Several notions of computability-theoretic reducibility between $\Pi^1_2$ principles have been studied. This paper contributes to the program of analyzing the behavior of versions of Ramsey’s Theorem and related principles under these notions. Among other results, we show that for each $n \geq 3$, there is an instance of RT$_{2n}^2$ all of whose solutions have PA degree over $\emptyset(n-2)$, and use this to show that König’s Lemma lies strictly between RT$_{2}^2$ and RT$_{3}^2$ under one of these notions. We also answer two questions raised by Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2016) on comparing versions of Ramsey’s Theorem and of the Thin Set Theorem with the same exponent but different numbers of colors.

Still on the topic of the effect of the number of colors on the computable aspects of Ramsey-theoretic properties, we show that for each $m \geq 2$, there is an $(m+1)$-coloring $c$ of $\mathbb{N}$ such that every $m$-coloring of $\mathbb{N}$ has an infinite homogeneous set that does not compute any infinite homogeneous set for $c$, and connect this result with the notion of infinite information reducibility introduced by Dzhafarov and Igusa (to appear). Next, we introduce and study a new notion that provides a uniform version of the idea of implication with respect to $\omega$-models of RCA$_0$, and related notions that allow us to count how many applications of a principle $P$ are needed to reduce another principle to $P$. Finally, we fill in a gap in the proof of Theorem 12.2 in Cholak, Jockusch, and Slaman (2001).

Keywords: Computable reducibility; uniform reducibility; Weihrauch reducibility; Ramsey’s Theorem; König’s Lemma.

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1. Introduction

Many of the mathematical principles that have been analyzed in the setting of reverse mathematics have the form

\[ \forall X [\Theta(X) \rightarrow \exists Y \Psi(X,Y)] , \]

where \( X \) and \( Y \) range over sets of natural numbers and \( \Theta \) and \( \Psi \) are arithmetic. We think of a true sentence of this form as a problem. An instance of this problem is an \( X \) such that \( \Theta(X) \) holds and a solution to this instance is a \( Y \) such that \( \Psi(X,Y) \) holds. From a computability-theoretic point of view, several ways of comparing the relative strength of two such problems have been studied. These are different ways to formalize the idea of reducing a problem \( P \) to a problem \( Q \), and each of them can give us interesting information within the broad program of analyzing the computability-theoretic content of mathematics.

Definition 1.1. Let \( P \) and \( Q \) be problems.

(1) Recall that a Turing ideal is a nonempty collection of sets closed under join and closed downward under Turing reducibility. A problem \( R \) holds in a Turing ideal \( I \) if for every instance \( X \in I \) of \( R \), there is a solution \( Y \in I \) to \( X \). We write \( P \leq_\omega Q \) to mean that for every Turing ideal \( I \), if \( Q \) holds in \( I \) then so does \( P \). In the language of reverse mathematics, this definition says that every \( \omega \)-model of RCA\(_0^+Q \) is a model of \( P \).

(2) We say that \( P \) is computably reducible to \( Q \), and write \( P \leq_c Q \), if for every instance \( X \) of \( P \), there is an \( X \)-computable instance \( \hat{X} \) of \( Q \) such that, for every solution \( \hat{Y} \) to \( \hat{X} \), there is an \( (X \oplus \hat{Y}) \)-computable solution to \( X \).

(3) We say that \( P \) is strongly computably reducible to \( Q \), and write \( P \leq_{sc} Q \), if for every instance \( X \) of \( P \), there is an \( X \)-computable instance \( \hat{X} \) of \( Q \) such that, for every solution \( \hat{Y} \) to \( \hat{X} \), there is a \( \hat{Y} \)-computable solution to \( X \).

(4) We say that \( P \) is Weihrauch reducible to \( Q \), and write \( P \leq_W Q \), if there are Turing functionals \( \Phi \) and \( \Psi \) such that, for every instance \( X \) of \( P \), the set \( \hat{X} = \Phi^X \) is an instance of \( Q \), and for every solution \( \hat{Y} \) to \( \hat{X} \), the set \( Y = \Psi^{X \oplus \hat{Y}} \) is a solution to \( X \).

(5) We say that \( P \) is strongly Weihrauch reducible to \( Q \), and write \( P \leq_{sW} Q \), if there are Turing functionals \( \Phi \) and \( \Psi \) such that, for every instance \( X \) of \( P \), the set \( \hat{X} = \Phi^X \) is an instance of \( Q \), and for every solution \( \hat{Y} \) to \( \hat{X} \), the set \( Y = \Psi^Y \) is a solution to \( X \).

Note that all of these notions are transitive.

Remark 1.2. The reducibility \( \leq_\omega \) is equivalent to a special case of the notion of computable entailment discussed by Shore [52], which is defined as follows: if \( \Phi, \Psi \) are sentences of second-order arithmetic, then \( \Psi \) computably entails \( \Phi \) if every \( \omega \)-model of RCA\(_0^+ \Psi \) is a model of \( \Phi \).

Weihrauch reducibility has also been called uniform reducibility in this context. It is a more general notion, first introduced by Weihrauch [59, 60] in the
context of computable analysis, and widely studied since. It was used for the kind of computability-theoretic comparison of mathematical principles we are engaged here by Gherardi and Marcone [22], Brattka and Gherardi [2, 3], and several other researchers since then. Dzhafarov [15] introduced the notion of computable reducibility. Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12] then uniformized this notion, yielding the definition of Weihrauch reducibility given above. This definition is different from the one given in the aforementioned papers coming from the computable analysis tradition, but it was proved to be equivalent to (a special case) of the latter in [12, Appendix A].

If $P$ and $Q$ are the trivial problems $\forall X \exists Y[Y = X]$ and $\forall X \exists Y[Y = \emptyset]$, respectively, then $P \leq_W Q$ but $P \not\leq_{sc} Q$. Intuitively, we would probably want to think of $P$ as reducible to any problem, so $\leq_{sc}$ and $\leq_{sw}$ may seem a bit suspect. Nevertheless, they can still give us useful information. To give an example, and for future use, we define some well-known versions of Ramsey’s Theorem and related concepts.

**Definition 1.3.** For a set $X$, let $[X]^n$ be the collection of $n$-element subsets of $X$. We refer to an element of $[X]^n$ as a $n$-tuple, although these elements are unordered. A $k$-coloring of $[X]^n$ is a map $c : [X]^n \to k$. We write $c(x_1, \ldots, x_n)$ for $c(\{x_1, \ldots, x_n\})$. A set $H \subseteq X$ is homogeneous for $c$ if there is an $i < k$ such that $c(s) = i$ for all $s \in [H]^n$. We also say that $H$ is homogeneous to $i$. A set $A \subseteq X$ is prehomogeneous for $c$ if the color of an element of $[A]^n$ depends only on its least $n - 1$ many elements.

**Ramsey’s Theorem for $n$-tuples and $k$ colors** $RT^n_k$ is the statement that every $k$-coloring of $[N]^n$ has an infinite homogeneous set. $RT^n_{\leq \infty}$ is the statement $\forall k RT^n_k$.

$RT$ is the statement $\forall n \forall k RT^n_k$.

A coloring $c : [N]^2 \to 2$ is stable if $\lim_y c(x, y)$ exists for all $x$ (in other words, for each $x$, there is an $i < 2$ such that $c(x, y) = i$ for all sufficiently large $y$). Stable Ramsey’s Theorem for Pairs $SRT^2_2$ is the statement that every stable $2$-coloring of $[N]^2$ has an infinite homogeneous set.

Let $D^2_2$ be the statement that for every stable $2$-coloring of pairs $c$ there is an infinite set $H$ that is limit-homogeneous in the sense that there is an $i < 2$ such that for all $x \in H$, if $y$ is a sufficiently large element of $H$ then $c(x, y) = i$. The principles $SRT^2_2$ and $D^2_2$ are closely related. Indeed, $D^2_2$ captures the way computability theorists generally think of $SRT^2_2$, by considering the task of computing an infinite homogeneous set given a computable stable $2$-coloring of pairs to be equivalent to that of computing a subset of $A$ or its complement $\overline{A}$ given a $\Delta^0_2$ set $A$. $SRT^2_2$ trivially implies $D^2_2$ (indeed, $D^2_2 \leq_{sw} SRT^2_2$), and it was shown by Chong, Lempp, and Yang [9, Theorem 1.4] that $RCA_0 \vdash D^2_2 \rightarrow SRT^2_2$. (Their proof involved showing that $D^2_2$ implies $\Sigma^0_2$-bounding over $RCA_0$, thus justifying a hidden use of $\Sigma^0_2$-bounding in the proof of [7, Lemma 7.10].) It is also easy to see that $SRT^2_2 \leq_{w} D^2_2$. However, Dzhafarov [16] has shown that $SRT^2_2 \not\leq_{w} D^2_2$ (see also [5, Corollary 6.12]) and
instance be an infinite set $c$, principles are equivalent, but we do have $\text{RT}_2^1$, answering this question might be a step toward answering the question in [16] to $\text{COH}$. The exact choice of coding does not affect any of our results.

Remark 1.4. In reverse mathematics, we work in the setting of second-order arithmetic, where the only first-order objects are natural numbers, and the only second-order objects are sets of natural numbers. Thus we must encode other first-order objects, such as tuples of natural numbers and strings, as natural numbers; and other second-order objects, such as colorings and trees, as sets of natural numbers. For the kinds of objects we consider in this paper, though, these codings are straightforward. Furthermore, in the computability-theoretic setting, we may argue informally, and use the fact that it is well understood what is meant by say, for example, that one coloring computes another. Thus, we regard instances and solutions of a problem $P$ as the actual objects mentioned in the (informal) statement of $P$. For example, an instance of $\text{RT}_2^1$ is a 2-coloring of $[\mathbb{N}]^2$. However, it will also be useful at times to be able to think of the instances and solutions of our problems as subsets of $\mathbb{N}$, so we identify all second-order objects with subsets of $\mathbb{N}$ (or, equivalently, elements of $2^\omega$) via some reasonable choice of coding (such as the ones in [54]). The exact choice of coding does not affect any of our results.

Remark 1.5. In almost all cases we consider, it is clear from the informal description of a problem what its instances and solutions are. The only exceptions are $\text{RT}_{\infty}^1$, $\text{RT}_{\infty}^2$, $\text{D}_{\infty}^2$, and $\text{RT}$. One way to formalize $\text{RT}_{\infty}^a$ is to have an instance be a function $f : [\mathbb{N}]^a \to \mathbb{N}$ such that $\text{rng}(f)$ is bounded (and a solution to this instance be an infinite set $H$ such that $f(H)$ is a singleton). Another is to have an instance consist of a number $k$ together with a function $f : [\mathbb{N}]^n \to k$. As the second notion seems closer to the intent of the principle, let us call it $\text{RT}_{\infty}^a$, and let us call the first notion $(\text{RT}_{\infty}^a)'$. In the nonuniform setting, the two principles are equivalent, but we do have $(\text{RT}_{\infty}^a)' \not\leq_w \text{RT}_{\infty}^a$, with a proof similar to that of Theorem 3.3 below. However, none of the results in this paper depend on which of these formalizations we adopt, so we will not treat $(\text{RT}_{\infty}^a)'$ as a separate principle. The same considerations apply to the cases of $\text{SRT}_{\infty}^2$, $\text{D}_{\infty}^2$, and $\text{RT}$. (See also Brattka and Rakotonaina [5], where $\text{RT}_{\infty}^n$ and $(\text{RT}_{\infty}^n)'$ are denoted by $\text{RT}_{n,+}$ and $\text{RT}_{n,ni}$, respectively.)

Dzhafarov [15] showed that $\text{COH} \not\leq_{sc} \text{D}^2_2$, and has recently improved this result in [16] to $\text{COH} \not\leq_{sc} \text{SRT}_2^2$. It is still an open question whether $\text{COH} \leq_C \text{SRT}_2^2$, or, equivalently, whether $\text{COH} \leq_C \text{D}_2^2$. Since $\text{RT}_2^2$ is equivalent to $\text{SRT}_2^2 + \text{COH}$ over RCA$_0$ (as shown by Mileti [41] and Jockusch and Lempp [unpublished], see [8, Item 2]), answering this question might be a step toward answering the question...
of whether $RT_2^2 \leq \omega SRT_2^2$. The latter question has become particularly interesting in light of the proof by Chong, Slaman, and Yang [10] that $RCA_0 + SRT_2^2 \not\vdash RT_2^2$. Note that Dzhafarov [16] has also shown that $COH \not\leq_W SRT_2^2$.

For any two problems $P$ and $Q$, we clearly have

$$P \leq \omega Q \iff P \leq_c Q \iff P \leq_{sc} Q \iff P \leq_{sw} Q.$$ 

No implications between these notions hold in general other than the ones shown or implied by transitivity of implication. We have already seen that $\leq_W$ does not imply $\leq_{sc}$. For the other cases, we can give examples using versions of Ramsey’s Theorem.

Here and below, we adopt standard computability-theoretic notation. In particular, we use $\downarrow$ to indicate that a computation converges, and for an object $X$ being computably approximated, we write $X[s]$ for the stage $s$ approximation to $X$; thus, for a Turing functional $\Phi$, we write $\Phi^X(n)[s]$ for the value of the computation of $\Phi$ with oracle $X$ on input $n$ at stage $s$ (which might of course be undefined if this computation has not yet converged).

If $P$ is $RT_2^1$ and $Q$ is $\forall X \exists Y[Y = X]$ then clearly $P \leq_{sc} Q$. However, $P \not\leq_W Q$. To see that this is the case, assume that $P \leq_W Q$ as witnessed by the functionals $\Phi$ and $\Psi$. Define $c : \mathbb{N} \rightarrow 2$ as follows. Start defining $c(s) = 0$ at each stage $s$ until we find $n \leq s$ such that $\Psi^{\Phi^\psi}(n)[s] = 1$ (adopting the usual convention that the use of this computation must be at most $s$). We must eventually find such $n$ and $s$, since $\Phi^\psi$ is a solution to itself as an instance of $Q$. Then define $c(t) = 1$ for all $t > s$. Any solution to $P$ must consist entirely of numbers greater than $s$, so $\Psi^{c_\psi \Phi^\psi}$ is not a solution to $P$. We will see nontrivial examples of this kind of phenomenon in Sec. 4.

Jockusch [30, Theorems 5.1 and 5.5] showed that for each $n, k \geq 2$, every computable $k$-coloring of $[\mathbb{N}]^n$ has an infinite $\Pi^0_n$ homogeneous set, but there is a computable $k$-coloring of $[\mathbb{N}]^n$ with no infinite $\Sigma^0_n$ homogeneous set. It follows that $RT_k^{n+1} \leq \Pi^0_n$. On the other hand, using results from Jockusch [30, Sec. 5], Simpson [53, Theorem III.7.6] showed that if $n \geq 3$ and $k \geq 2$, then $RT_k^n$ is equivalent to $ACA_0$ over $RCA_0$, so for such $n$ we have $RT_k^{n+1} \leq \Pi^0_n$. (RT$^2_k$, on the other hand, is weaker, not just reverse-mathematically but also in the sense of $\omega$-models, as shown by Seetapun in Seetapun and Slaman [51, Theorem 3.1].)

It is also worth remarking on the connections between these notions of reducibility and implication over $RCA_0$. If $RCA_0 + Q \vdash P$ then $P \leq_\omega Q$, but the example of $RT_k^{n+1}$ and $RT_k^n$, where $n \geq 3$, shows that it is not necessarily the case that $P \leq_c Q$. Conversely, even having $P \leq_{sw} Q$ is not enough to ensure that $RCA_0 + Q \vdash P$, although it is somewhat more difficult to find a counterexample. To do so, we consider the principle $\Pi^0_2 GA$, introduced by Hirschfeldt, Lange, and Shore [27], as a “miniaturization” of the principle $\Pi^0_2 G$ studied by Hirschfeldt, Shore, and Slaman [29].
Definition 1.6. Let $D$ be a property of binary strings. We write $\sigma \in D$ to mean that $D$ holds of $\sigma$. We say that $D$ is dense if for every $\sigma$ there is a $\tau \supseteq \sigma$ such that $\tau \in D$.

$\Pi^0_1 G$ is the principle stating that for any $X$ and any uniformly $\Pi^0_{1,X}$ collection $D_0, D_1, \ldots$ of dense properties of binary strings, there is a $G$ such that

$$\forall i \exists m \, \{ \{ g \mid m \in D_i \} \}.$$ 

$\Pi^0_1 GA$ is the principle stating that for any $X$ and any uniformly $\Pi^0_{1,X}$ collection $D_0, D_1, \ldots$ of dense properties of binary strings, there is a sequence $g_0, g_1, \ldots$ of sets such that

$$\forall i \exists m \exists t \forall u \geq t \, \{ \{ g_u \mid m \in D_i \} \}.$$ 

We can think of $\Pi^0_1 GA$ as a problem as follows. An instance of $\Pi^0_1 GA$ is a sequence $A_0, A_1, \ldots \subseteq \mathbb{N} \times 2^{\omega \omega}$ such that for every $\sigma$ and $i$, there is a $\tau \supseteq \sigma$ for which $\forall k \langle \{ k, \tau \} \in A_i \rangle$ holds. A solution to this instance is a sequence $g_0, g_1, \ldots$ of sets such that

$$\forall i \exists m \exists t \forall u > t \, \forall k \langle \{ g_u \mid m = g_t \mid m \in D_i \} \rangle.$$ 

It is easy to check that $\Pi^0_1 GA$ is uniformly computably true, i.e. that there is a uniform procedure to obtain $g_0, g_1, \ldots$ as above from $A_0, A_1, \ldots$. In particular, $\Pi^0_1 GA \leq_W WKL$. We will see in Proposition 4.9 that this fact implies that $\Pi^0_1 GA \leq_I WKL$. On the other hand, it is shown in [27, Theorem 3.3] that $\Pi^0_1 GA$ implies $\Sigma^0_2$-induction over RCA$_0$ together with $\Sigma^0_2$-bounding. By Hájek [24, Corollary 3.14], WKL is conservative over $\Sigma^0_2$-bounding for arithmetic statements, so RCA$_0 + WKL \nvdash \Pi^0_1 GA$.

In this paper, we consider several issues connected with the above notions of reducibility. In Sec. 2, motivated both by a question about computable reducibility and by one raised by Liu’s proof in [40] that RT$_2^2$ does not imply Weak König’s Lemma (WKL) over RCA$_0$, we show that for every $n \geq 3$, there is a computable 2-coloring of $[\mathbb{N}]^n$ such that any infinite homogeneous set has PA degree over $\emptyset^{(n-2)}$. Using this result we show that full König’s Lemma (KL) lies strictly between RT$_2^2$ and RT$_2^3$ under computable reducibility. We also analyze the limits of codability into solutions of (not necessarily computable) instances of Ramsey’s Theorem and use this analysis to show that WKL is not strongly computably reducible to RT, and prove several other results on the relationships between versions of Ramsey’s Theorem and König’s Lemma with respect to the notions of reducibility defined above.

In Sec. 3, we answer two questions raised by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12] on Weihrauch reducibility between versions of Ramsey’s Theorem and of the Thin Set Theorem with the same exponent but different numbers of colors. We also show that for each $m \geq 2$, there is an $(m + 1)$-coloring $c$ of $\mathbb{N}$ such that every $m$-coloring $d$ of $\mathbb{N}$ has an infinite homogeneous set that does not compute any infinite homogeneous set for $c$, which strengthens both a theorem of Dzhafarov...
Reduction between $\Pi^1_2$ principles

[15], who established this result under the assumption that $d$ is hyperarithmetic in $c$ (and hence showed that $RT^1_3 \not\leq_{sc} RT^1_2$), and one of Dzhafarov and Igusa [17] on the notion of infinite information reducibility introduced by them. Our result demonstrates that, while colorings of $\mathbb{N}$ might at first seem uninteresting, once we look at noncomputable colorings, surprising phenomena can emerge.

The difference between $P \leq c Q$ and $P \leq_\omega Q$ is that the former captures the idea of solving an instance of $P$ by reducing it to a single instance of $Q$, while the latter captures the idea of solving an instance of $P$ by reducing it to multiple instances of $Q$ (taken in parallel, in series, or in a combination of both modes). In Sec. 4, we make this interpretation of $\leq_\omega$ more precise, in a way that allows us to combine this general form of reduction with the idea of uniformity, to arrive at a uniform version of $\leq_\omega$, whose basic properties we study. We also look at ways to count the number of instances of $Q$ used in reducing $P$ to $Q$ (uniformly or not), and to extend our notions to non-$\omega$-models.

Section 5 is a summary of some of our results, their context, and some remaining open questions in the form of diagrams.

In the course of our work on this paper, we found a gap in the proof of the following result in Cholak, Jockusch, and Slaman [7, Theorem 12.2]: Let $n, k \geq 2$ and let $C_0, C_1, \ldots$ be such that $C_i \not\leq_T \emptyset^{(n-2)}$ for all $i$. Then each computable $k$-coloring of $\mathbb{N}^n$ has an infinite homogeneous set $H$ such that $H' \not\leq_T \emptyset^{(n)}$ and $C_i \not\leq_T H$ for all $i$. In Appendix A, we show how to fill in this gap.

We assume familiarity with the basic terminology and notation of computability theory and reverse mathematics, as found for instance in Soare [55] and Simpson [54], respectively. In Sec. 3.2, we will also need some basic facts about the hyperjump (see for instance Sacks [49]). We denote the $e$th $\{0, 1\}$-valued Turing functional in a fixed effective listing of such functionals by $\Phi_e$. A tree is a subset of $\mathbb{N}^{\lt\omega}$ that is closed downward under extension. A tree $T$ is finitely bounded if for each $n$ there is a $k$ such that $\sigma(n) < k$ for every $\sigma \in T$ of length $n + 1$. A binary tree is a tree that is a subset of $2^{\lt\omega}$.

Recall that KL is the statement that every infinite, finitely bounded tree has an infinite path, and Weak König’s Lemma (WKL) is the statement that every infinite binary tree has an infinite path. Recall also that a set has PA degree if it can compute a solution to each computable instance of WKL. More generally, we say that $X$ has PA degree over $Y$, and write $X \geq_Y$ if $X$ can compute a solution to each $Y$-computable instance of WKL. It is well known that $X \geq_Y$ if and only if $X$ computes a completion of the partial function $e \mapsto \Phi_e^X(e)$.

Weak Weak König’s Lemma (WWKL), which comes up often in the reverse mathematics of measure theory, states that if $T$ is a binary tree such that

$$\liminf_n \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n} > 0,$$

then $T$ has an infinite path.
2. Versions of Ramsey’s Theorem and König’s Lemma

In this section, we compare various versions of Ramsey’s Theorem (i.e. $\text{RT}_{n}^{k}$, $\text{SRT}_{k}^{2}$, $\text{D}_{k}^{2}$, $\text{RT}_{k}^{<\infty}$, $\text{SRT}_{<\infty}^{2}$, $\text{D}_{<\infty}^{2}$, RT, and COH) and König’s Lemma (i.e. KL, WKL, and WWKL). We leave a discussion of the case of $\text{RT}_{n}^{j}$ versus $\text{RT}_{k}^{j}$ for $j \neq k$ to the following section. For a summary of our results and how they fit into a more general picture, see the diagrams in Sec. 5. The key new technical result needed here is that there is a computable 2-coloring of triples such that every infinite homogeneous set has PA degree over $\emptyset'$. We begin with a reminder of results for $\leq_{\omega}$, summarized in Fig. 1. Since the number of colors does not matter in this case, the subscript $k$ here stands for any number greater than 1 or for $<\infty$. No other implications than the ones shown (or implied by transitivity) hold. For justifications of these implications and nonimplications, see [25]. There are two arrows with question marks, but they represent the same open question, since $\text{RT}_{2}^{2}$ is equivalent to $\text{SRT}_{2}^{2} + \text{COH}$ over $\text{RCA}_{0}$.

Liu [40] showed that $\text{RT}_{2}^{2}$ does not imply WKL over $\text{RCA}_{0}$. In fact he showed in his Corollary 1.6 that there is a $\omega$-model of $\text{RCA}_{0} + \text{RT}_{2}^{2}$ that does not contain any set of PA degree. It follows immediately that WKL $\not\leq_{\omega} \text{RT}_{2}^{2}$. It also follows immediately that for every computable 2-coloring of $[\mathbb{N}]^{2}$ there is an infinite homogeneous set whose degree is not PA. For coloring of exponent $n \geq 3$ it is natural to work over $\emptyset^{(n-2)}$, since it was shown in Jockusch [30, Lemma 5.9] that there is a computable 2-coloring of $[\mathbb{N}]^{n}$ with every infinite homogeneous set computing $\emptyset^{(n-2)}$. In particular, one might ask whether Liu’s result above could be extended by showing that for every $n \geq 2$ and every computable coloring of $[\mathbb{N}]^{n}$, there is an infinite homogeneous set whose degree is not PA over $\emptyset^{(n-2)}$.

A closely related question comes from considering the relationship between $\text{RT}_{2}^{2}$ and KL. These principles are both equivalent to $\text{ACA}_{0}$ over $\text{RCA}_{0}$. On the other hand, it is not difficult to see that every computable instance of KL has a $\Delta_{0}^{1}$ solution, while by Jockusch [30, Theorem 5.1], there are computable instances of
RT^3_2 with no Σ^0_3 solution, so RT^3_2 ⊈_c KL. We might then ask whether KL ≤_c RT^3_2. Suppose for the moment that this is so. By Jockusch, Lewis, and Remmel [31, Theorem 5] (and the existence of a nonempty Π^0_1 θ' set with all elements of PA degree over θ') there is a computable instance of KL with all solutions of PA degree over θ'. Hence, by our supposition, there must be a computable instance of RT^2_2 with all solutions of PA degree over θ'. Our next result shows that this is the case. We then use this result in relativized form to show in Corollary 2.4 that KL ≤_c RT^2_2 and in fact that KL ≤_W RT^2_2, a result which was obtained independently by Brattka and Rakotoniaina [5].

**Theorem 2.1.** There is a computable 2-coloring of [N]^3 such that any infinite prehomogeneous set has PA degree over θ'.

**Proof.** Let m < s < t. If for all ε < m we have Φ^θ'[ε][s](ε)[s] = Φ^θ'[t][ε][t] (which includes the possibility that both sides diverge) then let c(m, s, t) = 1. Otherwise, let c(m, s, t) = 0.

Suppose that A is prehomogeneous for c. Define a function g ≤_T A as follows.

Given x, let m ∈ A be such that x < m. Search for s, t ∈ A such that m < s < t and Φ^θ'[s][ε][s] = Φ^θ'[t][ε][t] for all ε < m. Such numbers must exist since there are only finitely many possibilities for the outcomes of these computations.

Let g(x) = Φ^θ'[s][ε][s] if the latter is defined, and otherwise let g(x) = 0. By prehomogeneity, Φ^θ'[s][ε][s] = Φ^θ'[u][ε][u] for all u > s in A, so if Φ^θ'[x](s)↓ then Φ^θ'[x](s) = Φ^θ'[s][ε][s] = g(x). Thus g is an A-computable completion of the partial function e → Φ^θ'[ε][e], and hence A ⊈_θ. □

In particular, there is a computable 2-coloring of [N]^3 such that any infinite homogeneous set has PA degree over θ'. We can extend this result to higher exponents as follows. As in [30], a coloring c is called unbalanced if all infinite homogeneous sets are homogeneous to the same color. Note that the coloring constructed in the proof of Theorem 2.1 is unbalanced.

**Corollary 2.2.** Let n ≥ 3. There is an unbalanced computable 2-coloring of [N]^n such that any infinite homogeneous set has PA degree over θ'^{(n-2)}.

**Proof.** We prove a strong relativized version by induction on n, specifically the following: For each n ≥ 3 and each set X there is an unbalanced X-computable 2-coloring of [N]^n such that for any infinite homogeneous set H, the set X ⊕ H has PA degree over X^{(n-2)}. The base case n = 3 is obtained by relativizing Theorem 2.1 to X. Now assume the result for n, where n ≥ 3, and let X be given. By the inductive hypothesis (for X') there is an unbalanced X'-computable 2-coloring c of [N]^n such that for any infinite homogeneous set H, the set X' ⊕ H has PA degree over X^{(n-1)}. By [30, Lemma 5.2], relative to X, there is an X-computable 2-coloring d of [N]^{n+1} such that every infinite set homogeneous for d is homogeneous for c. Furthermore, since c is unbalanced, the proof of [30, Lemma 5.2] shows that we
can take $d$ to be unbalanced. Next, by [30, Theorem 5.7], relative to $X$, there is an $X$-computable unbalanced 2-coloring $c$ of $[\mathbb{N}]^{n+1}$ such that, for every infinite homogeneous set $H$, the set $X \mathbin{\oplus} H$ computes $X^{(n-2)}$ and hence computes $X'$, since $n \geq 3$.

By [30, Lemma 5.10], relative to $X$, there is an unbalanced $X$-computable 2-coloring $f$ of $[\mathbb{N}]^{n+1}$ such that the infinite sets homogeneous for $f$ are precisely the infinite sets homogeneous for both $d$ and $e$. To complete the induction, it suffices to show that every infinite homogeneous set $H$ for $f$ is such that $X \mathbin{\oplus} H$ has PA degree over $X^{(n-1)}$. So assume that $H$ is homogeneous for $f$. It follows that $H$ is homogeneous for $d$ and hence for $c$, so $X' \mathbin{\oplus} H$ has PA degree over $X^{(n-1)}$. Since $H$ is homogeneous for $e$, the set $X \mathbin{\oplus} H$ computes $X'$. Hence $X \mathbin{\oplus} H \geq_T X' \mathbin{\oplus} H$, so $X \mathbin{\oplus} H$ has PA degree over $X^{(n-1)}$ as needed.

Thus we see that the analog of Liu’s Theorem for higher exponents fails for each exponent $n > 2$.

We now consider the relation between König’s Lemma and various versions of Ramsey’s Theorem. Our first result is a corollary to Theorem 2.1.

**Corollary 2.3** (due independently to Brattka and Rakotoniaina [5]).

$\mathsf{KL} \leq_w \mathsf{RT}_2^3$.

**Proof.** We show first that $\mathsf{KL} \leq_c \mathsf{RT}_2^3$. Let $X$ be any instance of $\mathsf{KL}$, so $X$ is an infinite, finitely branching tree. By relativizing Theorem 2.1 to $X$, we obtain an $X$-computable 2-coloring $c$ of $[\mathbb{N}]^3$ such that for any infinite homogeneous set $P$, the set $X \mathbin{\oplus} P$ has PA degree over $X'$. It now suffices to show that, for every such $P$, the set $X \mathbin{\oplus} P$ computes a solution to $X$, i.e. a path through $X$. This fact is clear because the paths through $X$ are bounded by a function $b \leq_T X'$, and thus $X$ is essentially a computable binary tree relative to $X'$. In more detail, let $b(n)$ be the greatest value of $\sigma(n)$ over all strings $\sigma \in X$ of length $n + 1$. (There are only finitely many such strings because $X$ is finitely branching, and at least one such string because $X$ is infinite.) Clearly $b \leq_T X'$. Thus $X$ is an infinite tree that is $X'$-computable and $X'$-computably bounded. As remarked in [35, p. 606], for every computable, computably bounded tree $T$, there is a computable tree $U \subseteq 2^{<\omega}$ such that the degrees of the paths through $T$ coincide with the degrees of the paths through $U$. Relativizing this remark to $X'$ yields the fact that every set $C$ of PA degree over $X'$ is such that $C \mathbin{\oplus} X'$ computes a path through $X$. In particular, if $P$ is an infinite homogeneous set for $c$, then $X \mathbin{\oplus} P$ is of PA degree over $X'$, so $X \mathbin{\oplus} P \mathbin{\oplus} X'$ computes a path through $X$. But $X \mathbin{\oplus} P \geq_T X'$ since $X \mathbin{\oplus} P$ is of PA degree over $X'$, so $X \mathbin{\oplus} P$ computes a path through $X$, as needed to complete the proof that $\mathsf{KL} \leq_c \mathsf{RT}_2^3$.

To show that $\mathsf{KL} \leq_w \mathsf{RT}_2^3$, observe that the coloring $c$ obtained from the proof of (the relativized version of) Theorem 2.1 is such that there is a uniform procedure taking infinite homogeneous sets for $c$ to completions of the partial function $e \mapsto \Phi^X_c(e)$. From such a completion we can uniformly obtain $X'$, and hence uniformly obtain a path through $X$.\[\Box\]
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The following corollary compares the computability-theoretic strength of KL with that of Ramsey’s Theorem for pairs and triples. It shows that the strength of KL should be, fancifully speaking, roughly that of RT\(^2\)\(^2\). Of course, \( P <_c Q \) means that \( P \leq_c Q \) and \( Q \not\leq_c P \).

**Corollary 2.4.** RT\(^2\)\(^2\) \( <_c \) KL \( <_c \) RT\(^3\)\(^2\).

**Proof.** The corollary we have just proved shows that KL \( \leq_c \) RT\(^3\)\(^2\), and we have already remarked that RT\(^3\)\(^2\) \( \not\leq_c \) KL. To see that RT\(^2\)\(^2\) \( \leq_c \) KL, note that in [30, Proposition 4.6], it is shown that for any computable 2-coloring of pairs there is a finitely branching computable tree \( T \) such that every path through \( T \) computes an infinite homogeneous set. (The fact that \( T \) is finitely branching is clear from the proof.) Relativizing the proof of this result shows that RT\(^2\)\(^2\) \( \leq_c \) KL. Since there is a computable instance of KL with all solutions computing \( \emptyset' \) (by [31, Corollary 5.1], for example) but no computable instance of RT\(^2\)\(^2\) with all solutions computing \( \emptyset' \) by Seetapun’s Theorem [51, Theorem 2.1], we have that KL \( \not\leq_c \) RT\(^2\)\(^2\).

The same proof also shows that RT\(^2\)\(^\infty\) \( <_c \) KL. We can improve this result by applying the following fact.

**Proposition 2.5.** For any problem \( P \), if \( P \leq_c KL \) then \( P \leq_{sc} KL \).

**Proof.** Let \( X \) be an instance of \( P \) and let \( T \) be an \( X \)-computable instance of KL such that for any solution \( Y \) to \( T \), there is an \( (X \oplus Y) \)-computable solution to \( X \). Let \( \hat{T} \) consist of all \( \sigma \) for which there is a \( \tau \in T \) with \( \sigma(2n) = \tau(n) \) for all \( 2n < |\sigma| \) and \( \sigma(2n+1) = X(n) \) for all \( 2n+1 < |\sigma| \). Then \( \hat{T} \) is also an \( X \)-computable instance of KL, and every solution \( \hat{Y} \) to \( \hat{T} \) computes \( X \oplus Y \) for some solution \( Y \) to \( T \), and hence computes a solution to \( X \).

**Corollary 2.6.** RT\(^2\)\(^\infty\) \( \leq_{sc} KL \).

On the other hand, RT\(^2\)\(^2\) \( \not\leq_W KL \). We will prove a stronger version of this result in part (2) of Theorem 2.10.

For WKL we have a considerably stronger version of Proposition 2.5.

**Proposition 2.7.** For any problem \( P \), if \( P \leq_\omega WKL \) then \( P \leq_{sc} WKL \).

**Proof.** Suppose that \( P \leq_\omega WKL \) and let \( X \) be an instance of \( P \). Let \( T \) be an \( X \)-computable infinite binary tree such that every infinite path on \( T \) has PA degree over \( X \). If \( Z \) is an infinite path on \( T \) then, by the work of Scott [50], there is a Turing ideal \( I \) in which WKL holds such that \( X \in I \) and \( I \) consists entirely of \( Z \)-computable sets. Since \( P \leq_\omega WKL \), \( X \) has a solution in \( I \) and hence a \( Z \)-computable solution.

We will prove a uniform version of Proposition 2.7 in Proposition 4.9.

The next result will be used to show that KL \( \not\leq_{sc} RT^2 \), and indeed WKL \( \not\leq_{sc} RT \).
Theorem 2.8. The following are equivalent for \( n, k \geq 2 \) and \( X \subseteq \mathbb{N} \).

(i) There is a \( k \)-coloring \( c \) of \( [\mathbb{N}]^n \) (not necessarily computable) such that every infinite homogeneous set computes \( X \).

(ii) \( X \) is hyperarithmetic.

Proof. To show that (i) implies (ii), assume (i). Given an infinite set \( A \), the restriction of \( c \) to \( A \) has an infinite homogeneous set \( H \subseteq A \). Since \( H \) is also homogeneous for \( c \), it computes \( X \). Thus every infinite set has a subset that computes \( X \), i.e. \( X \) is encodable. It follows from Solovay [56, Theorem 2.3] that \( X \) is hyperarithmetic.

The converse follows from the remark in Jockusch [30, p. 278] that there is a 2-coloring of \( [\mathbb{N}]^2 \) such that all infinite homogeneous sets compute all hyperarithmetic sets. We give a proof here for the convenience of the reader. By Jockusch and McLaughlin [32, Theorem 4.13], there is an increasing function \( f \) such that \( X \leq_T f \) and \( f \leq_T g \) for every function \( g \) that dominates \( f \). Now for \( x_1 < x_2 < \cdots < x_n \), let \( c(x_1, \ldots, x_n) = 1 \) if \( x_2 \geq f(x_1) \) and otherwise let \( c(x_1, \ldots, x_n) = 0 \). Then if \( H \) is any infinite homogeneous set for \( c \), it is easily seen that \( H \) is homogeneous to 1 and that \( p_H \) dominates \( f \), where \( p_H \) is the function that enumerates \( H \) in increasing order. It follows that \( X \leq_T f \leq_T p_H \leq_T H \).

Note that the above result fails for \( n = 1 \). Indeed, Dzhafarov and Jockusch showed in [18, Lemma 5.2(i)] that if (i) above holds for \( n = 1 \) and \( k = 2 \), then \( X \) is computable.

Corollary 2.9. WKL \( \not\leq_{sc} \text{RT} \).

Proof. Let \( X \) be any set that is not hyperarithmetic. Then WKL has an instance whose only solution is \( X \), but there is no instance of RT such that all solutions compute \( X \).

Monin and Patey [42] have recently extended this result from WKL to WWKL, answering a question stated in an earlier version of this paper. (The weaker version with \( \leq_{W} \) in place of \( \leq_{sc} \), which had also been a question in an earlier version of this paper, was answered independently by Brattka and Rakotoniaina [5].)

We finish this section with various facts about versions of Ramsey’s Theorem and of König’s Lemma. Part (4) of the following theorem was proved independently by Brattka and Rakotoniaina [5].

Theorem 2.10. (1) \( \text{RT}^1_1 \not\leq_{W} \text{WKL} \).

(2) \( D^2_k \not\leq_{W} \text{KL} \) (and hence \( \text{SRT}^2_k \not\leq_{W} \text{KL} \)).

(3) \( \text{RT}^1_k \leq_{W} D^2_k \) (and hence \( \text{RT}^1_k \leq_{W} \text{SRT}^2_k \)) for all \( k \geq 2 \). Similarly, \( \text{RT}^1_{\infty} \leq_{W} D^2_{\infty} \).

(4) \( \text{RT}^1_{k+1} \not\leq_{W} \text{SRT}^2_k \) (and hence \( \text{RT}^1_{k+1} \not\leq_{W} D^2_k \)) for all \( k \geq 2 \).

(5) \( \text{RT}^1_2 \not\leq_{W} \text{COH} \).
(10) RT

Proof. (1) Suppose RT

(2) Suppose D

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(6) COH \(\leq_{SW} KL\).

(7) RT

(8) RT

(9) RT

(10) RT

Proof. (1) Suppose RT

(2) Suppose D

Begin building \(c : N \to 2\) by letting \(c(s) = 0\) at stage \(s\), and let \(T^c\) be the infinite binary tree coded by \(\Phi^c\). Wait until we find an \(n\) and an \(s\) such that \(T^c[\sigma][s]\) for all \(\sigma \in 2^n\), and for each such \(\sigma\) for which \(\sigma \in T^c\), there is an \(m < s\) such that \(m \in \Psi^{\sigma}[s]\). Such \(n\) and \(s\) must eventually be found, as otherwise \(T^c\) is not total or, by compactness, it has an infinite path \(X\) for which \(\Psi^{\sigma}[X]\) is not an infinite set. When they are found, let \(c(t) = 1\) for all \(t \geq s\). Let \(X\) be an infinite path on \(T^c\). Since \(X\) extends some \(\sigma \in T^c \cap 2^n\), we thus ensure that \(\Psi^{\sigma}[X]\) is not an infinite homogeneous set for \(c\).

(2) Suppose D

Begin building \(c\) by letting \(c(s) = 0\) at stage \(s\), and let \(T^d\) be the infinite, finitely branching tree coded by \(\Phi^d\). Since we have access to \(\emptyset'\), we also have access to a function \(f\) such that \(T^d \subseteq f^{<\omega}\). Wait until we find an \(n\) and an \(s\) such that \(T^d[\sigma][s]\) for all \(\sigma \in f^n\), and for each such \(\sigma\) for which \(\sigma \in T^d\), there is an \(m < s\) such that \(m \in \Psi^{\sigma}[s]\). As in part (1), such \(n\) and \(s\) must eventually be found. When they are found, let \(c(t) = 1\) for all \(t \geq s\). Let \(X\) be an infinite path on \(T^d\). Since \(X\) extends some \(\sigma \in T^d \cap f^n\), we thus ensure that \(\Psi^{\sigma}[X]\) is not an infinite homogeneous set for \(c\), and hence is not an infinite limit-homogeneous set for \(c\).

(3) Given \(c : N \to k\), let \(d(x, y) = c(x)\) for all \(x < y\); then \(d\) is stable, and any set that is limit-homogeneous for \(d\) is homogeneous for \(c\).

(4) Suppose that RT

Suppose \(c : N \to (k + 1)\) in stages. Let \(d^c\) be the coloring of \([N]^2\) coded by \(\Phi^c\), and for \(x < s\) let \(e(x, s) = d^c(x, u)[s]\) for the largest \(u\) such that \(d^c(x, u)[s]\) (or undefined if there is no such \(u\)). At each stage \(s\), we have parameters \(i_j, F_j, n_j\) (which we call the \(j\)-parameters) for \(j < k\), all of which are initially undefined. Proceed as follows at stage \(s\) for each \(j < k\).

If the \(j\)-parameters are undefined then look for a finite set \(F\) with \(\max F < s\) such that \(d^c(x, y) = j\) for all \(x, y \in F\) with \(x < y\) and \(e(x, s) = j\) for all \(x \in F\),
and an $n < s$ such that $n \in \Psi^{e \oplus F}[s]$. If such $F$ and $n$ exist then let $F_j = F$, let $n_j = n$, and let $i_j = c(n)$.

If the $j$-parameters are defined and $c(x, s) \neq j$ for some $x \in F_j$, then redefine the $j$-parameters.

At the end of stage $s$, define $c(s)$ to be different from $i_j$ for each $i_j$ that is currently defined.

Now let $H$ be an infinite homogeneous set for $d^c$ and let $j$ be the color to which it is homogeneous. Then $\Psi^{e \oplus H}$ is an infinite homogeneous set for $c$. There are a finite $E \subset H$ and an $n$ such that $n \in \Psi^{e \oplus E}$, so a final $F_j$ and $n_j$ will be found. This $F_j$ is homogeneous to $j$ for $d^c$, and there is a $y$ such that $d^c(x, z) = j$ for all $x \in F_j$ and $z > y$, and $\Psi^{e \oplus F_j}(n_j) = 1$ with use at most $y$. Then $I = F_j \cup (H \upharpoonright y)$ is an infinite homogeneous set for $d^c$, and $n_j \in \Psi^{e \oplus I}$. However, $c(s) \neq i_j = c(n_j)$ for all sufficiently large $s$, so $\Psi^{e \oplus I}$ is not an infinite homogeneous set for $c$.

(5) Suppose $RT^1_2 \leq_W \text{COH}$ as witnessed by $\Phi$ and $\Psi$. Begin building $c : \mathbb{N} \to 2$ by letting $c(s) = 0$ at stage $s$, until we find a finite set $F$ and an $n$ such that $n \in \Psi^{e \oplus F}$. Then ensure that $c(n) = 0$ and $c(t) = 1$ for all sufficiently large $t$. Now, $\Phi^c$ is an instance of COH. No matter what this instance is, there is a solution $S$ to it such that $F \subset S$ and $\min S \setminus F$ is larger than the use of $\Psi^{e \oplus F}(n)$. Then $n \in \Psi^{e \oplus S}$, so $\Psi^{e \oplus S}$ is not an infinite homogeneous set for $c$.

This result can also be obtained from [4, Proposition 4.3], since, in the terminology of that paper, COH is densely realized, while $RT^1_2$ is indiscriminate. We will strengthen it in Sec. 4.3.

(6) Let $\mathcal{A} = (A_0, A_1, \ldots)$ be a sequence of sets. Let $A_i^1 = A_i$ and $A_i^0 = A \overline{A}_i$. Let $T$ be the binary tree consisting of all $\sigma$ such that $|\bigcap_{i < |\sigma|} A^{\pi(i)}_i| \geq |\sigma|$. It is easy to see that $T$ is infinite and $\mathcal{A}$-computable. By the relativized form of Jockusch, Lewis, and Remmela [31, Theorem 5], there is an $\mathcal{A}$-computable finitely branching tree $U$ such that the infinite paths on $U$ are in an effective, degree-preserving one-to-one correspondence with those of $T$. The definition of $U$ in that proof is uniform, so $U$ can be obtained uniformly from $\mathcal{A}$, and from an infinite path on $U$, we can uniformly obtain an infinite path $P$ on $T$. We have $|\bigcap_{i < n} A^{P(i)}_i| = \omega$ for all $n$, so from $\mathcal{A} \oplus P$ we can uniformly obtain a cohesive set $C = \{c_0, c_1, \ldots\}$ for $A$ by letting $c_0 = 0$ and letting $c_{n+1}$ be the least number greater than $c_n$ in $\bigcap_{i < n} A^{P(i)}_i$.

(7) Given a function $f : \mathbb{N} \to \mathbb{N}$, let $T$ consist of all sequences of the form $(i, n_0, (i, n_1), \ldots, (i, n_{k-1}))$ where $n_0 < \cdots < n_{k-1}$ are the first $k$ many numbers $n$ such that $f(n) = i$. Then $T$ is uniformly $f$-computable. If $\text{rng}(f)$ is bounded then $T$ is finitely branching and from any infinite path on $T$ we can compute an infinite homogeneous set for $f$ in a uniform way.

(8) Given a function $f : \mathbb{N} \to \mathbb{N}$, let $c : [\mathbb{N}]^2 \to 2$ be defined by letting $c(x, y) = 1$ if $f(x) = f(y)$ and $c(x, y) = 0$ otherwise. Suppose $H$ is homogeneous for $c$. If $f$ has bounded range, then $H$ cannot be homogeneous to $0$, so it is homogeneous to $1$ and hence also homogeneous for $f$.

(9) Given a function $f : \mathbb{N} \to \mathbb{N}$, let $A_i = \{n : f(n) = i\}$, and let $C$ be cohesive for $A_0, A_1, \ldots$. The $A_i$ are pairwise disjoint, and if $f$ has bounded range, then only
is a $\Sigma^0_k$ total.) Such a number exists since

$$n < \mu(C, [\sigma]) \geq \frac{1}{2},$$

where $[\sigma] = \{ X : \sigma \prec X \}$. For notational convenience we assume that $\sigma$ is the empty string, i.e. $\mu(C, \emptyset) > \frac{1}{2}$. We claim now that there is a computable function $f$ such that $[n, f(n)] \cap A \neq \emptyset$ for all $n$. The claim suffices to prove the lemma, since if $f$ is as in the claim, then it follows by induction on $n$ that $A$ has at least $n$ many elements less than $f^{(n)}(0)$ for each $n$, where $f^{(n)}$ is the $n$th iterate of $f$.

We now prove the claim. Given $n$, to calculate $f(n)$ search for a number $k$ such that

$$\mu(\{ X : \Phi^X \cap [n, k) \neq \emptyset \}) > \frac{1}{2}$$

and let $f(n)$ be the first such $k$ that is found. (In this definition, we identify $\Phi^X$ with $\{ j : \Phi^X(j) = 1 \}$ without assuming that $\Phi^X$ is total.) Such a number $k$ exists since

$$\frac{1}{2} < \mu(C) \leq \mu(\{ X : \Phi^X \text{ is infinite} \}) \leq \mu(\{ X : \exists j \{ j \in \Phi^X \wedge j \geq n \} \}) \leq \lim_{z \to \infty} \mu(\{ X : \exists j \leq z \{ j \in \Phi^X \wedge j \geq n \} \}),$$

where the final equation follows easily from the countable additivity of $\mu$. Thus all sufficiently large numbers $z$ meet the criterion to be chosen as $k$. Furthermore, $k$ can be found by effective search since the condition $\mu(\{ X : \Phi^X \cap [n, k) \neq \emptyset \}) > \frac{1}{2}$ is a $\Sigma^0_k$ predicate of $n$ and $k$. Hence $f$ is a (total) computable function. It remains to be shown that the interval $[n, f(n)]$ always contains at least one element of $A$. Since $C_e$ and $\{ X : \Phi^X \cap [n, k) \neq \emptyset \}$ each have measure greater than $\frac{1}{2}$, these two sets must have a nonempty intersection. Fix a set $X$ that belongs to both sets. Then $\Phi^X$ is a subset of $A$ that intersects the interval $[n, f(n)]$, so $A$ intersects this interval.

Now to complete the proof of (10), let $A$ be any set such that both $A$ and $A$ are hyperimmune, and let the coloring $c$ be the characteristic function of $A$. For $A$ we may choose any weakly 1-generic set, i.e. any set that meets every dense c.e. set of binary strings. By Kurtz [38, Corollaries 2.10 and 2.11], every hyperimmune
degree contains a weakly 1-generic set, and every such set is hyperimmune and has a hyperimmune complement.

3. The Number of Colors

3.1. Weihrauch reducibility and the number of colors

In [12, Theorem 3.1], Dorais, Dzhafarov, Hirst, Mileti, and Shafer showed that \( RT_k^n \lesssimW RT_j^n \) for all \( n \geq 1 \) and \( k > j \geq 2 \). They asked whether this result remains true for Weihrauch reducibility. We show here that it does. This result was obtained independently by Brattka and Rakotoniaina [5, 48]. During the final stages of preparation of this paper, we were informed that Patey [45] has proved the stronger result that \( RT_k^n \lesssimW RT_j^n \) for all \( n \geq 2 \) and \( k > j \geq 2 \) (with a proof considerably more involved than the one we give for our weaker result).

To motivate our proof, let us first consider the \( n = 1 \) case. Here the proof is easy: Assume for a contradiction that there is a Weihrauch reduction of \( RT_k^1 \) to \( RT_j^1 \) given by the functionals \( \Phi \) and \( \Psi \). We build a \( k \)-coloring \( c \) of \( \mathbb{N} \). For each \( i < j \), we search for a finite set \( F_i \) such that every element of \( F_i \) has \( \Phi^c \)-color \( i \) and \( \Psi^c \supseteq F_i \) \( (m_i) \) = 1, say with use \( u_i \), for some \( m_i \). Then we ensure that \( c(m) \neq c(m_i) \) for all sufficiently large \( m \). Since there are at most \( j \) values of \( i \) such that \( m_i \) is defined, and we have \( k \) colors available with \( k > i \), this is easy to arrange. Let \( i \) be such that infinitely many numbers have \( \Phi^c \)-color \( i \). Then \( F_i \) must eventually be found, since otherwise there is an infinite homogeneous set \( H \) for \( \Phi^c \) such that \( \Psi^c \upharpoonright H \) is not infinite. There is an infinite homogeneous set \( I^c \) for \( \Phi^c \) such that \( F_i \subset I^c \) and \( \min(I \setminus F_i) > u_i \). But \( m_i \in \Phi^c \supseteq I^c \), and \( c(m) \neq c(m_i) \) for all sufficiently large \( m \), so \( \Psi^c \upharpoonright I^c \) is not an infinite homogeneous set for \( c \), giving us the desired contradiction.

To lift this construction to higher exponents, we proceed as in the proof of part (2) of Theorem 2.10. Instead of constructing a computable coloring of \( n \)-tuples directly, we construct an \( \emptyset^{(n-1)} \)-computable coloring \( c \) of singletons. By the recursion theorem and the iterated form of the limit lemma, we may assume that we know an index for the reduction from \( \emptyset^{(n-1)} \) to \( c \), and hence know an index for a computable coloring \( d \) of \( n \)-tuples such that \( c(x) = \lim_{i} s_i \cdots s_{n-1} \), \( d(x, s_1, \ldots, s_{n-1}) \). Note that if an infinite set is homogeneous for \( d \) then it is homogeneous for \( c \).

Of course, we need to deal with the fact that \( \Phi^d \) is a coloring of \( n \)-tuples. But as we will see, we have access to an infinite \( \Pi_1^{0,\emptyset^{(n-1)}} \) set \( P \) such that the \( \Phi^d \)-color of a tuple in \( [P]^n \) depends only on its least element. Let \( e \) be the induced coloring of \( P \) (i.e. \( e(x) \) is the \( \Phi^d \)-color of a tuple in \( [P]^n \) with least element \( x \)). Then, as long as we stay inside \( P \), we can work with the coloring of singletons \( e \) in place of \( \Phi^d \), and proceed much as above, to build \( c \) so that, for some infinite homogeneous set \( H \) for \( e \), and hence for \( \Phi^d \), the set \( \Psi^d \upharpoonright H \) is not an infinite homogeneous set for \( c \), and hence is also not an infinite homogeneous set for \( d \).

There is a slight complication in that we have to approximate both \( e \) and \( P \), and hence might have to make several attempts at defining our \( m_i \) and \( F_i \) before we hit on suitable ones, but this fact will not affect the construction much.
Note that by Jockusch [30, Theorem 5.5], each computable 2-coloring \(d\) of \(n\)-tuples has an infinite \(\Pi^0_1\) homogeneous set \(P\). However, this fact does not serve our purposes here because this homogeneous set \(P\) is obtained by a nonuniform construction. By weakening the requirement of homogeneity, we can obtain uniformity using proofs from [30], as we now show.

**Lemma 3.1.** Let \(n \geq 1\) and \(k \geq 2\) and let \(X \subseteq \mathbb{N}\).

1. Let \(Y \gg X'\). There is an effective procedure that, from the index of an \(X\)-computable \(k\)-coloring \(c\) of \([\mathbb{N}]^n\), produces an index for a \(Y\)-computable infinite prehomogeneous set for \(c\). (Note that we are not claiming that the definition of this procedure is uniform with respect to \(Y\).)

2. There is an effective procedure that, from the index of an \(X\)-computable \(k\)-coloring \(c\) of \([\mathbb{N}]^n\), produces an index for a \(\Pi^0_1, X'\) infinite prehomogeneous set for \(c\).

**Proof.** For part (1), use the proof of [30, Lemma 5.4], together with the fact we have already noted that if \(Y \gg X'\) then each infinite \(X\)-computable finitely branching tree has a \(Y\)-computable path.

For part (2), if \(n = 2\) use the proof of [30, Theorem 4.2]. The set \(M\) constructed there is \(\Pi^0_2\), uniformly obtained, and prehomogeneous. An elaboration of the same argument gives the full result. However, we omit the proof since we need the result only for \(n = 2\) in the proof of the following lemma.

**Lemma 3.2.** Let \(n, k \geq 2\). Then there is an effective procedure that, from the index of a computable \(k\)-coloring of \([\mathbb{N}]^n\), produces an index for an infinite \(\Pi^0_1, \emptyset^{(n-1)}\) set \(P\) such that the color of a tuple in \([P]^n\) depends only on its least element.

**Proof.** For each \(i \leq n - 2\), let \(X_i \gg \emptyset^{(i)}\) be low over \(\emptyset^{(i)}\). Such sets exist by the low basis theorem. We first claim that, for each \(i \leq n - 2\), there is an effective procedure that, from an index of a computable \(k\)-coloring of \([\mathbb{N}]^n\), produces an index for an \(X_i\)-computable infinite set \(P_i\) such that the color of a tuple in \([P_i]^n\) depends only on its least \(n - i\) many elements.

We proceed by induction. The \(i = 0\) case is trivial. Now assume the claim holds for \(i - 1 \leq n - 3\). Applying the first part of Lemma 3.1 to the induced coloring of \([P_{i-1}]^{n-i+1}\) (with \(X = X_{i-1}\), noting that \(X'_{i-1} \equiv_T \emptyset^{(i)}\)) yields the desired procedure for obtaining \(P_i\).

Now applying the second part of Lemma 3.1 to the induced coloring of \([P_{n-2}]^2\) (with \(X = X_{n-2}\), noting that \(X'_{n-2} \equiv_T \emptyset^{(n-1)}\)) yields the desired procedure for obtaining \(P\).

**Theorem 3.3** (due independently to Patey [45] and Brattka and Rakotoniaina [5, 48]). Let \(n \geq 1\) and \(k > j \geq 2\). Then \(\text{RT}^0_k \not\leq_W \text{RT}^0_j\).
If there is no such pair, then proceed to Step 3. Otherwise, let

\[ F \]

Step 3.

Define \( c \) is defined, 

Assume for a contradiction that there is a Weihrauch reduction from \( \text{RT}_1^n \) to \( \text{RT}_0^m \) given by the functionals \( \Phi \) and \( \Psi \). We build an \( \emptyset^{(n-1)} \)-computable \( k \)-coloring of singletons \( c : \mathbb{N} \to k \). As mentioned above, we may assume that we know an index for the reduction from \( \emptyset^{(n-1)} \) to \( c \), and hence know an index for a computable coloring \( d : [\mathbb{N}]^n \to k \) such that \( c(x) = \lim_{i=1}^{\infty} d(x, s_1, \ldots, s_{n-1}) \). (If \( n = 1 \) then we just take \( d = c \).) Note that if an infinite set is homogeneous for \( d \) then it is homogeneous for \( c \). By Lemma 3.2, we have access to an infinite \( \Pi^0_{(n-1)} \) set \( P \) such that the \( \Phi^d \)-color of a tuple in \( [P]^n \) depends only on its least element. Let \( e \) be the induced coloring of \( P \). Let \( P_s \) be stage \( s \) of an \( \emptyset^{(n-1)} \)-computable approximation to \( P \). For \( x \in P_s \), let \( e_s(x) \) be the \( \Phi^d \)-color of the least element of \( [P_s]^n \) with first element \( x \) (in some fixed ordering of \( [\mathbb{N}]^n \)). We think of \( e_s \) as a coloring of \( P_s \). Note that \( e(x) = \lim_{i=1} e_s(x) \) for all \( x \in P \).

As discussed above, the idea is to build \( c \) as in the \( n = 1 \) case described above, but via a \( \emptyset^{(n-1)} \)-computable construction instead of a computable one, with \( \Phi^c \) replaced by \( e \), and working inside \( P \). Our task is to build \( c \) so that \( \Phi^d \) has an infinite homogeneous set \( H \) such that \( \Psi^{d \oplus H} \) is not an infinite homogeneous set for \( d \).

We have parameters \( m_0, \ldots, m_{j-1} \in \mathbb{N} \) and \( F_0, \ldots, F_{j-1} \subset \mathbb{N} \), with the latter being finite sets when defined. All parameters are initially undefined. Whenever \( m_i \) is defined, \( c(m_i) \) will also be defined. Let \( D_i \) be the \( i \)th finite set in a standard listing of finite sets.

At stage \( s \), we act as follows. There are three steps to this stage.

Step 1. For each \( i < j \), if \( F_i \) is defined and either \( F_i \not\subseteq P_s \) or \( e_s(x) \neq e_{s-1}(x) \) for some \( x \in F_i \), then undefine \( m_i \) and \( F_i \).

Step 2. Search for the least pair \( (i, l) \) with \( i < j \) and \( l < s \), if any, such that \( F_i \) is undefined and for \( F = D_l \),

1. \( F \subseteq P_s \),
2. \( e_s(x) = i \) for all \( x \in F \), and
3. \( \Psi^{d \oplus F}(m)[s] = 1 \) for some \( m \) such that \( c(m) \) is defined.

If there is no such pair, then proceed to Step 3. Otherwise, let \( F_i = F \) and let \( m_i = m \).

Step 3. Define \( c(s) \) to be the least number different from \( c(m_i) \) for each \( m_i \) that is defined.

There is an \( i < j \) such that \( e(x) = i \) for infinitely many \( x \in P \). It is then easy to see that there is a stage \( s \) at which \( F_i \) becomes defined and is never later undefined. Furthermore, \( F_i \subseteq H \) for some infinite \( H \subseteq P \) such that \( \min(H \setminus F_i) \) exceeds the use of the computation \( \Psi^{d \oplus F_i}(m_i) \) and \( H \) is homogeneous for \( e \), and hence for \( \Phi^d \). Then \( m_i \in \Psi^{d \oplus H} \), but for every sufficiently large \( M \), we have \( c(m) \neq c(m_i) \), so \( \Psi^{d \oplus H} \) is not an infinite homogeneous set for \( c \), and hence is not an infinite homogeneous set for \( d \).
Note that in the $n = 2$ case of the above proof, $d$ is stable, and $\Psi^{d\oplus H}$ is in fact not an infinite limit-homogeneous set for $d$. Thus we have the following result.

**Theorem 3.4 (due independently to Patey [45]).** If $k > j \geq 2$ then $D^2_k \not\leq_W RT^2_{k,j}$, and hence $SRT^2_k \not\leq_W RT^2_{j,k}$.

We can use our exponent-lifting technique to answer another question raised by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12]. In that paper, they studied the following Ramsey-theoretic principles, related to the Thin Set Theorem, whose reverse-mathematical analysis was introduced by Friedman [21] and further developed by Cholak, Giusto, Hirst, and Jockusch [6] and Wang [58], among others.

**Definition 3.5.** A set $S$ is thin for a $k$-coloring $c$ of $[\mathbb{N}]^n$ if there is an $i < k$ such that $f(s) \neq i$ for all $s \in [S]^n$. Let $TS^n_k$ be the statement that every $k$-coloring of $[\mathbb{N}]^n$ has an infinite thin set.

Note that $TS^2_k$ is the same as $RT^2_{k,j}$. Indeed, let $RT^2_{k,j}$ be the statement that for every $k$-coloring $c$ of $[\mathbb{N}]^n$, there is an infinite set $S$ such that the image of $[S]^n$ under $c$ consists of at most $j$ many colors. Then $RT^2_k$ becomes $RT^2_{k,1}$, and $TS^2_k$ becomes $RT^2_{k,k-1}$. The general reverse-mathematical study of this family of principles was suggested by J. Miller (see Montalbán [43, Sec. 2.2.3]) and has been pursued by Lepi, Miller and Ng (see [43]), Wang [58], and Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12].

Among several other results, Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12, Proposition 5.2 and Theorem 5.27] showed that if $k > j \geq 2$ and $n \geq 1$ then $TS^2_{k,j} \not\leq_W TS^2_k$ and $TS^2_j \not\leq_W TS^2_n$. They left open whether these results can be extended to prove the analog of the former result for higher exponents. As we will see, we can use the same exponent-lifting strategy as above to show that this is indeed the case. Patey [personal communication] has announced the stronger result that $TS^2_{k,j} \not\leq_W TS^2_k$ for $n \geq 2$ and $k > j \geq 2$.

To motivate our proof, let us review the proof in [12, Theorem 5.27] that $TS^2_{k,j} \not\leq_W TS^2_k$ for $k > j \geq 2$. Assume for a contradiction that there is a Weihrauch reduction of $TS^2_{k,j}$ to $TS^2_k$ given by the functionals $\Phi$ and $\Psi$. To obtain a contradiction, we define a computable $k$-coloring of singletons $c$ such that $\Phi^c$ (which is a $j$-coloring of singletons) has an infinite thin set $I$ such that $\Psi^{c\oplus I}$ is not a thin set for $c$. A key observation is that if $d$ is a $k$-coloring of $\mathbb{N}$ and $F$ is a finite set whose image under $d$ consists of at most $i \leq k - 2$ many colors, then $F$ is contained in an infinite set $I$ whose image under $d$ consists of at most $i + 1$ many colors, and hence is thin for $d$. For instance, let $I = F \cup H$, where $H$ is an infinite homogeneous set for $d$.

Our construction now proceeds as follows. We start building a $k$-coloring of singletons $c$ (say by setting $c(x) = 0$ for $x = 0, 1, \ldots$) while looking for a finite set $F_1$ such that the image of $F_1$ under $\Phi^c$ consists of only one color and $\Psi^{c\oplus F_1}(m_1)_{\downarrow} = 1$ for some $m_1$, say with use $u_1$. (In the foregoing, we require that any oracle information about $c$ uses only the part of $c$ already defined when the first such set $F_1$ is found.) We must eventually find such an $F_1$, since otherwise $c$ is total.
and hence there is an infinite homogeneous set \( I \) for \( \Phi^c \). (Then \( \Psi^{c \oplus I} \) is infinite, so we can take \( m_1 \) to be any element of \( \Psi^{c \oplus I} \) and \( F_1 \) to be a finite subset of \( I \) with \( \Psi^{c \oplus F_1}(m_1) = 1 \).) We then continue defining \( c \) and ensure that \( c(m) \neq c(m_1) \) for all sufficiently large \( m \). Now, assuming \( j > 2 \), we look for a finite set \( F_2 \supseteq F_1 \) such that \( \min(F_2 \setminus F_1) > u_1 \), the image of \( F_2 \) under \( \Phi^c \) consists of at most two colors, and \( \Psi^{c \oplus F_2}(m_2) = 1 \), say with use \( u_2 \), for some \( m_2 \) such that \( c(m_2) \neq c(m_1) \). Again, we must eventually find such an \( F_2 \) because, as noted above, \( F_1 \) is contained in an infinite set \( I \) with \( \min(I - F_1) > u_1 \) such that image of \( I \) under \( \Phi^c \) consists of at most two colors, and \( \Psi^{c \oplus I} \) must be infinite. We then ensure that \( c(m) \neq c(m_2) \) for all sufficiently large \( m \). Note that \( \Psi^{c \oplus F_2}(m_1) = 1 \) via the same computation as \( \Psi^{c \oplus F_1}(m_1) = 1 \), since \( F_2 \) and \( F_1 \) agree below \( u_1 \).

We continue in this way to define \( F_i \) and \( m_i \) for all \( i < j \), in such a way that the \( m_i \) all have different \( c \)-colors and \( \Psi^{c \oplus F_i}(m_i) = 1 \) with use \( u_i \), where \( \min(F_{i+1} \setminus F_i) > u_i \). The image of \( F_{j-1} \) under \( \Phi^c \) consists of at most \( j - 1 \leq k - 2 \) many colors, so it is contained in an infinite thin set \( I \) for \( \Phi^c \). We may assume that \( \min(I - F_{j-1}) > u_j \). Then \( \Psi^{c \oplus I}(m_i) = \Psi^{c \oplus F_i}(m_i) = 1 \) for all 0 < \( i < j \). But \( \Psi^{c \oplus I} \) must be infinite, so \( \Psi^{c \oplus I}(m) = 1 \) for some \( m \) such that \( c(m) \neq c(m_i) \) for all 0 < \( i < j \). Then the image of \( \Psi^{c \oplus I} \) under \( c \) consists of \( j \) many colors, so \( \Psi^{c \oplus I} \) is not a thin set for \( c \), which is a contradiction.

We are now ready to prove the following theorem.

**Theorem 3.6.** Let \( n \geq 1 \) and \( k > j \geq 2 \). Then \( \text{TS}_j^n \not\subseteq \text{TS}_k^n \).

**Proof.** Assume for a contradiction that there is a Weihrauch reduction from \( \text{TS}_k^n \) to \( \text{TS}_j^n \) given by the functionals \( \Phi \) and \( \Psi \). We build an \( \emptyset^{\langle n-1 \rangle} \)-computable \( k \)-coloring of singletons \( c : \mathbb{N} \to k \). Let \( d, P, e, P_s \), and \( e_s \) be as in the proof of Theorem 3.3. Note that if an infinite set is thin for \( d \) then it is thin for \( c \). As in the proof of Theorem 3.3, the idea is to build \( c \) as in the \( n = 1 \) case described above, but via an \( \emptyset^{\langle n-1 \rangle} \)-computable construction instead of a computable one, with \( \Phi^c \) replaced by \( c \), and working inside \( P \). Our task is to build \( c \) so that \( \Phi^d \) has an infinite thin set \( I \) such that \( \Psi^{d \oplus I} \) is not an infinite thin set for \( d \).

We have parameters \( m_1, \ldots, m_{j-1}, u_1, \ldots, u_{j-1} \in \mathbb{N} \) and \( F_0, \ldots, F_{j-1} \subseteq \mathbb{N} \), with the latter being finite sets when defined. We will always have \( F_0 = \emptyset \). All other parameters are initially undefined. Whenever \( m_i \) is defined, \( c(m_i) \) will also be defined. Let \( D_i \) be the \( i \)th finite set in a standard listing of finite sets.

At stage \( s \), we act as follows. There are three steps to this stage.

**Step 1.** If there exists \( i \) such that \( F_i \) is defined and either \( F_i \not\subseteq P_s \) or \( e_s(x) \neq e_{s-1}(x) \) for some \( x \in F_i \), then, for the least such \( i \) and all \( l \geq i \), undefine \( m_l, u_l \), and \( F_l \). If there is no such \( i \), do nothing.

**Step 2.** Let \( i \) be largest such that \( F_i \) is defined. If \( i = j - 1 \) then proceed to Step 3. Otherwise, search for the least \( l < s \), if any, such that, letting \( F = D_l \),

\[(1) \ F_l \subseteq F \subseteq P_s,\]

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Lemma 2.3 in [15] remains true when we add the condition that
must be an<br>
to ensure that for every $n$
In his proof that $\text{COH} \not\leq^* \Pi_2^0$
many intersections of the form
3.2. Strong computable reducibility, nonuniform infinite information reducibility, and the number of colors
In his proof that $\text{COH} \not\leq^* \Pi_2^0$, Dzhafarov [15] showed something stronger, namely that if $n \geq 2$ and $m < 2^n$, there is a sequence $R = (R_0, \ldots, R_{n-1})$ of sets such that for any partition $A_0, \ldots, A_{m-1}$ of $\mathbb{N}$ hyperarithmetical in $R$, there is an infinite subset of some $A_j$ that does not compute any cohesive set for $R$. There are $2^n$ many intersections of the form $\bigcap_{i < n} S_i$ with each $S_i$ equal to either $R_i$ or $\overline{R_i}$. As Dzhafarov [personal communication] has remarked, it is easy to adapt the proof in [15] to ensure that all but $m + 1$ many of these intersections are empty. (For those familiar with this proof, all that needs to be done is the following: Choose a set $I \subseteq 2^n$ of size $m + 1$. It is easy to restrict the set of conditions in the proof to ensure that for every $n$, there is a $\rho \in I$ such that $R_i(n) = \rho(i)$ for all $i < n$. Lemma 2.3 in [15] remains true when we add the condition that $u_0, \ldots, u_{n-1} \in I$. In the proof of [15, Lemma 2.4], we again restrict ourselves to such $u_0, \ldots, u_{n-1}$, and
may still conclude in the last paragraph that there are two different such n-tuples for which we can pick the same \( j < m \), since \( |I| > m \). Thus we can obtain an \((m + 1)\)-coloring \( c \) of \( \mathbb{N} \) from \( R \) such that every infinite homogeneous set for \( c \) is cohesive for \( R \). Of course, a partition \( A_0, \ldots, A_{m-1} \) of \( \mathbb{N} \) yields an \( m \)-coloring of \( \mathbb{N} \). Thus the proof in [15] yields the following result.

**Theorem 3.7 (Dzhafarov [15]).** Let \( 2 \leq j < k \). Then there is a \( c : \mathbb{N} \to k \) such that for every \( c \)-hyperarithmetic \( d : \mathbb{N} \to j \), there is an infinite homogeneous set for \( d \) that does not compute any infinite homogeneous set for \( c \). In particular, \( \text{RT}_j^2 \nless_{sc} \text{RT}_k^1 \).

In fact, since we can view a stable coloring of pairs \( c \) as a \( c' \)-computable coloring \( d \) of \( \mathbb{N} \), with limit-homogeneous sets for \( c \) corresponding to homogeneous sets for \( d \), Dzhafarov’s proof shows that if \( 2 \leq j < k \) then \( \text{RT}_k^1 \nless_{sc} \text{D}_j^c \). More recently, Dzhafarov, Patey, Solomon, and Westrick [19] have shown that, in fact, if \( 2 \leq j < k \) then \( \text{RT}_k^1 \nless_{sc} \text{SRT}_j^2 \), answering a question stated in an earlier version of this paper.

Montalbán [personal communication] asked whether the first statement in Theorem 3.7 might hold without any computability assumptions on \( d \). We will give a positive answer to this question in Theorem 3.9. (This theorem has been independently obtained by Patey [45], whose proof remarkably shows that \( c \) can be chosen to be low.) This result demonstrates a perhaps surprising difference in the degrees of homogeneous sets resulting from partitions of \( \mathbb{N} \) into \( j \) parts and into \( k \) parts for \( j \neq k \).

Our result is also connected with one of the several reducibilities Dzhafarov and Igusa [17] call “notions of robust information coding”. These include the generic-case and coarse reducibilities defined by Jockusch and Schupp [33], as well as several ones introduced in [17]. One of these is infinite information reducibility, denoted by \( \leq_{\text{ii}} \). A partial oracle for a set \( A \) is a set \( (A) \) such that if there is an \( s \) with \( \langle n, 0, i \rangle \in (A) \) then \( n \notin A \), and if there is such an \( s \) with \( \langle n, 1, i \rangle \in (A) \) then \( n \in A \). The domain of \((A)\) is the set of \( n \) such that \( \langle n, i, s \rangle \in (A) \) for some \( s \) and some \( i < 2 \).

We say that \( B \leq_{\text{ii}} A \) if there is a Turing functional \( \Gamma \) such that for every partial oracle \( (A) \) for \( A \) with infinite domain, \( \Gamma^{(A)} \) has infinite domain and is equal to \( B \) where defined.

As noted in [17], infinite information reducibility acts in the opposite way to what one might at first expect. For instance, as shown in [17, Proposition 3.6], if \( A \leq_{\text{ii}} B \) then \( B \leq_{\text{ii}} A \), and \( A \oplus B \) is the infimum of \( A \) and \( B \) under \( \leq_{\text{ii}} \). Another interesting feature of \( \text{ii} \)-reducibility is the existence of maximal pairs.

**Theorem 3.8 (Dzhafarov and Igusa [17, Theorem 5.1]).** There are sets \( B_0 \) and \( B_1 \) for which there is no set \( A \) such that \( B_0 \leq_{\text{ii}} A \) and \( B_1 \leq_{\text{ii}} A \).

One of the motivations of Dzhafarov and Igusa in introducing \( \text{ii} \)-reducibility was its potential connections with the computability-theoretic analysis of versions of Ramsey’s Theorem. In this regard, it is particularly interesting to examine the
nonuniform analog to ii-reducibility. We say that $B$ is nonuniformly infinite information reducible to $A$, and write $B \leq_{nii} A$, if each partial oracle $(A)$ for $A$ with infinite domain computes a partial function that has infinite domain and is equal to $B$ where defined. The connection with Ramsey’s Theorem is particularly apparent when we consider the following equivalent definition: $B \leq_{nii} A$ if every infinite subset of $A$ or $\overline{A}$ computes an infinite subset of $B$ or $\overline{B}$.

To see that these definitions are indeed equivalent, first suppose that the first definition holds of $A$ and $B$ and let $S$ be an infinite subset of $A$ or $\overline{A}$. Let $i = 1$ if $S \subseteq A$ and let $i = 0$ otherwise. Then $\{(n, i, 0) \mid n \in S\}$ is an $S$-computable partial oracle for $A$, so it computes a partial function $f$ that has infinite domain and is equal to $B$ where defined. Given $f$ we can compute an infinite subset of $B$ or $\overline{B}$. Now suppose that the second definition holds of $A$ and $B$ and let $(A)$ be a partial oracle for $A$ with infinite domain. From $(A)$ we can compute an infinite subset $S$ of $A$ or $\overline{A}$. From $S$ we can compute an infinite subset of $B$ or $\overline{B}$, and from such a subset we can compute a partial function with infinite domain that is equal to $B$ where defined.

Using ideas from its proof, we can extend Theorem 3.8 to nii-reducibility. Given a maximal pair $B_0, B_1$ for nii-reducibility, let $c : \mathbb{N} \rightarrow 4$ be defined by $c(n) = 2B_0(n) + B_1(n)$. Then $c$ is a 4-coloring of $\mathbb{N}$ such that for every 2-coloring $d$ of $\mathbb{N}$, there is an infinite homogeneous set for $d$ that does not compute any infinite homogeneous set for $c$. By working directly with colorings (but still using ideas from the proof of Theorem 3.8), we can make $c$ a 3-coloring, and generalize this result as follows.

**Theorem 3.9 (due independently to Patey [45]).** Let $m \geq 2$. There is a $c : \mathbb{N} \rightarrow (m + 1)$ such that for every $d : \mathbb{N} \rightarrow m$, there is an infinite homogeneous set for $d$ that does not compute any infinite homogeneous set for $c$.

**Proof.** We will show that any sufficiently Cohen generic $c : \mathbb{N} \rightarrow (m + 1)$ has this property. To determine a sufficient level of genericity, we begin with a few definitions based on the notion of $\alpha$-deduction for ordinals $\alpha$ in [17, Definition 5.5]. A Mathias condition is a pair $(F, I)$ such that $F$ is a finite set, $I$ is an infinite set, and $\max F < \min I$. We write $(F, I) \leq (G, J)$ if $I \subseteq J$ and $G \subseteq F \subseteq G \cup J$. Given a coloring $d : \mathbb{N} \rightarrow m$, the construction of an infinite homogeneous set for $d$ that does not compute any infinite homogeneous set for $c$ will use Mathias conditions. Thus we will need to adapt the notion of $\alpha$-deduction to work relative to a given infinite set $I$.

Let $\Phi_0, \Phi_1, \ldots$ be an effective listing of the $[0, m]$-valued Turing functionals. For a finite set $F$, we write $\Phi^F_c(n)\downarrow$ to mean that $\Phi^F_c(n)\downarrow$ with use no larger than $\max F + 1$ (where we take this max to be $-1$ for $F = \emptyset$). A set $X$ is compatible with the Mathias condition $(F, I)$ if $F \subseteq X \subseteq F \cup I$. Note that if $\Phi^F_c(n)\downarrow$ and $X$ is compatible with $(F, I)$ then $\Phi^X_c(n)\downarrow = \Phi^F_c(n)$.

**Definition 3.10.** Let $I$ be an infinite set and let $F$ be a finite set. We say that $F$ $\alpha$-deduces relative to $I$ that $\Phi_c(n) = i$ if either $\Phi^F_c(n)\downarrow = i$ or there are infinitely
many $k \in I$ such that $F \cup \{k\}$ $\beta$-deduces relative to $I$ that $\Phi_e(n) = i$ for some $\beta < \alpha$. We say that $F$ deduces relative to $I$ that $\Phi_e(n) = i$ if it $\alpha$-deduces relative to $I$ that $\Phi_e(n) = i$ for some $\alpha$.

Note that it is possible for $F$ to deduce relative to $I$ both that $\Phi_e(n) = i$ and that $\Phi_e(n) = j$ for some $i \neq j$. Note also that for each $I$ there is a countable ordinal $\gamma_I$ such that if $F$ deduces relative to $I$ that $\Phi_e(n) = i$, then $F$ $\alpha$-deduces relative to $I$ that $\Phi_e(n) = i$ for some $\alpha < \gamma_I$.

For each number $e$, finite set $F$, and infinite set $I$, define a partial function $f_{e,F,I}$ by letting $f_{e,F,I}(n) = i$ if and only if $F$ deduces relative to $I$ that $\Phi_e(n) = i$ but does not deduce relative to $I$ that $\Phi_e(n) = j$ for any $j \neq i$.

**Remark 3.11.** It is easy to show by transfinite induction on $\alpha$ that if $F$ $\alpha$-deduces relative to $I$ that $\Phi_e(n) = j$, then for every $k$ there is a finite set $G \subset I$ with $\min G > k$ such that $F^{G,n}_e$ deduces relative to $I$ that $\Phi_e(n) = i$. If $F$ 0-deduces relative to $I$ that $\Phi_e(n) = i$, then $F^{G,n}_e$ deduces relative to $I$ that $\Phi_e(n) = i$ for all $G \subset I$ such that $\min G > \max F$, so $F$ cannot deduce relative to $I$ that $\Phi_e(n) = j$ for $j \neq i$, and hence $f_{e,F,I}(n) = i$.

The basic reason for choosing our coloring $c$ to be generic is to ensure that, thinking of $c$ as a partition of $\mathbb{N}$, each part “knows nothing about” the other parts. In fact, we will need it to be the case that if $F$ and $G$ are disjoint subsets of $[0,m]$ such that $F \cup G \neq [0,m]$, then $c^{-1}(F)$ “knows nothing about” $c^{-1}(G)$, in the sense that (roughly speaking) we cannot use $c^{-1}(F)$ to compute an infinite subset of $c^{-1}(G)$, even with the help of some additional information, represented by a Turing ideal that we now define. (We will be more precise about the properties we need $c$ to satisfy below. Note that the requirement that $F \cup G \neq [0,m]$ is necessary, since $c^{-1}(F)$ can compute $c^{-1}([0,m] \setminus F)$.)

**Definition 3.12.** Let $\mathcal{I}$ be a countable Turing ideal such that for each $X \in \mathcal{I}$ there is a $Y \in \mathcal{I}$ with the following properties.

1. For any $X$-computable infinite set $I$ and any ordinal $\alpha \leq \gamma_I$, given numbers $e, n$, and $i$ and a finite set $F$, we can use $Y$ to compute whether $F$ $\beta$-deduces relative to $I$ that $\Phi_e(n) = i$ for some $\beta < \alpha$. (In particular, taking $\alpha = \gamma_I$, given numbers $e, n$, and $i$ and a finite set $F$, we can use $Y$ to compute whether $F$ deduces relative to $I$ that $\Phi_e(n) = i$, and hence we can use $Y$ to compute whether $f_{e,F,I}(n) = i$.)

2. For any $X$-computable infinite set $I$, given a number $e$ and finite sets $F$ and $G$, we can use $Y$ to compute whether $\text{dom } f_{e,F,I} \subseteq \text{dom } f_{e,G,I}$.

3. For any nonempty $\Pi^0_1$ subclass $C$ of $\omega^\omega$, we can use $Y$ to compute the range of some element of $C$.

Though we do not need this fact here, we note that we can take $\mathcal{I}$ to be the smallest hyperjump-ideal, i.e. the smallest Turing ideal closed under the hyperjump.

For sets $A$ and $B$, we say that $A$ is $(B \oplus \mathcal{I})$-computable if it is $(B \oplus X)$-computable for some $X \in \mathcal{I}$, that $A$ is $\mathcal{O}^B\mathcal{I}$-computable if it is $\mathcal{O}^B\mathcal{X}$-computable.
for some $X \in \mathcal{I}$ (where $O^Y$ is the hyperjump of $Y$), and that a subclass of $\omega^w$ is a $\Pi^0_1$ class if it is a $\Pi^0_X$ class for some $X \in \mathcal{I}$.

Let $c : \mathbb{N} \rightarrow (m+1)$ be sufficiently generic so that the following conditions hold. (The first condition was already mentioned above; the second will be needed in the proof of Lemma 3.17.)

(C1) Let $F, G \subseteq [0, m]$ be such that $F \cap G = \emptyset$ and $F \cup G \neq [0, m]$. Then $O^{c^{-1}(F) \oplus I}$ does not compute any infinite subset of $c^{-1}(G)$.

(C2) Let $D_0, D_1, \ldots$ be an $\mathcal{I}$-computable sequence of finite sets such that $\min D_n > n$ for all $n$. Then for each $i \leq m$ there are infinitely many $n$ such that $c(x) = i$ for all $x \in D_n$.

Note that, in particular, there is no infinite homogeneous set for $c$ in $\mathcal{I}$.

**Remark 3.13.** One way to see that such a $c$ exists is the following: Let $Z$ be a set computing all elements of $\mathcal{I}$. Then 1-genericity relative to $Z$ suffices to ensure that (C2) holds. Let $F$ and $G$ be as in (C1). Hyperarithmetic genericity relative to $O^Z$ suffices to ensure that there is no infinite subset of $c^{-1}(G)$ hyperarithmetic in $c^{-1}(F) \oplus O^Z$. A slight variation on Exercise IV.3.13 of Sacks [49] shows that if $c$ is sufficiently generic relative to $Z$ then $O^{c^{-1}(F) \oplus Z}$ is hyperarithmetically reducible to $c^{-1}(F) \oplus O^Z$. Thus, if $c$ is sufficiently generic then there is no infinite subset of $c^{-1}(G)$ computable (or even hyperarithmetic) in $O^{c^{-1}(F) \oplus Z}$.

Fix $d_0 : \mathbb{N} \rightarrow m$. We must show that there is an infinite homogeneous set for $d_0$ that does not compute any infinite homogeneous set for $c$. It will be convenient to “work inside” a certain infinite set $I \in \mathcal{I}$. The property we desire for $I$ is that for each $i < m$ either $d_0^{-1}(i) \cap I$ is empty or there is no infinite subset of $I \setminus d_0^{-1}(i)$ in $\mathcal{I}$. This property can be ensured by requiring that $|d(I)|$ (the number of colors assigned by $d$ to elements of $I$) be as small as possible as $I$ ranges over infinite sets in $\mathcal{I}$. Fix such a “minimal” $I$. Suppose that there are $l$ many values of $i < m$ such that $d_0^{-1}(i) \cap I$ is nonempty. By permuting the numbers less than $m$ we may assume that these $l$ many values of $i$ are exactly $0, 1, \ldots, l-1$, so that $d_0 | I$ is an $l$-coloring of $I$. Let $d = d_0 | I$. Then $d$ has the property that for each $i < l$ there is no infinite subset of $I \setminus d^{-1}(i)$ in $\mathcal{I}$. We may assume without loss of generality that $l \geq 2$ since clearly $l > 0$, while if $l = 1$ then $I$ is an infinite homogeneous set for $d_0$, and $I$ cannot compute an infinite homogeneous set for $c$, as there are no infinite homogeneous sets for $c$ in $\mathcal{I}$. Our goal now is to construct an infinite set $S \subset I$ that is homogeneous for $d$ and computes no infinite homogeneous set for $c$, which of course suffices to prove the theorem since $S$ is also homogeneous for $d_0$.

To give some intuition for our construction, let us consider the case where $l = 2$. We may assume that every infinite subset of $d^{-1}(1)$ computes an infinite homogeneous set for $c$, as otherwise we are done. It is enough to build an infinite set $S \subseteq d^{-1}(0)$ to satisfy the requirements

$$R_e : |\text{dom} \Phi^S_e| = \omega \Rightarrow \exists n [\Phi^S_e(n) \upharpoonright \neq c(n)],$$
as it then clearly follows that $S$ cannot compute an infinite homogeneous set for $c$. The idea is to build a sequence of Mathias conditions $(F_0, I_0) > (F_1, I_1) > \cdots$ beginning with $F_0 = \emptyset$ and $I_0 = \text{dom} \ d = I$, using $(F_{e+1}, I_{e+1})$ to satisfy $R_e$, and to let $S = \bigcup_e F_e$. (We can ensure that $S$ is infinite by also making $F_{e+1} \supseteq F_e$ for all $e$.) As in many Mathias forcing constructions, the easiest way to satisfy $R_e$ is to have a finite set $F \subseteq d^{-1}(0)$ compatible with $(F_e, I_e)$ and an $n$ such that $\Phi^F_e(n) \downarrow \neq c(n)$. We then set $F_{e+1} = F$ and let $I_{e+1}$ be the set of all elements of $I_e$ greater than $\max F$. We call this outcome a “finite win”. If there is no such finite set $F$, then we want to define $(F_{e+1}, I_{e+1}) < (F_e, I_e)$ so that for any $X$ compatible with $(F_{e+1}, I_{e+1})$, the domain of $\Phi^X_e$ is finite. (Notice that $S$ is such an $X$.)

As we will show in Lemma 3.14 of our more general setting, the assumption that we are not in the “finite win” case allows us to prove the following Correctness Lemma by transfinite induction: If the finite set $H \subseteq d^{-1}(0)$ is compatible with $(F_e, I_e)$, then $f_{e,H,I}(n) = c(n)$ when defined.

Let $\mathcal{C}$ be the $\Pi^0_1$ class of all $g \in \omega^\omega$ such that $g$ is strictly increasing, $\text{rng} \ g \subseteq I_e$, and $\text{dom} \ f_{e,F_e \cup H,I_e} \subseteq \text{dom} \ f_{e,F_e,I_e}$ for all finite $H \subseteq \text{rng} \ g$. If $\mathcal{C}$ is nonempty, then it has an element $g$ such that $Y = \text{rng} \ g$ is in $\mathcal{I}$. We claim that we can take $I_{e+1} = Y$ and satisfy $R_e$. Indeed, suppose that $F_e \subseteq X \subseteq F_e \cup Y$. It is easy to see using Remark 3.11 that if $\Phi^X_e(n) \downarrow$ then $n \in \text{dom} \ f_{e,F_e,I_e}$. However, the graph of $f_{e,F_e,I_e}$ is in $\mathcal{I}$, and by the Correctness Lemma, $f_{e,F_e,I_e}(n) = c(n)$ when defined, so $\text{dom} \ f_{e,F_e,I_e}$ cannot be finite, as otherwise there would be an infinite homogeneous set for $c$ in $\mathcal{I}$. Thus $\text{dom} \ \Phi^X_e$ is finite, as desired.

So we are left with arguing that $\mathcal{C}$ is nonempty. Assume for a contradiction that $\mathcal{C} = \emptyset$. As we will show in Lemma 3.15, it follows that there is a finite set $G \subseteq d^{-1}(0)$ compatible with $(F_e, I_e)$ such that $\text{dom} \ f_{e,G \cup \{k\},I_e} \not\subseteq \text{dom} \ f_{e,G,I_e}$ for infinitely many $k \in I_e$. Let $E$ be the set of such $k$. Note that $E \in \mathcal{I}$. For each $k \in E$, let $n_k$ be the least element of $\text{dom} \ f_{e,G \cup \{k\},I_e} \setminus \text{dom} \ f_{e,G,I_e}$. We claim that there is no $n$ such that $n_k = n$ for infinitely many $k \in E$. Assume for a contradiction that there is such an $n$. By the definition of deducibility, the fact that $n \in \text{dom} \ f_{e,G \cup \{k\},I_e}$ for infinitely many $k \in I_e$ implies that there is an $i$ such that $G$ deduces relative to $I_e$ that $\Phi_e(n) = i$. By the Correctness Lemma, the fact that $G$ is a subset of $d^{-1}(0)$ and is compatible with $(F_e, I_e)$ implies that $G$ cannot deduce relative to $I_e$ that $\Phi_e(n) = i$ unless $i = c(n)$. It follows that $n \in \text{dom} \ f_{e,G,I_e}$, which is a contradiction and hence establishes our claim.

It is now easy to see that there are an $i \leq m$ and an $\mathcal{I}$-computable sequence $k_0, k_1, \ldots \in E$ such that $n_{k_p} > p$ and $f_{e,G \cup \{k_p\},I_e}(n_{k_p}) = i$ for all $p$. Let $j \leq m$ be such that $j \neq i$. The sequence $n_{k_0}, n_{k_1}, \ldots$ is also $\mathcal{I}$-computable, so condition (C2), applied to the singletons $\{n_{k_0}\}, \{n_{k_1}\}, \ldots$, implies that there are infinitely many $p$ such that $c(n_{k_p}) = j$. Thus

$$R = \{k \in E : f_{e,G \cup \{k\},I_e}(n_k) = i \land c(n_k) = j\}$$

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is infinite. (In this case, we can argue that $R$ is infinite for some choice of $j \neq i$ without appealing to condition (C2), but that condition will be crucial in obtaining an analogous set in the full proof.)

If $k \in R$ then $G \cup \{k\}$ is compatible with $(F_e, I_e)$ but does not satisfy the conclusion of the Correctness Lemma, so it must be the case that $G \cup \{k\} \not\subseteq d^{-1}(0)$. Since $G \subseteq d^{-1}(0)$ and $l = 2$, it follows that $k \in d^{-1}(1)$. So $R$ is an infinite subset of $d^{-1}(1)$, and hence computes an infinite homogeneous set $H$ for $c$. If $H$ is not homogeneous to $j$ then we have our desired contradiction, since $R$ is $(c^{-1}(j) \oplus I)$-computable, but we cannot be sure that this is the case. Instead, we use the fact that $R$ cannot compute an infinite subset of $c^{-1}(i)$ (which follows from condition (C1)) and the fact that $R$ is $(c^{-1}(j) \oplus I)$-computable to analyze the following superset of $R$:

$$Q = \{k \in E : f_{e,G \cup \{k\}, I_e}(n_k) \downarrow = i \neq c(n_k)\}.$$  

The set $Q$ is $(c^{-1}(i) \oplus I)$-computable. Furthermore, $Q$ has an infinite subset that does not compute any infinite subset of $c^{-1}(i)$, namely $R$. The class of all such subsets of $Q$ is arithmetic relative to $c^{-1}(i) \oplus I$, so it contains an $\mathcal{O}^{c^{-1}(i) \oplus I}$-computable member $P$. By the same argument as for $R$, we have that $P$ is an infinite subset of $d^{-1}(1)$, and hence computes an infinite set $H$ that is homogeneous for $c$. Since $P$ does not compute any infinite subset of $c^{-1}(i)$, the set $H$ must be contained in $c^{-1}(i')$ for some $i' \neq i$. Thus there is an $\mathcal{O}^{c^{-1}(i) \oplus I}$-computable infinite subset of $c^{-1}(i')$, contradicting condition (C1).

Having given an outline of our proof in the case where $l = 2$, let us now turn to our full proof. Now we assume that every infinite subset of $d^{-1}(l - 1)$ computes an infinite homogeneous set for $c$, and want to make $S$ an infinite subset of $d^{-1}(i)$ for some $i \leq l - 2$. As candidates for $S$ we will build infinite sets $S_0, \ldots, S_{l - 2}$ with $S_i \subseteq d^{-1}(i)$. For each $i \leq l - 2$, we seek to satisfy the requirements

$$R_{i,e} : |\text{dom } \Phi^{S_i}| = \omega \Rightarrow \exists n \Phi^{S_i}(n) \downarrow \neq c(n).$$

Fix a bijection $\langle \ldots, \cdot \rangle$ between $\mathbb{N}^{l-1}$ and $\mathbb{N}$. At stage $\langle e_0, \ldots, e_{l-2} \rangle + 1$ of our construction, we will ensure the satisfaction of $R_{0,e_0} \vee \cdots \vee R_{l-2,e_{l-2}}$. We claim that it follows that there is an $i \leq l - 2$ such that every $R_{i,e}$ is satisfied. If $l = 2$ then the claim is clear. Otherwise, suppose that for each $i < l - 2$ there is an $e_i$ such that $R_{i,e_i}$ is never satisfied. Then at stage $\langle e_0, \ldots, e_{l-3}, e \rangle$, we necessarily satisfy $R_{l-2,e}$. Thus our claim holds. Furthermore, it is easy to see that if every $R_{i, e}$ is satisfied then $S_i$, which is an infinite homogeneous set for $d$, cannot compute an infinite homogeneous set for $c$.

We define sequences of Mathias conditions $(F^0_i, I_0) > (F^1_i, I_1) > \cdots$ for $i \leq l - 2$ by induction. (Notice that the finite parts of the conditions depend on $i$, but the infinite parts do not.) We will ensure that $F^i_s \subseteq d^{-1}(i)$, that $|F^{i+1}_s| > |F^s_s|$, and that $I_s \in I$. Our desired infinite subset $S_i$ of $d^{-1}(i)$ will be $\bigcup F^i_s$. We will begin by setting $I_0 = \text{dom } d = I$, which ensures that for each $s$, we have $I_s \subseteq \text{dom } d$, and
hence \(|I_s \cap d^{-1}(i)| = \omega\) for all \(i \leq l - 2\), by our assumption that there is no infinite subset of \(\text{dom} d \setminus d^{-1}(i)\) in \(\mathcal{I}\).

Let \(F_0^i = \emptyset\) and \(I_0 = \text{dom} \, d\). Suppose we have defined \(F_s^i \subseteq d^{-1}(i)\) and \(I_s \in \mathcal{I}\). Let \(e_0, \ldots, e_{l-2}\) be such that \(s = \langle e_0, \ldots, e_{l-2}\rangle\). We first ask whether there are an \(i \leq l - 2\), an \(n\), and a finite set \(G \subseteq d^{-1}(i)\) compatible with \((F_s^i, I_s)\) such that \(\Phi^G_e(n) \downarrow \neq c(n)\). If so, then, by adding an element to \(G\) if needed, we may assume that \(G \subseteq F_s^i\). In this case, proceed as follows. Let \(F_{s+1}^i = G\). For each \(j \leq l - 2\) such that \(j \neq i\), let \(k\) be the least element of \(I_s \cap d^{-1}(j)\) and let \(F_{j+1}^i = F_j^i \cup \{k\}\).

Let \(I_{s+1}\) be the restriction of \(I_s\) to numbers greater than \(\max(\bigcup_{j \leq l-2} F_{j+1}^i)\). Notice that \(I_{s+1} \in \mathcal{I}\), and for each \(j \leq l - 2\) (including \(i\)), \(F_{j+1}^i \subseteq d^{-1}(j)\) and \(|F_{j+1}^i| > |F_j^i|\), as required. We say that the conditions \((F_{s+1}^i, I_{s+1})\) are defined by finite extension.

Note that, in this case, \(R_{i_0, e_i}\) is satisfied.

If we are not in the above case then proceed as follows. We first prove a general version of the Correctness Lemma discussed in our outline of the case where \(l = 2\) above.

**Lemma 3.14.** For each \(i \leq l - 2\), if a finite set \(G \subseteq d^{-1}(i)\) compatible with \((F_s^i, I_s)\) deduces relative to \(I_s\) that \(\Phi^{G}_e(n) = j\), then \(c(n) = j\).

**Proof.** We prove by induction on \(\alpha < \gamma_L\) that if such a \(G\) \(\alpha\)-deduces relative to \(I_s\) that \(\Phi^{G}_e(n) = j\), then \(c(n) = j\). The statement holds for \(\alpha = 0\) because we have not defined the conditions \((F_{s+1}^i, I_{s+1})\) by finite extension. Now suppose it holds for all \(\beta < \alpha\), and let \(G \subseteq d^{-1}(i)\) be a finite set compatible with \((F_s^i, I_s)\) that \(\alpha\)-deduces relative to \(I_s\) that \(\Phi^{G}_e(n) = j\). Assume for a contradiction that \(c(n) \neq j\). Since we do not have \(\Phi^{G}_e(n) \downarrow = j\), there are infinitely many \(k \in I_s\) such that \(G \cup \{k\} \beta\)-deduces relative to \(I_s\) that \(\Phi^{G}_e(n) = j\) for some \(\beta < \alpha\). By the inductive hypothesis, none of these \(k\)'s can be in \(d^{-1}(i)\), and the set of all such \(k\)'s is in \(\mathcal{I}\). So there is an infinite subset of \(\text{dom} d \setminus d^{-1}(i)\) in \(\mathcal{I}\), contradicting our assumption on \(d\).

Now let \(C_s^i\) be the \(\Pi^0_1\) class of all \(g \in \omega^{\omega}\) such that \(g\) is strictly increasing, \(\text{rng}\, g \subseteq I_s\), and \(\text{dom}\, f_{e_i, F_s^i \cup H, I_s} \subseteq \text{dom}\, f_{e_i, F_j^i, I_s}\) for all finite \(H \subseteq \text{rng}\, g\). Note that if \(F_s^i \subseteq S \subseteq F_s^i \cup \text{rng}\, g\) for some element \(g\) of \(C_s^i\) and \(\Phi^{S}_e(n) \downarrow = j\), then \(\Phi^{F_s^i \cup H}_e(n) \downarrow = j\) for some finite \(H \subseteq \text{rng}\, g\), which means that \(F_s^i \cup H\) \(0\)-deduces relative to \(I_s\) that \(\Phi^{G}_e(n) = j\). It follows by Remark 3.11 that \(n \in \text{dom}\, f_{e_i, F_s^i \cup H, I_s}\), and hence that \(n \in \text{dom}\, f_{e_i, F_j^i, I_s}\).

We now show that there is at least one \(i \leq l - 2\) for which \(C_s^i\) is nonempty. We will then be able to take (a finite modification of) the range of an element of \(C_s^i\) as \(I_{s+1}\), which, as in our outline of the case where \(l = 2\), will allow us to satisfy \(R_{i_0, e_i}\) because the graph of \(f_{e_i, F_j^i, I_s}\) is in \(\mathcal{I}\), and hence cannot compute an infinite homogeneous set for \(c\). We begin with two auxiliary lemmas, which allow us to use the assumption that each \(C_s^i\) is empty to choose finite sets \(G_0, \ldots, G_{l-2}\) with properties analogous to the finite set \(G\) in the outline of the case where \(l = 2\). We
will also want a set $E$ analogous to the one in that outline. It will be important
that there is a single such $E$ that works for all $G_i$.

**Lemma 3.15.** Let $i \leq l - 2$ and let $D \subseteq I_s \cap d^{-1}(i)$ be infinite. Suppose that for
every finite set $G \subseteq d^{-1}(i)$ compatible with $(F^i_s, I_s)$, we have that $\text{dom } f_{e_i, G \cup \{k\}, I_s} \subseteq \text{dom } f_{e_i, G, I_s}$ for all but finitely many $k \in D$. Then $C_i^s$ is nonempty.

**Proof.** We define a function $g \in C_i^s$ by recursion. Suppose that $g(i)$ is defined for all $i < n$ and let $Y_n = \{g(i) : i < n\}$. Assume by induction that if $H \subseteq Y_n$ then $f_{e_i, F_i \cup H, I_s} \subseteq f_{e_i, F_i, I_s}$. By assumption, for each such $H$ we have $\text{dom } f_{e_i, F_i \cup H, I_s} \subseteq \text{dom } f_{e_i, F_i, I_s}$ for almost all $k \in D$. So there is a $k \in D$ such that $\text{dom } f_{e_i, F_i \cup H, I_s} \subseteq \text{dom } f_{e_i, F_i, I_s}$ for all $H \subseteq Y_n$ and $k > g(i)$ for all $i < n$. Let $g(n)$ be the least such $k$. Then $g \in C_i^s$ by construction.

**Lemma 3.16.** Suppose $C_i^s = \emptyset$ for all $i \leq l - 2$. Then there are an infinite set $E \subseteq I_s$ and finite sets $G_i \subseteq d^{-1}(i)$ for $i \leq l - 2$ such that $\text{dom } f_{e_i, G_i \cup \{k\}, I_s} \supseteq \text{dom } f_{e_i, G_i, I_s}$ for every $k \in E$ and $i \leq l - 2$.

**Proof.** By Lemma 3.15, there is a finite set $G_0 \subseteq d^{-1}(0)$ compatible with $(F^0_s, I_s)$ such that $\text{dom } f_{e_0, G_0 \cup \{k\}, I_s} \supseteq \text{dom } f_{e_0, G_0, I_s}$ for infinitely many $k \in I_s \cap d^{-1}(0)$. Then there is an infinite set $E_0 \subseteq I_s$ in $\mathcal{I}$ such that $\text{dom } f_{e_0, G_0 \cup \{k\}, I_s} \supseteq \text{dom } f_{e_0, G_0, I_s}$ for all $k \in E_0$. Now, $E_0 \cap d^{-1}(1)$ must be infinite, so again by Lemma 3.15, there is a finite set $G_1 \subseteq d^{-1}(1)$ compatible with $(F^1_s, I_s)$ such that $\text{dom } f_{e_1, G_1 \cup \{k\}, I_s} \supseteq \text{dom } f_{e_1, G_1, I_s}$ for infinitely many $k \in E_0 \cap d^{-1}(1)$. Again, there is an infinite set $E_1 \subseteq E_0$ in $\mathcal{I}$ such that $\text{dom } f_{e_1, G_1 \cup \{k\}, I_s} \supseteq \text{dom } f_{e_1, G_1, I_s}$ for all $k \in E_1$. We can continue in this manner to define, for each $i \leq l - 2$, a finite set $G_i \subseteq d^{-1}(i)$ compatible with $(F^i_s, I_s)$ and an infinite set $E_i \subseteq I_s$ with $E_i \subseteq E_{i-1}$ such that $\text{dom } f_{e_i, G_i \cup \{k\}, I_s} \supseteq \text{dom } f_{e_i, G_i, I_s}$ for all $k \in E_i$. We have $E_0 \supseteq E_1 \supseteq \cdots \supseteq E_{l-2}$, so the lemma follows by taking $E = E_{l-2}$.

**Lemma 3.17.** There is an $i \leq l - 2$ such that $C_i^s$ is nonempty.

**Proof.** Assume otherwise, and let $E$ and $G_0, \ldots, G_{l-2}$ be as in Lemma 3.16. Our argument will be similar to that in our outline of the case where $l = 2$. We will define sets $R$, $Q$, and $P$ analogous to the ones in that outline. The main difference is that the single color $i$ in the definitions of $R$ and $Q$ will now be replaced by a set $F$ of up to $l - 1$ many colors.

For $k \in E$ and $i \leq l - 2$, let $n^k_i$ be the least element of $\text{dom } f_{e_i, G_i \cup \{k\}, I_s} \setminus \text{dom } f_{e_i, G_i, I_s}$. Fix $i \leq l - 2$. We claim that there is no $n$ such that $n^k_i = n$ for infinitely many $k \in E$. Assume for a contradiction that there is such an $n$. By the definition of deducibility, the fact that $n \in \text{dom } f_{e_i, G_i \cup \{k\}, I_s}$ for infinitely many $k \in I_s$ implies that there is a $j$ such that $G_i$ deduces relative to $I_s$ that $\Phi_{e_0}(n) = j$. By Lemma 3.14, $G_i$ cannot deduce relative to $I_s$ that $\Phi_{e_0}(n) = j$ unless $j = c(n)$.
It follows that \( n \in \text{dom } f_{e_i, G_i, I_i} \), which is a contradiction and hence establishes our claim.

It is now easy to see that there are \( j_0, \ldots, j_{l-2} \leq m \) and an \( I \)-computable sequence \( k_0, k_1, \ldots \in E \) such that \( n_{k_p} > \overline{p} \) and \( f_{e_i, G_i \cup \{ k_p \}, I_i}(n_{k_p}) = j_i \) for all \( p \) and \( i \leq l - 2 \). Let \( F = \{ j_0, \ldots, j_{l-2} \} \) and let \( j \leq m \) be such that \( j \notin F \). Applying condition (C2) to the sets \( \{ n_{k_p} : i \leq l - 2 \} \), we see that there are infinitely many \( p \) such that \( c(n_{k_p}) = j \) for all \( i \leq l - 2 \). Thus

\[
R = \{ k \in E : (\forall i \leq l - 2)[f_{e_i, G_i \cup \{ k \}, I_i}(n_k) \in F \land c(n_k) = j] \}
\]

is infinite. Since \( R \) is \((e^{-1}(j) \oplus I)\)-computable, \(|F| \leq l - 1 \leq m - 1 \), and \( j \notin F \), condition (C1) implies that \( R \) cannot compute an infinite subset of \( c^{-1}(F) \).

Now let

\[
Q = \{ k \in E : (\forall i \leq l - 2)[f_{e_i, G_i \cup \{ k \}, I_i}(n_k) \in F \land c(n_k) \notin F] \}.
\]

Then \( Q \) is \((c^{-1}(F) \oplus I)\)-computable. Furthermore, \( Q \) has an infinite subset that does not compute any infinite subset of \( c^{-1}(F) \), namely \( Q \). The class of all such subsets of \( Q \) is arithmetic relative to \( c^{-1}(F) \oplus I \), so it contains an \( O^{c^{-1}(F) \oplus I} \)-computable member \( P \). Let \( k \in P \). For each \( i \leq l - 2 \), we have that \( f_{e_i, G_i \cup \{ k \}, I_i}(n_k) \neq c(n_k) \), which implies, by Lemma 3.14, that \( G_i \cup \{ k \} \not\subset d^{-1}(i) \), and hence that \( k \not\in d^{-1}(i) \). Thus \( k \in d^{-1}(l - 1) \). So \( P \) is an infinite subset of \( d^{-1}(l - 1) \), and hence computes an infinite homogeneous set \( H \) for \( c \). Since \( P \) does not compute any infinite subset of \( c^{-1}(F) \), the set \( H \) must be contained in \( c^{-1}(j') \) for some \( j' \notin F \). So there is an \( O^{c^{-1}(F) \oplus I} \)-computable infinite subset of \( c^{-1}(j') \), which contradicts condition (C1).

Let \( i \leq l - 2 \) be such that \( C_i \) is nonempty, and let \( Y \in I \) be the range of an element of \( C_i \). Note that \( Y \cap d^{-1}(j) \) is infinite for all \( j \leq l - 2 \). For each \( j \leq l - 2 \), let \( k \) be the least element of \( Y \cap d^{-1}(j) \) and let \( F_{j+1} = F_j \cup \{ k \} \). Let \( I_{j+1} \) be the restriction of \( Y \) to numbers greater than \( \max(\bigcup_{j \leq l - 2} F_{j+1}) \). Notice that \( I_{j+1} \in I \), and for each \( j \leq l - 2 \), we have \( F_{j+1} \subset d^{-1}(j) \) and \( |F_{j+1}| > |F_j| \), as required.

We have completed the definition of our Mathias conditions. Now, for each \( i \leq l - 2 \), let \( S_i = \bigcup_s F_s \). We have ensured that \( |F_{j+1}| > |F_j| \) and that \( F_s \subset d^{-1}(i) \) for all \( s \), so each \( S_i \) is an infinite subset of \( d^{-1}(i) \). As argued above, it is enough to show that for each \( s = (e_0, \ldots, e_{l-2}) \), we satisfy at least one requirement \( R_{e_i} \).

Assume for a contradiction that we do not. Then it cannot be the case that the conditions \((F_{j+1}, I_{j+1})\) were defined by finite extension, so there is an \( i \leq l - 2 \) such that \( F_s \subset S_i \subset F_s \cup \text{rng } g \) for some element \( g \) of \( C_s \), which, as noted above, implies that \( \Phi_{e_i}^S(n) \) \( n \in \text{dom } f_{e_i, F_s \cup I_s} \). Since we are assuming that \( R_{e_i} \) is not satisfied, there are infinitely many such \( n \), so \( \text{dom } f_{e_i, F_s \cup I_s} \) is infinite. Since \( f_{e_i, F_s \cup I_s}(n) = c(n) \) when defined and the graph of \( f_{e_i, F_s \cup I_s} \) is in \( I \), there is an infinite homogeneous set for \( c \) in \( I \), contradicting the choice of \( c \).

Letting \( c \) be as in the theorem with \( m = 2 \) and defining \( B_0 = c^{-1}(\{0, 1\}) \) and \( B_1 = c^{-1}(\{0, 2\}) \) yields the following strengthening of Theorem 3.8.
Corollary 3.18. There exist sets $B_0$ and $B_1$ for which there is no set $A$ such that $B_0 \leq_{nii} A$ and $B_1 \leq_{nii} A$.

The sets $B_0$ and $B_1$ constructed in the proof of Theorem 3.8 are quite complicated, and the ones in the above corollary even more so. As mentioned above, Patey [45] has shown that, remarkably, the coloring $c$ in Theorem 3.9 can in fact be taken to be low, and hence so can the sets $B_0$ and $B_1$ in Corollary 3.18. We do not have any nontrivial lower bounds on the complexity of maximal pairs in either the ii or the nii case. Indeed, Dzhafarov and Igusa [17, Question 5.9] asked whether being mutually 1-random or mutually 1-generic is enough to ensure that a pair of sets is maximal with respect to ii-reducibility, and we may ask the same question for nii-reducibility. Even this level of complexity might conceivably not be necessary.

4. Reductions to Multiple Instances of a Problem

4.1. Reduction games

As mentioned in Sec. 1, $P \leq_{\omega} Q$ can be thought of as meaning that solutions to $P$ can be obtained from solutions to multiple instances of $Q$. We can make this intuition precise by thinking of reductions between problems as games.

Definition 4.1. For problems $P$ and $Q$, the reduction game $G(Q \rightarrow P)$ is a two-player game that proceeds as follows.

On the first move, Player 1 plays an instance $X_0$ of $P$, and Player 2 either plays an $X_0$-computable solution to $X_0$ and declares victory, in which case the game ends, or responds with an $X_0$-computable instance $Y_1$ of $Q$. If Player 2 cannot move (which might happen if there is no $X_0$-computable instance of $Q$), then Player 1 wins, and the game ends.

For $n > 1$, on the $n$th move (if the game has not yet ended), Player 1 plays a solution $X_{n-1}$ to the instance $Y_{n-1}$ of $Q$. Then Player 2 either plays a $(\bigoplus_{i<n} X_i)$-computable solution to $X_0$ and declares victory, in which case again the game ends, or plays a $(\bigoplus_{i<n} X_i)$-computable instance $Y_n$ of $Q$.

Player 2 wins this play of the game if it ever declares victory. Otherwise, Player 1 wins.

Proposition 4.2. If $P \leq_{\omega} Q$ then Player 2 has a winning strategy for $G(Q \rightarrow P)$. Otherwise, Player 1 has a winning strategy for $G(Q \rightarrow P)$. (Note that these strategies do not have to be effective in any way.)

Proof. Suppose that $P \leq_{\omega} Q$. Player 2 then plays according to the following strategy. At the $n$th move, if there is a legal winning move, Player 2 makes it. Otherwise, it lets $Y_n, Y_{n+1}, \ldots$ be all instances of $Q$ computable in the join of the moves of the game so far. For the least pair $\langle m, i \rangle$ with $m \leq n$ for which Player 2 has not yet acted, it then acts by playing $Y_{m,i}$ (to which Player 1 must reply with a solution to $Y_{m,i}$). Suppose Player 2 never has a winning move, and Player 1 never
fails to have a legal move. Then the Turing ideal generated by the moves in the game is an \( \omega \)-model of \( Q \), and hence is a model of \( P \). But then there must be some finite number of moves whose join computes a solution to Player 1’s first move, which gives Player 2 a winning move. Thus Player 2 must eventually win the game no matter what Player 1 does.

Now suppose that \( P \not\leq_\omega Q \) and let \( S \) be an \( \omega \)-model of \( \text{RCA}_0 + Q \) that is not a model of \( P \). Since \( S \) is a Turing ideal, as long as Player 1’s moves stay inside \( S \), so must Player 2’s moves. Furthermore, the fact that \( S \) is a model of \( Q \) implies that, as long as Player 2’s moves stay inside \( S \), Player 1 will always be able to reply with moves that stay inside \( S \). So Player 1 can simply begin by playing an instance \( X_0 \in S \) of \( P \) that has no solution in \( S \), and then keep playing elements of \( S \), which ensures that the game never ends (unless there is no \( X_0 \)-computable instance of \( Q \), in which case Player 2 loses on the first move).

**4.2. Generalized Weihrauch reducibility**

We can now define a generalized notion of Weihrauch reduction by considering computable strategies. We assume that we have defined the join operation for finitely many sets so that we can determine \( n \) from \( \bigoplus_{i \leq n} X_i \), say by letting \( \bigoplus_{i \leq n} X_i = \{ n \} \oplus \{ (i, k) : i \leq n \land k \in X_i \} \). In general, when we write \( X_0 \oplus X_1 \) we mean \( \{ 2n : n \in X_0 \} \cup \{ 2n + 1 : n \in X_1 \} \) as usual. However, we will sometimes write \( X_0 \oplus X_1 \) for \( \bigoplus_{i \leq 1} X_i \), when it is clear that this is what we mean. Similarly, we will write simply \( X_0 \) instead of \( \bigoplus_{i \leq 0} X_i \).

**Definition 4.3.** A computable strategy for Player 2 in a reduction game is a Turing functional that, given the join of Player 1’s first \( n \) moves as an oracle, outputs Player 2’s \( n \)th move. More precisely, the strategy is a functional \( \Phi \) such that, if \( Z \) is the join of Player 1’s first \( n \) moves, then \( \Phi^Z = V \oplus Y \), where \( Y \) is Player 2’s \( n \)th move, and \( V \) is \( \{ 0 \} \) if Player 2 declares victory at this move and \( \emptyset \) otherwise. We write \( \Phi^X \) for \( \{ n : 2n + 1 \in \Phi^X \} \) (so if \( Z \) is the join of Player 1’s first \( n \) moves, then \( \Phi^Z \) is Player 2’s \( n \)th move). This strategy is winning if it enables Player 2 to win no matter what Player 1 does.

We say that \( P \) is Weihrauch (or uniformly) reducible to \( Q \) in the generalized sense, and write \( P \leq_{\text{WH}} Q \), if Player 2 has a computable winning strategy in \( G(Q \rightarrow P) \).

For example, consider RT\(^\text{1}\)\(^n\) and RT\(^\text{2}\)\(^n\). While RT\(^\text{1}\)\(^n\) \(\leq_\omega\) RT\(^\text{2}\)\(^n\), we have seen in Theorem 3.3 that RT\(^\text{1}\)\(^n\) \(\not\leq_\omega\) RT\(^\text{2}\)\(^n\). However, the procedure for obtaining solutions to instances of RT\(^\text{1}\)\(^n\) using RT\(^\text{2}\)\(^n\) is clearly uniform, though it does require two applications of RT\(^\text{2}\)\(^n\). Indeed, we do have RT\(^\text{1}\)\(^n\) \(\leq_{\text{WH}}\) RT\(^\text{2}\)\(^n\). Here is Player 2’s strategy in \( G(\text{RT}^n_1 \rightarrow \text{RT}^n_2) \): Player 1 plays a coloring \( c : [N]^n \rightarrow 3 \). Then Player 2 plays the \( c \)-computable coloring \( d : [N]^n \rightarrow 2 \) defined by \( d(s) = 0 \) if \( c(s) = 2 \) and \( d(s) = 1 \) otherwise. Next, Player 1 plays an infinite homogeneous set \( H \) for \( d \). From \( c \oplus H \), Player 2 can computably determine to which color \( H \) is homogeneous. If that color
and the strategy in a homogeneous set for the induced 2-coloring of $[\mathbb{N}]^n$. If the color is 1, then Player 2 plays the 2-coloring $c$ is also homogeneous for $G$. The argument can easily be adapted to show that $RT_{k}^{n} \leq_{gW} RT_{j}^{n}$ for any $k > j \geq 2$.

**Proposition 4.4.** The relation $\leq_{gW}$ is transitive.

**Proof.** Let $\Phi$ and $\Psi$ be computable winning strategies for Player 2 in $G(Q \rightarrow P)$ and $G(R \rightarrow Q)$, respectively. Then Player 2 has the following computable winning strategy in $G(R \rightarrow P)$. Player 1 plays the instance $X_0$ of $P$. If $\Phi^{X_0}(0) = 1$ then $\hat{\Phi}^{X_0}$ is a solution to $X_0$, so player 2 plays it and wins. Otherwise, $\hat{\Phi}^{X_0}$ is an instance $Y_1$ of $Q$. Player 2 can then begin a play of $G(R \rightarrow Q)$ by pretending that Player 1 has played $Y_1$ on its first move, and play this game according to $\Psi$ until it obtains a solution $X_1$ to $Y$, which must happen no matter how player 1 replies. If $\Phi^{X_0} \oplus^{X_1}(0) = 1$ then $\hat{\Phi}^{X_0} \oplus^{X_1}$ is a solution to $X_0$, so player 2 plays it and wins. Otherwise, $\hat{\Phi}^{X_0} \oplus^{X_1}$ is an instance $Y_2$ of $Q$. Again, Player 2 can play $G(R \rightarrow Q)$ according to $\Psi$ pretending that Player 1 has played $Y_2$ on its first move. Continuing in this way, Player 2 must eventually find an $n$ such that $\Phi^{\oplus_{\leq n} X_0}(0) = 1$. □

Clearly, if $P \leq_{W} Q$ then $P \leq_{gW} Q$, and if $P \leq_{gW} Q$ then $P \leq_{\omega} Q$, so we have the following picture.

$$
\begin{align*}
P \leq_{gW} Q & \quad P \leq_{W} Q \\
P \leq_{\omega} Q & \quad P \leq_{c} Q \\
P \leq_{sc} Q & \quad P \leq_{gW} Q.
\end{align*}
$$

Versions of Ramsey’s Theorem for different exponents provide interesting witnesses to the fact that $\leq_{gW}$ does not imply $\leq_{c}$. (We will see an example showing that $\leq_{sc}$ does not imply $\leq_{gW}$ in Corollary 4.10.) Consider for instance $RT_{2}^{1}$ and $RT_{3}^{2}$. As discussed in Sec. 1, $RT_{2}^{1} \not\leq_{c} RT_{2}^{3}$. On the other hand, let $PRE$ be the principle stating that for every $k, n \geq 2$, every $k$-coloring of $[\mathbb{N}]^n$ has an infinite prehomogeneous set. The construction in proof of Lemma 5.4 of Jockusch [30] gives us the first part of Lemma 3.1 also shows that $PRE \leq_{W} KL$. We have already seen that $KL \leq_{W} RT_{2}^{1}$, so given an instance $c$ of $RT_{2}^{1}$, we can use one application of $RT_{2}^{3}$ to obtain an infinite prehomogeneous set $A$ for $c$, then another to obtain an infinite homogeneous set for the induced 2-coloring of $[A]^3$, which will also be homogeneous for $c$. This procedure is uniform, so $RT_{2}^{1} \leq_{gW} RT_{2}^{3}$. A similar argument establishes the following fact.

**Proposition 4.5.** $RT \leq_{gW} RT_{k}^{n}$ for all $n \geq 3$ and $k \geq 2$.

Similarly, given an instance $c$ of $RT_{2}^{3}$, we can use one application of $PRE$ to obtain an infinite prehomogeneous set $A$ for $c$, then another to obtain an infinite
prehomogeneous set $B$ for the induced 2-coloring on $|A|^2$, then an application of RT\(_1\) to obtain an infinite homogeneous set for the induced 2-coloring on $B$, which will also be homogeneous for $c$. Since RT\(_1\) \leq W KL, we see that RT\(_2\) \leq W KL. By Proposition 4.5, we have the following result.

**Proposition 4.6.** RT \leq W KL.

Thus we see that the equivalences represented by the top line of Fig. 1 also hold for \leq W. The same is true of the bottom line of that figure, a fact worth noting in light of part (4) of Theorem 2.10, and of the similar equivalences for exponent 2.

**Proposition 4.7.**

1. RT\(_{1,\infty}\) \leq W RT\(_2\).
2. (S)RT\(_{2,\infty}\) \leq W (S)RT\(_2\) and D\(_{2,\infty}\) \leq W D\(_2\).

**Proof.** For (1), let $f : \mathbb{N} \rightarrow \mathbb{N}$ have bounded range. Let $c_0 : \mathbb{N} \rightarrow 2$ be defined by letting $c_0(n) = 0$ if and only if $f(n) = 0$, and let $H_0$ be an infinite homogeneous set for $c_0$. If $H_0$ is homogeneous to 0 then it is homogeneous for $f$. Otherwise, let $c_1 : H_0 \rightarrow 2$ be defined by letting $c_0(n) = 0$ if and only if $f(n) = 1$, and let $H_1$ be an infinite homogeneous set for $c_1$. If $H_1$ is homogeneous to 0 then it is homogeneous for $f$. Otherwise, continue in this way to define $c_2, c_3, \ldots$ and $H_2, H_3, \ldots$. Eventually we must reach an $n$ such that $H_n$ is homogeneous to 0, and hence homogeneous for $f$.

The argument for (2) is essentially the same. \(\square\)

In light of Dzhafarov’s result [16] that SRT\(_2\) \not\leq W D\(_2\), the following fact is also of interest.

**Proposition 4.8.** SRT\(_2\) \leq W D\(_2\).

**Proof.** Let $c : [\mathbb{N}]^2 \rightarrow 2$ be stable. With one application of D\(_2\), we can obtain an infinite limit-homogeneous set $H$ for $c$. Let $a \in H$ and let $d(n) = c(a, a + n + 1)$. With one application of RT\(_1\), we can obtain an infinite homogeneous set $I$ for $d$. Let $n \in I$ and let $i = c(a, a + n + 1)$. Then $\lim_n c(a, s) = i$, so $\lim_n c(x, s) = i$ for all $x \in H$, and hence we can define an infinite homogeneous set $b_0, b_1, \ldots$ for $c$ by letting $b_0$ be the least element of $H$ and letting $b_{k+1}$ be the least element of $H$ greater than $b_k$ such that $c(b_j, b_{k+1}) = i$ for all $j \leq k$. This construction is uniform, and RT\(_2\) \leq W D\(_2\), so SRT\(_2\) \leq W D\(_2\). \(\square\)

We will summarize the relationships between versions of RT and KL under \leq W in Sec. 5.

The following notions play an important role in the work of Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12]. For problems $P_1$ and $P_1$, let $(P_2, P_1)$ be the problem whose instances are pairs $(X_0, X_1)$ where each $X_i$ is an instance of $P_i$, and such that a solution to such an instance is a pair $(Y_0, Y_1)$ where each $Y_i$ is a solution to $X_i$. Let $P^m = (P, P)$. We can similarly define $P^n$ for any $n$, and $P^\omega$, which is also sometimes denoted by Seq$P$ (the sequential version of $P$), also known as the...
parallelization of $P$) to avoid too many superscripts. We also have $P^\prec \omega$, where an instance is any instance of $P^n$ for $n \in \omega$. Lemma 3.3 of [12] illustrates the fact that $P_0 \leq_W Q$ and $P_1 \leq_W Q$ does not necessarily imply that $(P_0, P_1) \leq_W Q$; indeed, it is even possible to have $P^2 \not\leq_W P$. On the other hand, if $P_0 \leq_W Q$ and $P_1 \leq_{\text{WKL}} Q$ then clearly $(P_0, P_1) \leq_{\text{WKL}} Q$. In particular, we always have $P^n \leq_{\text{WKL}} P$, and even $P^\omega \leq_{\text{WKL}} P$. It is not necessarily the case that $P^\omega \leq_{\text{WKL}} P$, however. In fact, we can even have $P^\omega \not\leq_\omega P$. For example, as shown in [12, Lemma 3.2], there is a computable instance of SeqRT $^2_1$ such that every solution computes $\emptyset''$, but as shown by Seetapun in [51, Theorem 3.1], there is an $\omega$-model of RCA $_0 + \text{RT}^2_1$ that does not contain $\emptyset$.

It is more difficult to give examples of principles $P$ and $Q$ such that $P \leq_\omega Q$ but $P \not\leq_{\text{WKL}} Q$. We will discuss one class of examples in the following subsection. A different kind of example is given by the following proposition, which is a uniform version of Proposition 2.7.

**Proposition 4.9.** Let $P$ be a problem. If $P \leq_{\text{WKL}} P'$ then $P \leq_{\text{WKL}} P'$.

**Proof.** For a set $Z$, let $Z[i] = \{n : (i, n) \in Z\}$. It is straightforward to define computable functions $f$, $g$, and $h$, and a functional $T$, such that $T^X$ is a binary tree whose infinite paths are exactly the sets $Z$ with the following properties.

1. $Z[0] = X$.
2. $Z[i(j)] = Z[i] \uplus Z[j]$ for all $i, j$.
3. If $\Phi^Z_{\prec}(i)$ is total then $Z^\prec(e, i) = \Phi^Z_{\prec}(i)$.
4. If $Z[i]$ codes an infinite binary tree then $Z[h(i)]$ is an infinite path on that tree.

Now suppose that $P \leq_{\text{WKL}} P'$ and let $\Phi_e$ be a winning strategy for Player 2 in $G(\text{WKL} \rightarrow P)$. Given an instance $X$ of $P$, let $Z$ be an infinite path on $T^X$. Proceed as follows to obtain a solution to $X$ uniformly from $Z$. Let $i_0 = 0$. Suppose we have defined $i_0, \ldots, i_n$. Using $f$ and $g$, we can compute a $j$ such that $Z[j] = \bigoplus_{k \leq n} Z[\hat{k}]$. Wait for $\Phi^Z_{\prec}(j)$ to converge. When it does, if it converges to 1 then output $\Phi_e^Z_{\prec}(i)$.

(Remember the hat notation from Definition 4.3.) Otherwise, let $i_{n+1} = h(g(e, j))$, so that $Z[h(i)]$ is a solution to the instance $\hat{\Phi}^Z_{\prec}(i)$ of WKL. It is easy to see that this procedure eventually ends, producing a solution to $X$.

Combining the above proposition with part (1) of Theorem 2.10 yields the following result.

**Corollary 4.10.** $\text{RT}^1_2 \not\leq_{\text{WKL}} P$.

Since clearly $\text{RT}^1_2 \leq_{\text{sc}} \text{WKL}$, we see that $\leq_{\text{sc}}$ does not imply $\leq_{\text{WKL}}$.

### 4.3. Diagonalizability

In most cases, when one shows that $P \leq_\omega Q$, one does so via a generalized Weihrauch reduction. However, several examples where this is not the case fit a
general pattern we now describe. (The definitions we give below are not the absolutely most general ones we could give, but they suffice for our examples.) For \( \sigma_0, \ldots, \sigma_n \in 2^{<\omega} \) and a functional \( \Phi \), we write \( \Phi^{\leq n} \sigma_i(k) \downarrow = j \) to mean that 
\[ \Phi^{|n|} \langle (i, k) \rangle \leq n \land \sigma_i(k) = 1 \} (k) \downarrow = j \] via a computation that does not query the oracle on \( 2(i, k) + 1 \) for any \( i \leq n \) and \( k \geq |\sigma_i| \). The point is that then, for any \( X_0 \succ \sigma_0, \ldots, X_n \succ \sigma_n \), we have \( \Phi^{\leq n} X_i(k) \downarrow = j \).

Let us begin with an example. We saw in Theorem 2.10(5) that \( RT_2^1 \not\leq_\w COH \). The proof of this result uses two properties of these problems: the ability to use instances of \( RT_2^1 \) to diagonalize against a supposed infinite homogeneous set, and the fact that for every instance \( X \) of \( COH \), every string in \( 2^{<\omega} \) can be extended to a solution to \( X \). It turns out that these properties suffice to show that, in fact, \( RT_2^1 \not\leq_\w COH \), as we now discuss.

Suppose that Player 2 has a computable winning strategy \( \Phi \) for the game \( G(COH \rightarrow RT_2^1) \), and build a coloring \( c : \mathbb{N} \rightarrow 2 \) as follows. (As mentioned in Remark 1.4, we identify \( c \) with a subset of \( \mathbb{N} \) coding it. We say that a binary string \( \sigma \) is an initial segment of \( c \), and write \( \sigma \prec c \), if \( \sigma \) is an initial segment of this set.)

Start by letting \( c(s) = 0 \) at stage \( s \) until we find \( \sigma_0, \ldots, \sigma_n \in 2^{<\omega} \) such that

1. \( \sigma_0 \prec c \),
2. \( \Phi^{\leq n} \sigma_i(0) \downarrow = 0 \) for all \( j < n \),
3. \( \Phi^{\leq n} \sigma_i(0) \downarrow = 1 \), and
4. \( \Phi^{\leq n} X_i(k) \downarrow = 1 \) for some \( k \).

If such strings are found, then ensure that \( c(k) = 0 \) and \( c(t) = 1 \) for all sufficiently large \( t \).

Consider a play of \( G(COH \rightarrow RT_2^1) \) where Player 1 plays \( c \), then Player 2 plays the instance \( Y_0 \) of \( COH \) determined by \( \Phi \), then Player 1 plays a solution \( X_1 \) to \( Y_0 \), and so on, until Player 2 plays an infinite homogeneous set \( X_n \) for \( c \) (still determined by \( \Phi \)). For simplicity of notation, let \( X_0 = c \). Then \( \Phi^{\leq n} X_i(0) \downarrow = 1 \) and \( \Phi^{\leq n} X_i(0) \downarrow = 0 \) for all \( j < n \). Furthermore, there is a \( k \) such that \( \Phi^{\leq n} X_i(k) \downarrow = 1 \). Thus \( \sigma_0, \ldots, \sigma_n \) as above must eventually be found.

The crucial point now is that there are \( X_0 \succ \sigma_0, \ldots, X_n \succ \sigma_n \) such that \( X_0 = c \) and if \( \Phi^{\leq n} X_i \) is an instance of \( COH \) then \( X_{j+1} \) is a solution to this instance. This is the case simply because for every instance \( Z \) of \( COH \), every string can be extended to a solution to \( Z \). Now \( X_0, \ldots, X_n \) are valid moves by Player 1 in a game in which Player 2 plays according to \( \Phi \), and by the choice of \( \sigma_0, \ldots, \sigma_n \),

1. \( \Phi^{\leq n} X_i(0) \downarrow = 0 \) for all \( j < n \),
2. \( \Phi^{\leq n} X_i(0) \downarrow = 1 \), and
3. \( \Phi^{\leq n} X_i(k) \downarrow = 1 \), where \( k \) is as above.

Then \( \Phi^{\leq n} X_i \) must be an infinite homogeneous set for \( c \). But \( c(k) = 0 \), while \( c(t) = 1 \) for all sufficiently large \( t \), so we have a contradiction. Thus \( RT_2^1 \not\leq_\w COH \).
We did not use any properties of COH in this argument other than the fact that for every instance $X$ of COH, every string can be extended to a solution to $X$. (In the terminology of [4], COH is densely realized.) We could also adapt the argument to the following more general situation (where we recall from Remark 1.4 the fact that we identify instances and solutions to problems with subsets of $\mathbb{N}$).

**Definition 4.11.** A problem $P$ is undiagonalizable if there is a uniform procedure that, given an instance $X$ of $P$ and a string $\sigma$, decides whether $\sigma$ can be extended to a solution to $X$.

Let us now discuss the essential property of $\text{RT}_1^2$ used in the above argument. We think of $\sigma_0, \ldots, \sigma_n$ as forming a finite approximation to (Player 1’s moves in) a possible run of $G(\text{COH} \rightarrow \text{RT}_1^2)$ where Player 1 begins by playing the coloring $c$ we are building and Player 2 plays according to $\Phi$. Then $k$ is an element that is necessarily included in Player 2’s final move in that run. We get a contradiction by ensuring that there is no infinite homogeneous set for $c$ containing $k$. Suppose we were working with $\text{TS}_1^3$ instead of $\text{RT}_1^2$. Then finding a single $k$ would not be enough. We would need a two-step process instead: First start defining $c(s) = 0$ at stage $s$ and find a finite approximation $F_0$ to a possible run of $G(\text{COH} \rightarrow \text{TS}_1^3)$ where Player 1 begins by playing the coloring $c$ we are building and Player 2 plays according to $\Phi$, and a corresponding $k_0$ necessarily included in Player 2’s final move in that run, with $c(k_0) = 0$. Then start defining $c(s) = 1$ at stage $s$ and find a further finite approximation $F_1$ extending $F_0$ and a corresponding $k_1$ with $c(k_1) = 1$. At that point, by letting $c(t) = 2$ for all sufficiently large $t$, we ensure that there is no infinite thin set for $c$ containing both $k_0$ and $k_1$.

In general, we can think of this construction as producing a sequence $F_0 \subset F_1 \subset \cdots$ of approximations to runs of a game $G(Q \rightarrow P)$. (In the above cases, we can stop after one or two such approximations, but for the general case we think of the construction as finding infinitely many approximations.) Each $F_i$ determines some part of Player 2’s last move. We can easily adapt the above construction so that these parts are longer and longer initial segments, so that their union is a set. We think of this set as $\Psi^X$, where $X$ is the instance of $P$ that we construct, and $\Psi$ has the property that $\Psi^X$ is total and infinite for all instances $Y$ of $P$. The key property of $P$ that makes the construction work is captured by the following definition. (If solutions to instances of $P$ are not themselves subsets of $\mathbb{N}$, we think of them as coded by such sets, as mentioned in Remark 1.4. We then say that such a solution is infinite if the set coding it is infinite, which for the problems considered in this paper is always the case.)

**Definition 4.12.** A problem $P$ has diagonalization opportunities if all of its solutions are infinite and, given any Turing functional $\Psi$ such that $\Psi^Y$ is total and infinite for all instances $Y$ of $P$, there are a $\tau$ and an instance $X \succ \tau$ of $P$ such that no solution to $X$ extends $\Psi^X$.

We now have the following general result.
Theorem 4.13. Let $Q$ be undiagonalizable and $P$ have diagonalization opportunities. Then $P \nleq_{gW} Q$.

Proof. In this proof, we regard all instances and solutions as subsets of $\mathbb{N}$. Write $X^{[i]}$ for $\{n : (i, n) \in X\}$. Assume for a contradiction that $P \leq_{gW} Q$, so that, by Proposition 4.2, Player 2 has a computable winning strategy $\Phi$ for $G(Q \rightarrow P)$. Say that $R$ is a possible run if there is a legal run of this game where Player 1 plays $R^{[0]}, \ldots, R^{[k]}$. Player 2 plays according to its winning strategy, and the game ends after exactly $k + 1$ many moves. From a possible run $R$, we can determine the corresponding $k$, and hence compute $\tilde{\Phi}:\Theta_{<k} \rightarrow R^{[i]}$, which is a solution to $R^{[0]}$. Thus there is a Turing functional $\Gamma$ such that if $R$ is a possible run then $\Gamma^R$ is a solution to $R^{[0]}$.

For $s = (\sigma_0, \ldots, \sigma_n)$ and $s' = (\sigma'_0, \ldots, \sigma'_n)$, write $s \preceq s'$ to mean that $n' \geq n$ and $\sigma_i \preceq \sigma'_i$ for all $i \leq n$, and write $\Gamma^s$ for $\Gamma^{\Theta_{<n} \sigma_i}$.

Let $\Theta$ be a Turing functional such that if $Y$ is an instance of $Q$ then $\Theta^Y$ is the set of all $\sigma$ that can be extended to a solution to $Y$. For a set $X$, let $S_X$ be the set of all sequences $s = (\sigma_0, \ldots, \sigma_n)$ for which there is a $k \leq n$ such that

1. $\sigma_0 \prec X$,
2. $\Phi^{s_{i+1}}(0) \preceq 0$ for all $i < k$,
3. $\Phi^{s_{k+1}}(0) = 1$, and
4. $\sigma_{i+1} \in \Theta^{R^{s_{i+1}}}$ for all $i < k$.

We say that a run $R$ extends $s$ if $R^{[i]} \succ \sigma_i$ for all $i \leq n$. Note that if $X$ is an instance of $P$ then $S_X$ is uniformly $X$-c.e., every element of $S_X$ can be extended to a possible run $R$ with $R^{[0]} = X$, and for every possible run $R$ with $R^{[0]} = X$, every sufficiently long initial segment of $R$ is in $S_X$.

Let $\Psi$ be the functional such that $\Psi^X$ is defined as follows. Search for an $s_0 \in S_X$ and an $n_0$ such that $\Gamma^{n_0}(m) \preceq 1$ for all $m \leq n_0$ and $\Gamma^{n_0}(n_0) = 1$. If such an $s_0$ and $n_0$ are found then let $\Psi^X(m) = \Gamma^{n_0}(m)$ for all $m \leq n_0$ and search for an $s_1 \geq s_0$ in $S_X$ and an $n_1 > n_0$ such that $\Gamma^{s_1}(m) \preceq 1$ for all $m \leq n_1$ and $\Gamma^{s_1}(n_1) = 1$. If such an $s_1$ and $n_1$ are found then let $\Psi^X(m) = \Gamma^{s_1}(m)$ for all $m \in [n_0, n_1)$. Continue to define the $s_i$ and $n_i$ in this way. If $X$ is an instance of $P$ then each $s_i$ can be extended to a possible run $R$ such that $R^{[0]} = X$, and $\Gamma^R$ must be total and infinite, so $s_{i+1}$ and $n_{i+1}$ will be found. Thus in this case $\Psi^X$ is total and infinite.

Let $\tau$ be as in Definition 4.12 and let $X \succ \tau$ be an instance of $P$ such that no solution to $X$ extends $\Psi^\tau$. Letting $s_i$ be as above, there is an $i$ such that $\Psi^\tau \preceq \Gamma^{s_i}$. There is a possible run $R$ extending $s_i$ such that $R^{[0]} = X$. Then $\Gamma^R \succ \Psi^\tau$, so $\Gamma^R$ is not a solution to $R^{[0]}$, which is a contradiction. □

Remark 4.14. Brattka, Hendtlass, and Kreuzer [4] defined the notion of $\omega$-indiscriminative multi-valued functions on represented spaces using the all or co-unique choice operation ACC$\mathbb{N}$. The latter can be restated as a problem $P_{ACC\mathbb{N}}$. 

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Reduction between $\Pi_1^2$ principles

with infinite solutions, for example by letting an instance be either $\emptyset$ or a singleton $\{\langle n, k \rangle\}$, and letting a solution to an instance $X$ be any set $Y$ of the form $\{\langle n, k \rangle : k \in \mathbb{N}\}$ such that $X \cap Y = \emptyset$. Then a problem $Q$ is $\omega$-indiscriminative if and only if $P_{ACC_N} \leq_W Q$. Brattka, Hendtlass, and Kreuzer [4, Proposition 4.3] showed that if $Q$ is densely realized (i.e. if for every instance $X$ of $Q$, every $\sigma$ can be extended to a solution to $X$) then $Q$ is $\omega$-indiscriminative. It is easy to see that $P_{ACC_N}$ has diagonalization opportunities, so Theorem 4.13 extends that result in the context of $\Pi_1^2$ principles.

The class of densely realized problems mentioned above includes all problems where the class of solutions to any given instance is a collection of subsets of $\mathbb{N}$ closed under finite difference, for example $n$-RAND, the statement that for every $X$, there is a set that is $n$-random relative to $X$, and indeed RAND, the statement that for every $X$ there is a set that is arithmetically random (i.e. $n$-random for every $n$) relative to $X$. (See Downey and Hirschfeldt [13] for more on algorithmic randomness.)

A different kind of example is given by the Atomic Model Theorem (AMT), which states that every complete atomic theory has an atomic model. Given such a theory $T$, we can uniformly determine whether $\sigma$ is an initial segment of a model of $T$, in which case $\sigma$ is also an initial segment of an atomic model of $T$. Thus AMT is undiagonalizable. The same is true of the related principles OPT and AST considered in Hirschfeldt, Shore, and Slaman [29].

As we have seen, an example of a problem with diagonalization opportunities is $RT_1^2$. Thus, although $RT_1^2$ is provable in RCA$_0$, it is not Weihrauch reducible in the generalized sense to any undiagonalizable problem, such as RAND, COH, or AMT. (Of course, the same holds of $RT_n^k$ for any $n \geq 1$ and $k \geq 2$.)

**Corollary 4.15.** If $Q$ is undiagonalizable, then $RT_1^2 \nleq_{gW} Q$.

Several other examples of problems with diagonalization opportunities can be found among the consequences of $RT_2^2$, for example $SRT_2^2$ and the principles ADS, CAC, SADS, and SCAC studied in Hirschfeldt and Shore [28]. The following are a few more examples.

Recall the principle WWKL, defined at the end of Sec. 1. Let $\Psi$ be as in Definition 4.12 and let $\tau$ be an initial segment of the full binary tree such that $\Psi^{\tau}(0)$ exists. Let $T$ be a binary tree extending $\tau$ such that $A$ is a path on $T$ if and only if $A(0) \neq \Psi^A(0)$. Then $T$ is an instance of WWKL such that no solution to $T$ extends $\Psi^T$, so WWKL has diagonalization opportunities.

Let $\Theta_e$ be the $e$th Turing functional in a fixed effective listing of such functionals. The Diagonally Nonrecursive Principle (DNR) states that for every $X$, there is a function $f$ such that $f(e) \neq \Theta^X_e(e)$ for all $e$. Let $\Psi$ be as in Definition 4.12, and let $e$ be such that $\Theta_e = \Psi$. Let $\tau$ be any string such that $\Psi^\tau(e)$ and let $X \succ \tau$. Then $\Psi^\tau(e) = \Psi^X(e) = \Theta^X_e(e)$, so no solution to $X$ extends $\Psi^\tau$, and hence DNR has diagonalization opportunities.
Giusto and Simpson [23, Lemma 6.18] showed that WWKL implies DNR over RCA₀. Their proof in fact shows that DNR \( \leq_s W \) WWKL. Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1] showed that WWKL \( \nleq_\omega \) DNR. Yu and Simpson [61, Sec. 2] showed that WKL \( \nleq_\omega \) WWKL.

As shown by Kučera [37, Lemma 3], for every \( \Pi^0_1 \) class \( C \) of positive measure and every 1-random set \( R \), there is a final segment of \( R \) contained in \( C \). Relativizing this result, we see that WWKL \( \leq \) \( sc \) 1-RAND. Kučera [37, also Theorem 8.8.1] also showed that every 1-random set computes a function \( f \) as in the definition of DNR, and his proof shows that in fact DNR \( \leq \) \( sc \) 1-RAND. On the other hand, Brattka, Hendtlass, and Kreuzer [4, Proposition 7.3] showed that DNR \( \nleq \) W 1-RAND.

Indeed, the above remarks show that the following holds.

**Corollary 4.16.** WWKL \( \nleq_\omega \) gW RAND and DNR \( \nleq_\omega \) gW RAND.

A function \( f : [\mathbb{N}]^n \rightarrow \mathbb{N} \) is \( k \)-bounded if \( |f^{-1}(i)| \leq k \) for all \( i \). A set \( R \) is a rainbow for \( f \) if \( R \) is infinite and \( f \) is injective on \( [R]^n \). The Rainbow Ramsey Theorem for \( n \) and \( k \), RRT\(^n_k\), states that every \( k \)-bounded function on \( [\mathbb{N}]^n \) has a rainbow. Csima and Mileti [11, Theorem 3.1] showed that RRT\(^2_\omega\) \( \leq \) 2-RAND. However, RRT\(^2_\omega\) has diagonalization opportunities: Let \( \Psi \) be as in Definition 4.12. Let \( \tau \) be an initial segment of an injective function on \( [\mathbb{N}]^2 \) such that \( \Psi \upharpoonright \tau \) contains at least two elements \( x \) and \( y \). Let \( f \succ \tau \) be a 2-bounded function on \( [\mathbb{N}]^2 \) such that \( f(x, z) = f(y, z) \) for all sufficiently large \( z \). Then no infinite rainbow for \( f \) can extend \( \Psi \upharpoonright \tau \). Thus we have the following result.

**Corollary 4.17.** RRT\(^2_\omega\) \( \nleq_\omega \) gW RAND.

One may argue that in the above examples, the real issue is not effectivity but continuity. That is, we can define the notion of a continuous strategy by replacing the Turing functional in Definition 4.3 by any continuous map \( 2^\omega \rightarrow 2^\omega \), and use that to define a notion of “continuously reducible in the generalized sense”. If we then redefine the notion of diagonalizable instance by replacing “any Turing functional \( \Psi \)” by “any continuous \( \Psi \),” we can redo all the work in this subsection for this weaker notion of reducibility in essentially the same way. However, a continuous map \( 2^\omega \rightarrow 2^\omega \) is just a Turing functional relative to some oracle, so working in the cone above this oracle, there is no difference between the two notions of reducibility. Thus it appears that, for natural problems like the ones discussed above, there is no real difference between these two approaches.

### 4.4. Counting instances

Another thing we can do with our reduction games is calibrate the exact number of instances of \( Q \) needed to obtain \( P \) (uniformly or not). Let us write \( P \leq_{\omega}^n Q \) if Player 2 has a winning strategy in \( G(Q \rightarrow P) \) that guarantees victory in \( n + 1 \) or fewer moves, and \( P \leq_{\omega}^{<\omega} Q \) if Player 2 has a computable winning strategy in \( G(Q \rightarrow P) \) that guarantees victory in \( n + 1 \) or fewer moves. Notice that \( P \leq_0^0 Q \) if
and only if $P$ is computably true (i.e. every instance $X$ of $P$ has an $X$-computable solution), while $P \leq^0_{e.d.} Q$ if and only if $P$ is uniformly computably true (i.e. there is a procedure for obtaining a solution to an instance $X$ of $P$ from $X$). $RT_1^3$ is an example of a principle that is computably true but not uniformly so. Similarly, $P \leq^1_{e.d.} Q$ if and only if $P \leq_c Q$, and $P \leq^1_{s.d.} Q$ if and only if $P \leq_W Q$.

For instance, the proof of Proposition 4.18 shows that $SRT^g_2 \leq^2_{s.d.} D^2_2$. We also have the following more detailed analysis of the equivalence, under both $\leq_{e.d.}$ and $\leq_{s.d.}$, of $KL$, $RT^g_k$ for $n \geq 3$ and $k \geq 2$, $RT^g_{n,\infty}$ for $n \geq 3$, and $RT$.

**Proposition 4.18.** Let $n \geq 1$. Then $RT^{n,1}_{\omega,\infty} \leq^1_{\omega,\infty} KL$ but (for $n > 1$) $RT^{n,2}_{\omega,\infty} \nleq^2_{\omega,\infty} KL$. Thus $RT \nleq^0_{KL}$ and $RT \nleq^0_{KL} RT^{n,\infty}_{\omega,\infty}$ for all $k$ and all $n \geq 1$.

**Proof.** For $n = 1$ we are just stating that $RT^{1,1}_{\omega,\infty}$ is computably true, so let $n \geq 2$. As mentioned before Proposition 4.5, $PRE \leq_W KL$, where $PRE$ is the principle stating that for every $n, k \geq 2$, every $k$-coloring of $[N]^n$ has an infinite prehomogeneous set. With $n - 1$ applications of $PRE$, we can reduce a coloring $c$ of $[N]^n$ to a coloring of $N$. This coloring has a computable infinite homogeneous set, which is also homogeneous for $c$. Thus $RT^{n,1}_{\omega,\infty} \leq^1_{\omega,\infty} KL$.

For the second statement, we use the fact that every instance $Y$ of $KL$ has a solution $Z$ such that $Y \oplus Z$ is low over $Y'$. (This fact follows from the relativized form of the low basis theorem, since every $Y$-computable finitely bounded tree is $Y'$-computably bounded.) From this fact, it follows easily by induction that in the game $G(KL \rightarrow RT^{n,2}_2)$, if Player 1 plays a computable instance of $RT^{n,2}_2$ on its first move, then it is always possible for Player 1 to ensure that Player 2’s $k$th move is low over $\emptyset^{(k-1)}$. By Jockusch [30, Theorem 5.1], there is a computable instance of $RT^{n,2}_2$ with no $\emptyset^{(n-1)}$-computable solution, so if Player 1 begins with such an instance and plays as above, Player 2 cannot win until move $n$. Thus $RT^{n,2}_2 \nleq^2_{\omega,\infty} KL$.

It follows from Proposition 4.18 that $RT^{n,2}_2 \nleq^2_{\omega,\infty} KL$. In the positive direction, we have the following fact.

**Proposition 4.19.** Let $n \geq 1$. Then $RT^{n,1}_{\omega,\infty} \leq^1_{e.d.} KL$.

**Proof.** As in the previous proof, with $n-1$ applications of $PRE$, we can reduce a coloring $c$ of $[N]^n$ to a coloring of $N$, in a uniform way, and by Theorem 2.10(7), $RT^{1,1}_{\omega,\infty} \leq_{e.d.} KL$.

It is easy to see that $RT^{1,2}_2 \nleq^1_{e.d.} KL$, and by Theorem 2.10(2), $RT^{2,2}_2 \nleq^1_{e.d.} KL$. However, we do not know whether $RT^{n,1}_{\omega,\infty} \leq_{e.d.} KL$ for $n \geq 3$, where $k \geq 2$ may be $<\omega$.

**Theorem 4.20.** Let $n \geq 3$, let $k \geq 2$, and let $j \geq 1$, and suppose that

\[
m \in (n + (j - 1)(n - 2), n + j(n - 2)].
\]

Then $RT^m_k \leq^j_{\omega,\infty} RT^m_k$ but $RT^m_k \nleq^j_{\omega,\infty} RT^m_k$. 

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Proof. We begin by showing that \( RT^m_k \leq^+ \aleph_1 \) \( RT^m_k \), and then discuss uniformity.
Suppose that in the game \( G(RT^m_k \rightarrow RT^m_k) \), Player 1 begins by playing an instance \( c \) of \( RT^m_k \). By the relativized form of Lemma 5.9 of Jockush [30], for each \( A \) there is an \( A \)-computable instance \( Y \) of \( RT^m_k \) such that for any solution \( Z \), we have \( A^{(n-2)} \leq_T A \oplus Z \). Thus Player 2 can force Player 1 to play a set \( Z_2 \) on its second move such that \( c^{(n-2)} \leq_T c \oplus Z_2 \), then force it to play a set \( Z_3 \) on its third move such that \( c^{(2(n-2))} \leq_T c \oplus Z_2 \oplus Z_3 \), and so on. So Player 2 can gain access to \( c^{((j-1)(n-2))} \) before its \( j \)th move. By the relativized form of Corollary 2.2, with one more move, Player 2 can gain access to a set of PA degree over \( c^{(j(n-2))} \). By the proof of Lemma 3.2, Player 2 has access to an infinite set \( P \) such that the \( c \)-color of a tuple in \([P]^m\) depends only on its least \( m - j(n - 2) \leq n \) many colors.
With one more move, Player 2 can force Player 1 to play a solution to the induced coloring of \([P]^n\), which is also a solution to \( c \). Player 2 then plays this set as its \((j + 2)\)nd move and wins.

We now need to show that the above strategy for Player 2 is computable. We claim that the proof of the relativized form of Corollary 2.2 can be carried out uniformly, or, more precisely, that there are Turing functionals \( \Phi \) and \( \Psi \) such that for each \( X \), the set \( \Phi^X \) codes a computable 2-coloring of \([N]^n\) such that if \( H \) is an infinite homogeneous set for this coloring, then \( \Psi^X \oplus H \) is a completion of the partial function \( e \mapsto \Phi^X_e(n-2)(e) \). Assume the claim for now. Then, since \( X^{(n-2)} \) can be computed uniformly from such a completion, Player 2's first \( j \) many moves under the above strategy can be performed computably. Player 2's \( j \)th move is a completion of \( e \mapsto \Phi^X_e(n-2)(e) \). From such a completion, we can uniformly compute completions of \( e \mapsto \Phi^X_e(i)(i) \) for each \( i \leq j(n - 2) \), which allows us to carry out the construction in the proof of Lemma 3.2 uniformly to obtain a set \( P \) as above. (The key point here is that Lemma 5.4 of [30] is proved by building a finitely branching tree all of whose paths are infinite prehomogeneous sets for a given coloring \( d \). This tree is built uniformly from \( d \), and a path on such a tree can be obtained uniformly from a completion of \( e \mapsto \Phi^d_e(e) \).) Now Player 2's final two moves can also be made computably.

It remains to prove the claim. We repeat the argument in the proof of Corollary 2.2, but now paying attention to uniformity. As in that proof, we will ensure that the coloring coded by \( \Phi^X \) is unbalanced. In fact, we will ensure that every infinite homogeneous set for that coloring is homogeneous to 1; we say that the coloring is unbalanced toward 1. For \( n = 3 \), the existence of the functionals \( \Phi \) and \( \Psi \) follows from the proof of Lemma 3.1. Now let \( n \geq 3 \) and assume by induction that we have Turing functionals \( \hat{\Phi} \) and \( \hat{\Psi} \) such that for each \( Y \), the set \( \hat{\Phi}^Y \) codes a 2-coloring \( c^Y \) of \([N]^n\) that is unbalanced toward 1 and such that if \( H \) is an infinite homogeneous set for \( c^Y \), then \( \hat{\Psi}^Y \oplus H \) is a completion of the partial function \( e \mapsto \Phi^Y_e(n-2)(e) \). In particular, for each \( X \), if \( H \) is an infinite homogeneous set for \( c^X \), then \( \hat{\Psi}^X \oplus H \) is a completion of \( e \mapsto \Phi_e^X(n-1)(e) \). The argument in the proof of Lemma 5.2 of [30] shows that we can uniformly obtain an \( X \)-computable 2-coloring \( d^X \) of \([N]^{n+1}\) that is unbalanced toward 1 and such that every infinite set homogeneous for \( d^X \) is
homogeneous for $c^X$. Since $n \geq 3$, we can also transform $e^X$ into a 2-coloring $e^X$ of $[N]^{n+1}$ that is unbalanced toward 1 and such that we can uniformly compute $X'$ from $X \oplus H$ for any infinite homogeneous set $H$ for $e^X$. (We just let the $e^X$-color of an $(n+1)$-tuple be the $c^X$-color of its $n$ many least elements.) By the proof of Lemma 5.10 of [30], from $d^X$ and $e^X$ we can uniformly obtain a 2-coloring $f^X$ of $[N]^{n+1}$ that is unbalanced toward 1 and such that the infinite homogeneous sets for $f^X$ are precisely the infinite sets homogeneous for both $d^X$ and $e^X$. Let $\Phi^X$ code $f^X$. Given an infinite homogeneous set $H$ for $f^X$, the fact that $H$ is homogeneous for $d^X$, and hence for $e^X$, implies that $\Psi^X \oplus H$ is a completion of $c \mapsto \Phi^X(\hat{c})$. But since $H$ is homogeneous for $e^X$, we can uniformly compute $X'$ from $X \oplus H$. Thus we can define a Turing functional $\Psi$ such that, for any such $H$, the set $\Psi^X \oplus H$ is a completion of $c \mapsto \Phi^X(\hat{c})$.

To show that $RT^X_{\omega^2} \not\leq_1 RT^X_{\bar{2}}$, we use the relativized form of Theorem 12.1 of Cholak, Jockusch, and Slaman [7], which states that every instance $Y$ of $RT^Y_{\omega^2}$ has a solution $Z$ such that $(Y \oplus Z)^n \leq_T Y^{(n)}$. From this result, it follows easily by induction that in the game $G(RT^Y_{\omega^2} \rightarrow RT^Z_{\omega^2})$, if Player 1 plays a computable instance of $RT^Y_{\omega^2}$ on its first move, then it is always possible for Player 1 to ensure that Player 2’s ith move has $\emptyset((n+i-2)(n-2))-\text{computable double jump}$. As mentioned above, there is a computable instance of $RT^Z_{\omega^2}$ such that every solution $Z$ computes $\emptyset^{(m-2)}$, and hence has $Z^n \geq_T \emptyset^{(m)} \geq_T \emptyset^{(n+j-1)(n-2)}$. If Player 1 begins with such an instance and plays as above, Player 2 cannot win until move $j+2$.

We turn now to comparing versions of Ramsey’s Theorem with fixed size of tuples but different numbers of colors. We have seen that $RT^m_3 \leq_{gW} RT^2_2$ but $RT^3_3 \not\leq_{gW} RT^2_2$. In the $n = 1$ case at least, this fact has the following generalization.

**Theorem 4.21.** Let $j \geq 2$. If $k \in \{j^m, j^{m+1}\}$ then $RT^k_j \leq_{gW} RT^1_j$ but $RT^k_j \not\leq_{gW} RT^1_j$. Similarly, $RT^1_j \not\leq_{gW} RT^m_j$ for all $m$.

**Proof.** For the positive direction, we use the usual reduction from $RT^1_{j^{m+1}}$ to $RT^1_j$, which we can think of inductively. The base case is $m = 0$, for which a single application of $RT^1_j$ suffices. Given a $j^{m+1}$-coloring $c$, we partition its colors into $j$ many sets $G_0, \ldots, G_{j-1}$, each of size $j^m$, and let $d$ be the $j$-coloring defined by letting $d(x) = i$ if $c(x) \in G_i$. Solving this instance of $RT^1_j$ gives us an infinite homogeneous set $H$ for $d$, and $c \upharpoonright H$ is a $j^m$-coloring (which we can encode as a coloring of $N$), so by induction we can find an infinite homogeneous set for $c \upharpoonright H$ (and hence for $c$) with $m$ applications of $RT^1_j$. This procedure is clearly uniform (both in the instances of $RT^1_{j^{m+1}}$ and in $m$), so we conclude that $RT^1_{j^{m+1}} \leq_{gW} RT^1_j$, and hence $RT^1_k \leq_{gW} RT^1_j$ for all $k \leq j^{m+1}$.

For the negative direction, let $\Phi$ be a computable strategy for Player 2 in $G(RT^1_j \rightarrow RT^1_k)$, where $k > j^m$. Recall that if $X$ is the join of Player 1’s first $m$ moves in a run of this game, then $\Phi^X$ is of the form $V \oplus \Phi^X$, where $V = \{0\}$ if Player 2 declares victory on the $m$th move, in which case $\Phi^X$ is a solution to
Player 1’s first move, and \( V = \emptyset \) otherwise, in which case \( \hat{\Phi}^X \) codes a \( j \)-coloring of \( \mathbb{N} \). We write \( d^X \) for this coloring.

Build a \( k \)-coloring \( c \) of \( \mathbb{N} \) as follows. We have auxiliary sets \( A_{i_0,\ldots,i_l} \), where \( l < m \) and \( i_p < j \) for each \( p \leq l \). Begin defining \( c \) arbitrarily. For uniformity of notation, let \( A_0 = c \), and start computing \( \Phi^{A_k} \). If we find that \( \Phi^{A_k}(0) = 1 \) (which indicates that Player 2 declares victory on the first move) then wait until we find an \( x \) such that \( \Phi^{A_k}(x) = 1 \) and ensure that \( c(s) \neq c(x) \) for all sufficiently large \( s \).

Otherwise, once we find that \( \Phi^{A_k}(0) = 0 \), start putting all \( x \) such that \( d^{A_k}(x) = i \) into \( A_i \). Once we have started building a set \( A_{i_0,\ldots,i_l} \), if we find that \( \Phi(\exists j \geq i_1+1 \ A_{i_0,\ldots,i_l-1}(0)) = 1 \) and \( \Phi(\exists j \geq i_1+1 \ A_{i_0,\ldots,i_l-1}(x)) = 1 \) for some \( x \), ensure that \( c(s) \neq c(x) \) for all sufficiently large \( s \).

On the other hand, if we find that \( \Phi(\exists j \leq i_1+1 \ A_{i_0,\ldots,i_l-1}(0)) = 0 \), and \( l < x-1 \), then start putting all \( x \) such that \( d^{\hat{\Phi}}(0) = 0 \), and for some \( m \), \( \Phi(\exists j \leq i_1+1 \ A_{i_0,\ldots,i_l-1}(x)) = i \) into \( A_{i_0,\ldots,i_l-1} \).

The total number of colors that we can ever want to avoid in this construction is \( j^m \), so we never run out of colors. Now consider the following run of our game, where Player 2 plays according to \( \Phi \). Player 1 begins by playing \( A_0 = c \). By construction, Player 2 cannot declare victory on the first move, so the \( A_i \) are built. There is an \( i_0 < j \) such that \( A_{i_0} \) is an infinite homogeneous set for \( \Phi^{A_k} \), and Player 1 plays \( A_{i_0} \).

Again by construction, Player 2 cannot declare victory on this move, so the \( A_{i_0,i_j} \) are built. Again, there is an \( i_1 < j \) such that \( A_{i_0,i_1} \) is an infinite homogeneous set for \( \Phi^{A_k} \), and Player 1 plays \( A_{i_0,i_1} \). Continuing in this way, we have a run of our game in which Player 2, playing according to \( \Phi \), cannot declare victory before move \( m + 2 \).

Since \( \Phi \) is arbitrary, \( RT^1_k \nleq_m RT^1_j \), which of course implies that \( RT^1_k \nleq_m RT^1_j \).

The positive part of the above proof works for higher exponents, so we have the following result.

**Proposition 4.22.** Let \( j, n \geq 2 \). If \( k \in (j^m, j^{m+1}] \) then \( RT^m_k \nleq_m RT^m_j \).

The exponent-lifting technique of Sec. 3.1 does not seem immediately applicable to establishing corresponding lower bounds on the \( m \) such that \( RT^m_k \leq_m RT^m_j \) for \( n \geq 2 \) and \( k > j \geq 2 \), since some of the sets \( A_{i_0,\ldots,i_l} \) in the proof of Theorem 4.21 might be infinite, and hence, if we try to perform a version of the construction in that proof inside a set \( P \) as in the proof of Theorem 3.3, we might get infinitely many convergent computations \( \hat{\Phi}(\exists j \leq i_1+1 \ A_{i_0,\ldots,i_l-1}(\hat{\Phi}(\exists j \leq i_1+1 \ A_{i_0,\ldots,i_l-1})(x))) \neq 1 \).

Patey [46] gave an exact characterization of the least \( m \) such that \( (S)RT^m_k \leq_m RT^m_j \) for \( n \geq 2 \) and \( k > j \geq 2 \), answering a question stated in an earlier version of this paper. For \( n \geq 3 \), this \( m \) is 2, independent of \( j \) and \( k \). The \( n = 2 \) case is more complicated: Let \( \pi(x,y) \) be the least \( a \geq 1 \) such that \( x = ay - b \) for some \( b < y \).

For \( j \geq 2 \), let \( m_{1,j} = 0 \), and for \( k > 1 \), let \( m_{k,j} = m_{\pi(k,j),j} + 1 \). Then \( m_{k,j} \) is the least \( m \) such that \( (S)RT^m_k \leq_m RT^m_j \).
Remark 4.23. Another way to define $\leq^n W$ is to use the operation • discussed in Dorais, Dzhafarov, Hirst, Mileti, and Shafer [12, Sec. 5.2]. For problems $P$ and $Q$ and a fixed Turing functional $\Theta$, the problem $Q \cdot P$ is defined as follows. A set $A$ is an instance of $Q \cdot P$ if $A$ is an instance of $P$ and, for every solution $B$ to $A$ as an instance of $P$, we have that $\Theta^{A \oplus B}$ is an instance of $Q$. (In practice, $\Theta$ will normally be chosen so that every instance $A$ of $P$ has the latter property.) A solution to an instance $A$ of $Q \cdot P$ is a pair $(B, C)$ such that $B$ is a solution to $A$ as an instance of $P$, and $C$ is a solution to the instance $\Theta^{A \oplus B}$ of $Q$. (See [12] for a discussion of the relationship of this notion with function composition in the context of computable analysis, and the compositional product in the Weihrauch lattice.)

For notational simplicity, let us consider the $n = 2$ case. It is easy to check that if $P \leq W Q \cdot Q$ then $P \leq^2 W Q$: Suppose that $\Phi$ and $\Psi$ witness that $P \leq W Q \cdot Q$. In the game $G(Q \rightarrow P)$, once $\Phi$ plays an instance $X$ of $P$, $\Psi$ responds with $\Phi^X$. $\Phi$ then plays a solution $Z$ to $\Phi^X$, thought of as an instance of $Q$. $\Phi$ can then play $\Theta^{\Phi^X \oplus Z}$, which is an instance of $Q$. $\Phi$ then plays a solution $W$ to this instance. Now $(Z, W)$ is a solution to $\Phi^X$ as an instance of $Q \cdot Q$, so $\Psi(Z, W)$ is a solution to $P$. $\Phi$ can thus play $\Psi(Z, W)$ on its third move and declare victory.

In the other direction, we need a slight technical adjustment. Suppose that $P \leq^2 W Q$ and let $\Gamma$ be Player 2’s computable winning strategy for $G(Q \rightarrow P)$. Let $\tilde{Q}$ be the problem whose instances are pairs $(X, Y)$ where $X$ is any set and $Y$ is an instance of $Q$ such that a solution to such an instance is a pair $(X, Z)$ where $Z$ is a solution to $Y$. Let $\Phi$ be a Turing functional that, on oracle $X$, returns $(X, \Gamma^X)$. Let $\Psi$ be a Turing functional that, on oracle $(X, Y) \oplus (X, Z)$, returns $\Gamma^X \oplus Z$. Let $\Theta$ be a Turing functional that, on oracle $X \oplus (Z, W)$, returns $\Gamma^X \oplus Z \oplus W$. Define • using $\Theta$. Let $X$ be an instance of $P$. Then $\Phi^X$ is an instance of $Q \cdot \tilde{Q}$. A solution to this instance has the form $(Z, W)$, where $Z$ is a solution to $(X, \Phi^X)$ as an instance of $\tilde{Q}$, and $W$ is a solution to the instance $\Gamma^X \oplus Z$ of $Q$. Then $\Psi(X, W)$ is a solution to $X$, as it is equal to $\Gamma^X \oplus Z \oplus W$, which is Player 2’s third, and therefore winning, move in a run of $G(Q \rightarrow P)$ played according to $\Gamma$. Thus $\Phi$ and $\Psi$ witness that $P \leq W Q \cdot \tilde{Q}$.

A similar analysis can be made for larger values of $n$. Of course, the real power of the • operation comes when considering multiple principles. For instance, saying that $RT^2_2 \leq W SRT^2_2 \cdot COH$ gives us more information than saying that $RT^2_2 \leq^2 W SRT^2_2 \land COH$. On the other hand, we cannot use the • operation to define $\leq^2 W$ in general, as there are principles $P$ and $Q$ such that $P \leq W Q$ but $P \not\leq^2 W Q$ for all $n$.

4.5. Non-$\omega$-models

We need not limit our game-based approach to $\omega$-models. Let $\Sigma^0_1$-PA be first-order Peano arithmetic with the induction scheme restricted to $\Sigma^0_1$ formulas. For a first-order model $M$ of $\Sigma^0_1$-PA and $X_0, \ldots, X_n \subseteq M$, let $M[X_0, \ldots, X_n]$ be the
second-order model whose first-order part is $M$ and whose second-order part consists of all subsets of $M$ that are $\Delta^1_0$-definable over $M$ with $X_0, \ldots, X_n$ as additional parameters. Let $P$ be of the form $\forall X[\Theta(X) \rightarrow \exists Y \Psi(X,Y)]$, with $\Theta$ and $\Psi$ arithmetic. An instance of $P$ over $M$ is an $X \subseteq M$ such that $\Theta(X)$ holds in $M[X]$, and a solution to $X$ (over $M$) is a $Y \subseteq M$ such that $\Psi(X,Y)$ holds in $M[X,Y]$.

**Definition 4.24.** For problems $P$ and $Q$, the generalized reduction game $\hat{G}(Q \rightarrow P)$ is a two-player game that proceeds as follows. If at any point one of the players does not have a legal move, then the game ends with a victory for the other player.

On the first move, Player 1 plays a countable first-order model $M$ of $\Sigma^1_0$-PA and an instance $X_0$ of $P$ over $M$ with $M[X_0] \models RCA_0$, and Player 2 either plays a solution to $X_0$ in $M[X_0]$ and declares victory, in which case the game ends, or responds with an instance $Y_0$ of $Q$ in $M[X_0]$.

For $n > 1$, on the $n$th move (if the game has not yet ended), Player 1 plays a solution $X_{n-1}$ to the instance $Y_{n-1}$ of $Q$ with $M[X_0, \ldots, X_{n-1}] \models RCA_0$. Then Player 2 either plays a solution to $X_{n-1}$ in $M[X_0, \ldots, X_{n-1}]$ and declares victory, in which case again the game ends, or plays an instance $Y_n$ of $Q$ in $M[X_0, \ldots, X_{n-1}]$.

Player 2 wins this play of the game if it ever declares victory, or if Player 1 has no legal move at some point in the game. Otherwise, Player 1 wins.

It is easy to adapt the proof of Proposition 4.2 to show that if $\text{RCA}_0 + Q \vdash P$ then Player 2 has a winning strategy in $\hat{G}(Q \rightarrow P)$, while otherwise Player 1 has a winning strategy in $\hat{G}(Q \rightarrow P)$. We can use our generalized games to count applications as above, by writing $\text{RCA}_0 + Q \vdash^k P$ to mean that Player 2 has a winning strategy in $\hat{G}(Q \rightarrow P)$ that guarantees a win in at most $k+1$ many moves.

This notion seems particularly interesting in connection with principles equivalent to $\text{ACA}_0$, such as $\text{RT}^n_k$ for $n \geq 3$ and $k \geq 2$. For example, our discussion above transfers to non-$\omega$-models to show that $\text{RCA}_0 + \text{RT}^3_2 \vdash^2 \text{RT}^2_2$ but $\text{RCA}_0 + \text{RT}^3_2 \nvdash^2 \text{RT}^2_4$. It is well known that adding $\Sigma^0_1$-comprehension to $\text{RCA}_0$ yields $\text{ACA}_0$, but our notion allows us to count how many applications of $\Sigma^0_1$-comprehension are needed in a particular proof in $\text{ACA}_0$. For example, Jockusch [30, Theorems 5.1 and 5.5] showed that for $n \geq 2$, every instance $X$ of $\text{RT}^n_k$ has a solution computable in $X^{(n)}$, but there is a computable instance with no solution computable in $\psi^{(n-1)}$. The proof of the first of these facts carries through in $\text{RCA}_0$ to show that $\text{RT}^n_k$ can be proved in $\text{RCA}_0$ together with a single application of the principle $\forall X \exists Y [Y = X^{(n)}]$, from which it follows easily that $\text{RCA}_0 + \Sigma^0_1\text{-CA} \vdash^n \text{RT}^n_k$. The second fact above implies that $\text{RT}^n_k \nvdash^{n-1} \Sigma^0_1\text{-CA}$, and hence $\text{RCA}_0 + \Sigma^0_1\text{-CA} \nvdash^{n-1} \text{RT}^n_k$.

A simple example showing that $P \leq^n_0 Q$ (or even $P \leq^n_{\mathbb{R}_\omega} Q$) and $\text{RCA}_0 + Q \vdash P$ do not together imply $\text{RCA}_0 + Q \vdash^n P$ is given by the principles $\Pi^0_1G$ and $\Pi^0_1G$ from Definition 1.6. It is easy to see that $\Pi^0_1\text{GA}$ is uniformly computably true, so in particular $\Pi^0_1\text{GA} \leq^n_0 \Pi^0_1\text{G}$. Of course, we also have $\text{RCA}_0 + \Pi^0_1\text{G} \vdash^n \Pi^0_1\text{GA}$. However, as shown in [27, Theorem 3.3], $\text{RCA}_0 \not\vdash \Pi^0_1\text{GA}$. Let $\mathcal{M}$ be a countable
model of $\text{RCA}_0 + \neg \Pi_1^0 \text{GA}$, let $M$ be the first-order part of $\mathcal{M}$, and let $X \in M$ be an instance of $\Pi_1^0 \text{GA}$ over $M$ with no solution in $\mathcal{M}$. In the game $\hat{G}(\Pi_1^0 \text{G} \rightarrow \Pi_1^0 \text{GA})$, Player 1 can begin by playing $M$ and $X$, and Player 2 cannot then win on its first move. Thus $\text{RCA}_0 + \Pi_1^0 \text{G} \not\vdash \Pi_1^0 \text{GA}$.

It would of course be interesting to have natural examples for larger $n$ of problems $P$ and $Q$ such that $\text{RCA}_0 + Q \vdash P$ and $P \leq^w_n Q$, but $\text{RCA}_0 + Q \not\vdash^n P$.

It is also straightforward to define the notion of a computable strategy for Player 2 in a generalized reduction game (using $\Delta^0_1$-definable functionals), and hence a notion of $gW$-reducibility that is not restricted to $\omega$-models, which we may denote by $\text{RCA}_0 + Q \vdash^w W P$, as well as the corresponding instance-counting version $\text{RCA} + Q \vdash^n W P$. In this context, $\text{RCA}_0 + Q \vdash^{n+1} P$ and $\text{RCA}_0 + Q \vdash^{1+1}_W P$ are the analogs to $P \leq^n Q$ and $P \leq^n W Q$, respectively. We leave the further study of these notions to future work.

5. Relationships Between Versions of RT and KL: A Summary

We summarize the relationships between some versions of RT and KL under $\leq_\omega$, $\leq_\epsilon$, $\leq_{sc}$, $\leq_{sw}$, and $\leq_{gW}$ in Figs. 2–7. (We do not include the Thin Set Theorem, as we have not considered it beyond Theorem 3.6.) No other implications than the ones shown (or implied by transitivity) hold. Dotted arrows represent hierarchies of principles. For example, the dotted arrow in Fig. 3 represents the fact that $\text{RT}_2^1 < \text{RT}_3^2 < \text{RT}_5^3 < \cdots < \text{RT}$. We list several open questions, old and new, at the end of this section.

Figure 2 extends Fig. 1. Since the number of colors does not matter in this case, the subscript $k$ here stands for any number greater than 1 or for $<\infty$. For justifications of the implications and nonimplications in this figure, see [25].
In Fig. 3, we leave out $\text{RT}_k^n$, $\text{SRT}_k^n$, and $D_2^k$ for $n \geq 2$ and $k > 2$. As noted above, Patey [45] has shown that $\text{RT}_k^n \nleq_c \text{RT}_j^n$ for all $n \geq 2$ and $k > j \geq 2$. His argument also shows that $\text{SRT}_k^2 \nleq_c \text{RT}_j^2$ for all $k > j \geq 2$. In this figure, all implications are obvious or follow from those in Fig. 5, except for that between KL and $\text{RT}_2^2$, which is established in Corollary 2.4; and that between $D_2^2$ and $\text{SRT}_2^2$, which follows from the same proof that shows that $D_2^2$ implies $\text{SRT}_2^2$ over RCA$_0$ (see Cholak, Jockusch, and Slaman [7, Lemma 7.10]). All nonimplications in this figure follow either from those in Fig. 2 or from computability-theoretic results of Jockusch [30].

In Fig. 4 too, we leave out $\text{RT}_k^n$, $\text{SRT}_k^n$, and $D_2^k$ for $n \geq 2$ and $k > 2$; Patey’s results also apply here. As mentioned following Theorem 3.7, Dzhafarov, Patey, Solomon, and Westrick [19] showed that if $2 \leq j < k$ then $\text{RT}_k^1 \nleq_{sc} \text{SRT}_j^2$. They also showed that COH $\nleq_{sc} \text{SRT}_{<\infty}^2$. In this figure, all implications are obvious or follow from those in Fig. 6, except for those between

1. KL and $\text{RT}_2^3$, which is established in Corollary 2.6,
2. WKL and $\text{RT}_{<\infty}^1$, which follows from Proposition 2.7,
3. COH and $\text{RT}_{<\infty}^1$, which is established in Theorem 2.10(9).
Reduction between $\Pi^1_2$ principles

Fig. 4. Versions of RT and KL under $\leq_{sc}$. 
All nonimplications in this figure follow from those in Fig. 3, except for those between

1. RT (and hence $\text{RT}_2^4$ and $\text{RT}_2^3$) and WWKL, which is due to Monin and Patey [42] (strengthening the nonimplication between RT and (W)KL established in Corollary 2.9),

2. $\text{SRT}_2^3$ and COH, which is due to Dzhafarov [16] (strengthening the nonimplication between $D_2^2$ and COH that he proved in [15]),

3. $\text{SRT}_2^3$ and $\text{RT}_3^1$ (and hence $\text{RT}_{<\infty}^1$), which is due to Dzhafarov, Patey, Solomon, and Westrick [19] (strengthening the nonimplication between $D_2^2$ and $\text{RT}_3^1$ proved by Dzhafarov [15]),

Fig. 5. Versions of RT and KL under $\leq_W$. 
Reduction between $\Pi^1_2$ principles

(4) $D^2_2$ and $\text{SRT}^2_2$, which is due to Dzhafarov [16],

(5) WWKL (and hence DNR) and $\text{RT}^1_3$ (and hence $\text{RT}^1_3$ and $\text{RT}^1_{<\infty}$), which is established in Theorem 2.10(10), and

(6) $\text{RT}^j_1$ and $\text{RT}^k_1$ for $k > j \geq 2$, which is Theorem 3.7, again due to Dzhafarov [15].

In Fig. 5, we again leave out $\text{RT}^n_k$, $\text{SRT}^2_k$, and $D^2_k$ for $n \geq 2$ and $k > 2$; see

Theorems 3.3 and 3.4. In this figure, all implications are obvious or follow from

those in Fig. 6, except for the one between $\text{RT}^2_3$ and KL, which is established in

Fig. 6. Versions of RT and KL under $\leq_{\text{W}}$. 

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Corollary 2.3. All nonimplications in this figure follow from those in Fig. 3, except for those between

1. KL and $D_2^2$ (and therefore $SRT_2^2$), which is established in Theorem 2.10(2),
2. $SRT_2^2$ (and therefore $D_2^2$) and COH, which is due to Dzhafarov [16],
3. $SRT_2^2$ (and therefore $D_2^2$) and $RT_1^3$ (and therefore $RT_{<\infty}^1$), which is established in Theorem 2.10(4),
4. $D_2^2$ and $SRT_2^2$, which is due to Dzhafarov [16],
5. COH and $RT_1^3$ (and therefore $RT_1^3$ and $RT_{<\infty}^1$), which is established in Theorem 2.10(5),
6. WKL (and therefore WWKL and DNR) and $RT_1^3$ (and therefore $RT_1^3$ and $RT_{<\infty}^1$), which is established in Theorem 2.10(1), and
7. $RT_1^3$ and $RT_k^3$ for $k > j \geq 2$, which is the easy case of Theorem 3.3.

In Fig. 6 we once more leave out $RT_n^k$, $SRT_n^k$, and $D_n^2$ for $n \geq 2$ and $k > 2$; again see Theorems 3.3 and 3.4. In this figure, all implications are obvious except for those between

1. $RT_2^2$ and $RT_{<\infty}^1$, which is established in Theorem 2.10(8),
2. $RT_2^2$ and COH, which follows from the proof of Theorems 12.4 and 12.5 of Cholak, Jockusch, and Slaman [7], as noted also by Brattka and Rakotoniaina [5],
3. KL and $RT_{<\infty}^1$, which is established in Theorem 2.10(7),
4. KL and COH, which is established in Theorem 2.10(6),
Reduction between $\Pi^1_2$ principles

(5) $D^2_k$ and $RT^1_2$, which is established in Theorem 2.10(3),

(6) $D^2_k$ and DNR, which follows from the proof of Theorem 2.3 of Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [26] (see also [5, Corollary 5.25]), and

(7) WWKL and DNR, which follows from the proof of Lemma 6.18 of Giusto and Simpson [23].

All nonimplications in this figure follow from those in Figs. 4 and 5.

In light of Propositions 4.5 and 4.7, in Fig. 7, $k$ stands for any number greater than 1 or for $<\infty$. All implications in this figure follow from those in Fig. 5 except for the equivalences on the top and bottom lines, which are established in Propositions 4.5–4.7; and the implication between $D^2_k$ and $SRT^2_k$, which is established in Proposition 4.8. All nonimplications in this figure follow from the ones in Fig. 2, except for those between

(1) WKL (and therefore WWKL and DNR) and $RT^1_2$, which is established in Corollary 4.10, and

(2) COH and $RT^1_1$, which is established at the beginning of Sec. 4.3.

Figure 3 shows that most of the implications in Fig. 2 are in fact $\leq_1\omega$-reductions (or $\leq_0\omega$-reductions in the case of reductions to $RT^1_k$). The exceptions are

(1) some of the reductions witnessing the equivalences represented by the top line of Fig. 2 (see Proposition 4.18 and Theorem 4.20), and

(2) the ones between principles with the same exponent (greater than 1) and different numbers of colors (by Patey [45]).

Figure 5 shows that most of the implications in Fig. 7 are in fact $\leq_{gW}$-reductions. The exceptions are

(1) some of the reductions witnessing the equivalences represented by the top line of Fig. 7 (again see Proposition 4.18 and Theorem 4.20),

(2) those between principles with the same exponent and different numbers of colors (see Theorems 3.3 and 3.4), and

(3) the one between $D^2_k$ and $SRT^2_k$ (see the comment on the work of Dzhafarov [16] in Definition 1.3, and the paragraph preceding Proposition 4.18).

The following questions remain open.

Question 5.1. Is $RT^2_2 \leq_\omega SRT^2_2$? (Equivalently, is $COH \leq_\omega SRT^2_2$?)

Question 5.2. Is $RT^2_2 \leq_{gW} SRT^2_2$? (As in the previous question, this question is equivalent to asking whether $COH \leq_{gW} SRT^2_2$, since we can obtain $RT^2_2$ uniformly from $COH + SRT^2_2$, by the proof of [7, Lemma 7.11]).
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Question 5.3. Is \( \text{COH} \leq_c \text{SRT}_2^2 \)?

Question 5.4. Let \( n \geq 3 \) and \( k \geq 2 \), where \( k \) may also be \( < \infty \). Is \( \text{RT}^n_k \leq_{eW} \text{RT}^n_j \)? (See the mention of the work of Patey [46] following Proposition 4.22 for the \( \leq_m^\omega \) case.)

Question 5.5. Let \( n \geq 2 \) and \( k > j \geq 2 \), where \( k \) may also be \( < \infty \). For which \( m \) do we have \( \text{RT}^m_k \leq_{eW} \text{RT}^m_j \)?

There are also many principles related to Ramsey’s Theorem not shown in the diagrams in this section. In some cases, their positions in these diagrams follow from known results (or from the proofs of these results), but there are likely several interesting questions involved in filling out these diagrams. For more information on some of these principles, see [14, 25].

Another area for further work is the development of useful extensions of the reducibilities discussed here to principles with more complicated syntactic forms than the \( \Pi^1_2 \) principles we have considered.

Appendix A. On a Proof by Cholak, Jockusch, and Slaman

The following result is stated as Theorem 12.2 of Cholak, Jockusch, and Slaman [7].

Theorem A.1. Let \( n, k \geq 2 \) and let \( C_0, C_1, \ldots \) be sets such that \( C_i \nleq_T \varnothing^{(n-2)} \) for all \( i \). Each computable \( k \)-coloring of \( [\mathbb{N}]^n \) has an infinite homogeneous set \( H \) such that \( H' \nleq_T \varnothing^{(n)} \) and \( C_i \nleq_T H \) for all \( i \).

The proof given in that paper is by induction on \( n \). The base case \( n = 2 \) of that proof is correct, but the inductive case has a flaw. The argument for \( n + 1 \) fixes a computable coloring \( c \) of \( [\mathbb{N}]^{n+1} \) and \( C_0, C_1, \ldots \) such that \( C_i \nleq_T \varnothing^{(n-1)} \) for all \( i \). It then chooses an infinite prehomogeneous set \( A \) for this coloring with \( A' \leq T \varnothing'' \), and claims that \( C_i \nleq_T A^{(n-2)} \) for each \( i \). This claim is correct if \( n \geq 3 \), but for \( n = 2 \) it becomes a claim that \( C_i \nleq_T A \) for all \( i \). However, the only condition on \( C_i \) in this case is that \( C_i \nleq_T \varnothing' \). There are computable colorings of \( [\mathbb{N}]^3 \) with no infinite \( \varnothing' \)-computable prehomogeneous sets, as may be seen easily from the existence of a computable 2-coloring of pairs with no infinite \( \varnothing' \)-computable homogeneous sets [30, Theorem 3.1]. Taking \( C_i = A \) for such a coloring shows that the claim does not hold in general.

To correct this argument, it is enough to prove the theorem for \( n = 3 \), as then the inductive argument can proceed as before. We give a proof based on that of the \( n = 2 \) case, beginning with an auxiliary lemma, which is an extension of [7, Theorem 12.1], and is proved by a similar argument. We write \( \text{deg}(X) \) for the (Turing) degree of a set \( X \).

Lemma A.2. Let \( k, n \geq 2 \), let \( e \) be a computable \( k \)-coloring of \( [\mathbb{N}]^n \), and let \( p \) be a degree such that \( p \gg \varnothing^{(n-1)} \). Then there is an infinite homogeneous set \( H \) for \( e \) such that \( \text{deg}(H)' \leq p \).


**Lemma A.3.** Let $k \geq 2$ and let $C_0, C_1, \ldots$ be sets such that $C_i \not\subseteq_T \emptyset'$ for all $i$. Each computable $k$-coloring of $[N]^3$ has an infinite homogeneous set $H$ such that $H \not\subseteq_T \emptyset''$ and $C_i \not\subseteq_T H$ for all $i$.

**Proof.** Fix a $k$-coloring $c$ of $[N]^3$. Let $c_i = \deg(C_i)$. Let $d_0 = 0''$. If $d_i \neq 0''$ then let $e_1 = c_i$; otherwise let $e_1 = 0''$. Let $d_{i+1} = d_i \lor e_1$. By Kleene and Post [36], Lacombe [39], and Spector [57] (see Odifreddi [44, Theorem V.4.3]), the ideal generated by the $d_i$ has an exact pair $f, g$ such that the $e_i$ are uniformly computable in both $f$ and $g$. (More precisely, there are $E_i \in e_i$ such that the $E_i$ are uniformly computable from any element of $f$, and $E_i \in e_i$ such that the $E_i$ are uniformly computable from any element of $g$.) Since $0''$ is not in this ideal, at least one of $f$ and $g$ is not above $0''$, say $g \neq 0''$. Note that, since $d_0 = 0''$, we have $g \geq 0''$. Also, $e_i \neq 0'$ for all $i$. By Posner and Robinson [47, Theorem 3] relativized to $0'$, there is a $d \geq 0'$ such that $g = d' = d' \lor e_i$ for all $i$. By Friedberg [20], there is an $a$ such that $a' = d$. Then $a'' = g$, so

$$a'' \neq 0''.$$

If $e_1 = c_i$ then $(a' \lor c_i)' = (d \lor e_1)' = g' \geq 0''$. If $e_1 \neq c_i$ then $(a' \lor c_i)' \geq a'' \lor c_i = g \lor c_i \geq d_i \lor c_i \geq 0''$. Thus

$$(a' \lor c_i)' \geq 0''$$

for all $i$. These two displayed properties of $a$ are all that we will use below.

Let $p$ be a degree that is PA over $a''$ and hyperimmune-free over $a''$. By Jockusch [30, Lemma 5.9] relative to $a$, there is an $a$-computable 2-coloring $d$ of $[N]^3$ such that for any infinite homogeneous set $H$ for $d$, we have $\deg(H) \lor a \geq a'$. We can combine the colorings $c$ and $d$ into an $a$-computable $2k$-coloring $e$ of $[N]^3$ such that any homogeneous set for $e$ is also homogeneous for both $c$ and $d$. (Just let $e(s) = 2e(s) + d(s)$.) By the $n = 3$ case of Lemma A.2 (relativized to $a$), there is an infinite homogeneous set $H$ for $e$ such that $(\deg(H) \lor a) \leq p$. We claim that $H$ satisfies the conclusion of Lemma A.3. Let $b = \deg(H) \lor a$. Then $b \geq a'$ and $b'$ is hyperimmune-free over $a''$. 


If \( b' \geq 0'' \) then, since we also have \( b' \geq a'' \lor 0'' \) and hence \( a'' \lor 0'' \) is hyperimmune-free over \( a'' \). But \( a'' \lor 0'' \) is c.e. relative to \( a'' \), so \( a'' \geq 0''' \), which is not the case. Hence \( a'' \geq 0''' \), which is not the case. Thus \( c_i \not\geq b \) for all \( i \), and hence \( C_i \not\geq T \) for all \( i \). This completes the proof of Lemma A.3 and hence also the proof of Theorem A.1.

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References

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