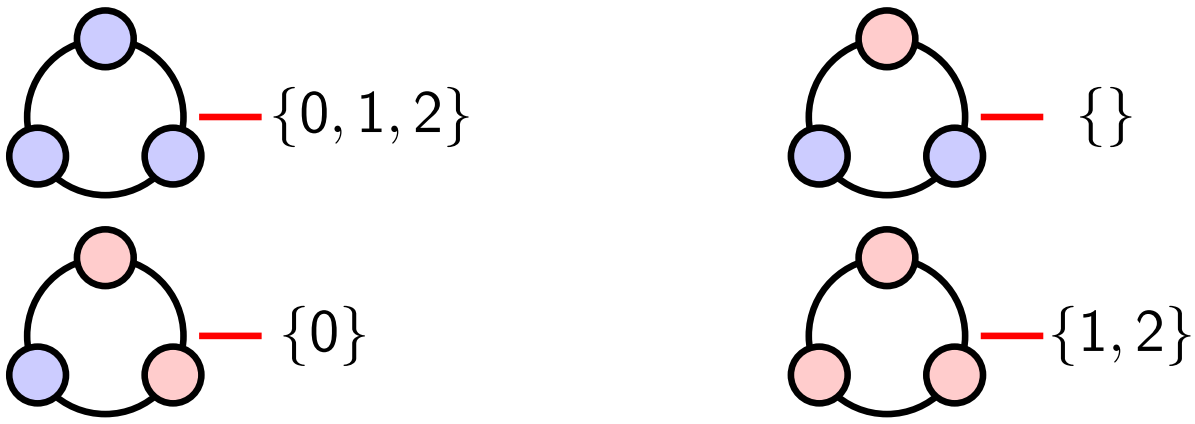
Q: How are those two sets related?

A: They are both secretly a union of products of cyclic groups.

What is next?

Bijection that preserves the number of blue beads?



Possible leads to answering this mystery:

- Number of necklaces with k blue beads is as an evaluation of an arithmetic Tutte Polynomial (Ardila-Castilo-Henley '15).
- Also the number of components in a chamber of the coroot toric arrangement of type A (Aguiar, C. '17).

Why stop at two colors?

Theorem

If q and n are coprime, then:

$$\begin{array}{c}\#\text{ of necklaces of length}\\ n\text{ with }q\text{ colors}\end{array} = \begin{array}{c}\#\text{ of multi-subsets of }\mathbb{Z}_n\text{ with}\\ \text{mult. } < q \text{ that sums to } 0,\end{array}$$

and is equal to

$$\sum_{I\subseteq\{1,\ldots,m\}} \frac{\gcd(n,(s_i)_{i\in I})}{n} \prod_{i\in I}(q^{\ell_i}-1).$$

The proof is bijective if q is prime power.

Bijective proof for the rest of the values of q?

Preprint: arXiv:1802.03507
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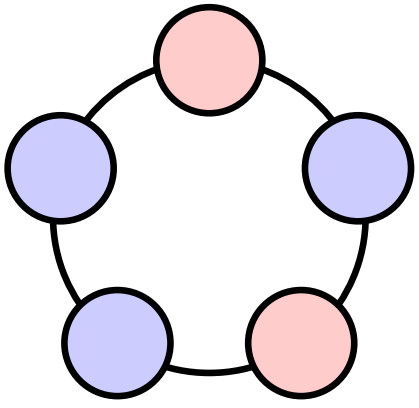
THANK YOU!



$\Rightarrow \{4, 13, 7, 1, 10\} \Rightarrow \{6, 9, 7, 5, 8\} \Rightarrow \{1, 4\}$

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