Topological aspects of the Medvedev lattice

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Abstract

We study the Medvedev degrees of mass problems with distinguished topological properties, such as denseness, closedness, or discreteness. We investigate the sublattices generated by these degrees; the prime ideal generated by the dense degrees and its complement, a prime filter; the filter generated by the nonzero closed degrees and the filter generated by the nonzero discrete degrees. We give a complete picture of the relationships of inclusion holding between these sublattices, these filters, and this ideal. We show that the sublattice of the closed Medvedev degrees is not a Brouwer algebra. We investigate the dense degrees of mass problems that are closed under Turing equivalence, and we prove that the dense degrees form an automorphism base for the Medvedev lattice. The results hold for both the Medvedev lattice on the Baire space and the Medvedev lattice on the Cantor space.

1 Introduction

The Medvedev lattice $\mathfrak{M}$ was introduced by Medvedev, [8], in order to provide a computational semantics for constructive propositional logic: see for instance [8], [16], [19], [21]. On the other hand, Medvedev reducibility, whose degree structure gives rise to the Medvedev lattice, provides a novel computability-theoretic reducibility paradigm which is interesting in its own right. For instance, it is not difficult to see that classical degree structures such as the Turing degrees and the

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enumeration degrees can be embedded into the Medvedev lattice. But, perhaps more interestingly, Medvedev reducibility is a reducibility on sets of reals, not just reals, so its domain of discourse is a very interesting mathematical structure, namely the Baire space (or the Cantor space, if we confine ourselves to 0-1-valued functions). As such, Medvedev reducibility and the Medvedev lattice make it possible, from a computability-theoretic point of view, to discuss and study properties of topological objects such as closed, dense and discrete sets of reals. In this paper we investigate the sublattices generated by the closed, dense and discrete Medvedev degrees; we study the prime ideal generated by the dense degrees and its complement, a prime filter; we also study the filter generated by the nonzero closed degrees, and the filter generated by the nonzero discrete degrees. We describe the relationships of inclusion holding between these sublattices, these filters, and this ideal. We show that the sublattice of the closed Medvedev degrees is not a Brouwer algebra. Among other things, we also investigate the dense degrees of mass problems that are closed under Turing equivalence, and we prove that the dense degrees form an automorphism base for the Medvedev lattice. The results hold both of the Medvedev lattice on the Baire space and of the Medvedev lattice on the Cantor space.

In the rest of this section, we briefly introduce notions, terminology and background material that are relevant to this paper. For other unexplained notions and terminology, the reader is referred to any standard textbook, such as [3], [7], [11], or [17]. Let ω^ω denote the collection of all functions from the set ω of natural numbers into itself. If A is any set, by A^* we denote the set of all finite sequences, or *strings*, of elements of A. We use standard notations and terminology for strings. In particular, λ denotes the empty string; if $\sigma, \tau \in A^*$ and f is a function $f : \omega \longrightarrow A$ then: $\sigma\hat{\ }\tau$ denotes the concatenation of σ and τ; $|\sigma|$ denotes the length of σ; $\sigma\hat{\ }f$ denotes the function

$$\sigma\hat{\ }f(x) = \begin{cases} \sigma(x), & \text{if } x < |\sigma|, \\ f(x - |\sigma|), & \text{otherwise;} \end{cases}$$

we use $\sigma \subseteq \tau$ to denote that σ is an initial segment of τ, and $\sigma \subset f$ to denote that the string σ is an initial segment of the function f; we write $\sigma|\tau$ to mean that σ and τ are $\subseteq$-incomparable. We let $\langle a_0 \cdots a_{n-1}\rangle$ denote the string of length n formed by the elements $a_0, \cdots, a_{n-1}$ of A.

For every string $\sigma \in \omega^*$, let

$$[[\sigma]] = \{f \in \omega^\omega : \sigma \subset f\}.$$

The *Baire topology* on ω^ω is the topology generated by the basic open neighborhoods $\{[[\sigma]] : \sigma \in \omega^*\}$. With this topology, ω^ω is called the *Baire space*. In view of this topology, finite strings will be sometimes also called *basic neighborhoods*, or *intervals*. Notice that each $[[\sigma]]$ is also closed. The Baire space is a complete metric space. Restriction to the set 2^ω of 0-1-valued functions gives the *Cantor space*, which up to homeomorphism can be characterized as the unique space that is compact, Hausdorff, without isolated points, and with a countable base (provided by $\{[[\sigma]] : \sigma \in 2^*\}$, where $2 = \{0,1\}$) of clopen (i.e. closed and open) sets. For properties of the Baire space and the Cantor space see for instance [6].

Let φ_n^f be the partial function computed with oracle f by the oracle Turing machine with index n (see [17] for an introduction to oracle Turing machines). In the following, by a *Turing functional* on the Baire space we mean a partial mapping $\Psi : \omega^\omega \longrightarrow \omega^\omega$ for which there exists some n such that $\text{domain}(\Psi) = \{f : \varphi_n^f \text{ total}\}$, and for every $f \in \text{domain}(\Psi)$ and $x \in \omega$, $\Psi(f)(x) = \varphi_n^f(x)$. This also gives us an effective listing $\{\Psi_n\}_{n\in\omega}$ of the Turing functionals. Notice that a Turing functional can also be viewed as a total mapping from ω^ω to the set of partial functions from ω to ω: if $\Psi = \Psi_n$, then $f \notin \text{domain}(\Psi)$ means that there is some x such that $\varphi_n^f(x) \uparrow$. In this sense we sometimes use the expression"$\Psi(f)$ is partial" to mean that $f \notin \text{domain}(\psi)$. If $\Psi = \Psi_n$ is a Turing functional and σ is a string, then $\Psi(\sigma)$ denotes the partial function $\Psi(\sigma)(x) = \varphi_n^\sigma(x)$: for details see [17, Definition 1.7]. When necessary we can also regard Ψ as a mapping from strings to strings by identifying $\Psi(\sigma)$ with the longest string $\rho \subseteq \Psi(\sigma)$ which is an initial segment of the partial function $\Psi(\sigma)$. Turing functionals on the Cantor space, and related definitions, are defined similarly.

A *mass problem* is any subset $\mathcal{A} \subseteq \omega^\omega$. On mass problems one can define ([8]; see also [11], section XIII.7) the following reducibility relation $\leq$,

$$\mathcal{A} \leq \mathcal{B} \Leftrightarrow (\exists n)[\mathcal{B} \subseteq \text{domain}(\Psi_n) \text{ and } \Psi_n(\mathcal{B}) \subseteq \mathcal{A}],$$

i.e. $\mathcal{A} \leq \mathcal{B}$ if there is a uniform Turing reduction procedure by means of which any function in $\mathcal{B}$ computes some function of $\mathcal{A}$. Let $\equiv$ denote the equivalence relation generated by $\leq$. The $\equiv$-equivalence class of a mass problem $\mathcal{A}$, denoted by $\deg_{\mathrm{M}}(\mathcal{A})$, is called the *Medvedev degree* (or simply, the M-*degree*) of $\mathcal{A}$, or the *degree of difficulty* of $\mathcal{A}$. We use boldface capital letters $\mathbf{A}, \mathbf{B}, \ldots$ to denote Medvedev degrees.

The collection of Medvedev degrees is not only a partial order, but in fact:

Theorem 1.1 ([8]) *The Medvedev degrees form a distributive lattice with* $0, 1$*, denoted by* $\mathfrak{M}$*.*

Proof. We sketch the proof for later reference. We define the following operations $\vee$ and $\wedge$ on mass problems $\mathcal{A}, \mathcal{B}$:

1. $\mathcal{A} \vee \mathcal{B} = \{f \oplus g : f \in \mathcal{A} \text{ and } g \in \mathcal{B}\}$, where, given partial functions φ and ψ, we let
$$\varphi \oplus \psi(x) = \begin{cases} \varphi(y), & \text{if } x = 2y, \\ \psi(y), & \text{if } x = 2y+1; \end{cases}$$

2. $\mathcal{A} \wedge \mathcal{B} = \langle 0 \rangle \hat{} \mathcal{A} \cup \langle 1 \rangle \hat{} \mathcal{B}$, where for a given string σ and a given mass problem $\mathcal{A}$,
$$\sigma \hat{} \mathcal{A} = \{\sigma \hat{} f : f \in \mathcal{A}\}.$$

It is now easy to see (for an easy reference, see for instance [11, Section XIII.7]) that these operations on mass problems can be used to define the operations of supremum $\vee$, and infimum $\wedge$, on Medvedev degrees as

$$\deg_{\mathrm{M}}(\mathcal{A}) \vee \deg_{\mathrm{M}}(\mathcal{B}) = \deg_{\mathrm{M}}(\mathcal{A} \vee \mathcal{B}),$$
$$\deg_{\mathrm{M}}(\mathcal{A}) \wedge \deg_{\mathrm{M}}(\mathcal{B}) = \deg_{\mathrm{M}}(\mathcal{A} \wedge \mathcal{B}).$$

These operations distribute over each other: for instance, one can easily check that, for all mass problems $\mathcal{A}, \mathcal{B}, \mathcal{C}$,

$$\mathcal{A} \vee (\mathcal{B} \wedge \mathcal{C}) \equiv (\mathcal{A} \vee \mathcal{B}) \wedge (\mathcal{A} \vee \mathcal{C}).$$

Finally, the least degree $\mathbf{0}$ is defined as $\mathbf{0} = \deg_{\mathrm{M}}(\mathcal{A})$, where $\mathcal{A}$ contains some computable function; and the greatest degree $\mathbf{1}$ is defined as $\mathbf{1} = \deg_{\mathrm{M}}(\emptyset)$. $\square$

It is known that $\mathfrak{M}$ has cardinality $2^{2^{\aleph_0}}$ (this result was obtained independently by Ershov [5] and Platek [10]), although every Medvedev degree different from the greatest one has large cardinality, as shown by the following observation, first noted By Terwijn [23, Proposition 4.1]:

Lemma 1.2 ([23]) *Every* M*-degree different from* $\mathbf{1}$ *contains* $2^{2^{\aleph_0}}$ *distinct mass problems.*

Proof. Let $\mathcal{A}$ be any nonempty mass problem. For every mass problem $\mathcal{B}$ we have by absorption that $\mathcal{A} \equiv (\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{A}$; since different $\mathcal{B}$'s produce different mass problems of the form $(\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{A}$, the claim is proved. $\square$

We recall that a Brouwer algebra is a distributive lattice $\mathfrak{L}$ with $0, 1$, equipped with a binary operation $\rightarrow$, satisfying for all $a, b \in \mathfrak{L}$,

$$a \rightarrow b = \min \{c : b \leq a \vee c\}.$$

Lemma 1.3 ([8]) $\mathfrak{M}$ *is a Brouwer algebra.*

Proof. If $\mathcal{A}, \mathcal{B}$ are mass problems, then define a new mass problem

$$\mathcal{A} \rightarrow \mathcal{B} = \{\langle z \rangle \hat{\ } g : (\forall f \in \mathcal{A})[\Psi_z(f \oplus g) \in \mathcal{B}]\}.$$

It is easy to see that $\mathcal{B} \leq (\mathcal{A} \rightarrow \mathcal{B}) \vee \mathcal{A}$, and $\mathcal{B} \leq \mathcal{C} \vee \mathcal{A}$ if and only if $(\mathcal{A} \rightarrow \mathcal{B}) \leq \mathcal{C}$. Thus we can extend to degrees by defining

$$\deg_{\mathrm{M}}(\mathcal{A}) \rightarrow \deg_{\mathrm{M}}(\mathcal{B}) = \deg_{\mathrm{M}}(\mathcal{A} \rightarrow \mathcal{B}),$$

thus showing that $\mathfrak{M}$ is a Brouwer algebra. $\square$

Definition 1.4 An M-degree in $\mathfrak{M}$ is called *closed* (respectively: *dense*, *discrete*) if it contains a mass problem that is closed (respectively: dense, discrete) in the Baire space.

The following two lemmas record known facts. (By a sublattice, we mean also that the least element and the greatest element are preserved.)

Lemma 1.5 ([2]) *The closed Medvedev degrees form a sublattice,* $\mathfrak{M}_{\mathrm{Cl}}$.

Dyment, [4], shows that there is no greatest dense Medvedev degree. However:

Lemma 1.6 ([4]) *The dense Medvedev degrees, with the addition of* $\mathbf{1}$*, form a sublattice,* $\mathfrak{M}_{\mathrm{De}}$*. The discrete Medvedev degrees form a sublattice,* $\mathfrak{M}_{\mathrm{Di}}$.

If we restrict our attention to mass problems that are subsets of the Cantor space, and use for reduction on these mass problems Turing functionals on the Cantor space, we get in a similar way a Brouwer algebra, denoted by $\mathfrak{M}^{0,1}$, called the *Medvedev lattice on the Cantor space.*

The Medvedev degree in $\mathfrak{M}^{01}$ of a mass problem $\mathcal{A} \subseteq 2^{\omega}$ will be denoted by $\deg_{\mathrm{M}^{01}}(\mathcal{A})$. Similarly to the Medvedev lattice on the Baire space, an M-degree in $\mathfrak{M}^{01}$ is called *closed* (respectively: *dense*, *discrete*) if it contains a mass problem that is closed (respectively: dense, discrete) in the Cantor space. Thus, in analogy with the Baire space, one can consider the sublattices formed by the closed (dense, discrete) degrees, denoted by $\mathfrak{M}^{0,1}_{\mathrm{Cl}}$ ($\mathfrak{M}^{0,1}_{\mathrm{De}}$, $\mathfrak{M}^{0,1}_{\mathrm{Di}}$): in order to get a lattice with a greatest element, for $\mathfrak{M}^{0,1}_{\mathrm{De}}$, in analogy with the Baire space case, one has to add the top element $\mathbf{1}$.

For every $f \in \omega^{\omega}$ define (viewing a function as an infinite sequence)

$$f^{01} = \langle 1^{f(0)}01^{f(1)}01^{f(2)}\cdots\rangle$$

where 1^x denotes $\underbrace{1\cdots 1}_{x \text{ times}}$. Notice that the assignment $f \mapsto f^{01}$ is a Turing functional and provides a homeomorphism between ω^{ω} and the subset $\mathcal{G}$ of the Cantor space,

$$\mathcal{G} = \{g \in 2^{\omega} : (\exists^{\infty} x)[g(x) = 0]\}\,.$$

Given $\mathcal{A} \subseteq \omega^{\omega}$, define the mass problem $\mathcal{A}^{01} \subseteq 2^{\omega}$, as $\mathcal{A}^{01} = \{f^{01} : f \in \mathcal{A}\}$; and given $\mathcal{B} \subseteq \mathcal{G}$, define $\mathcal{B}^{\omega} = \{f \in \omega^{\omega} : f^{01} \in \mathcal{B}\}$.

Theorem 1.7 *The embedding* $\iota : \mathfrak{M}^{01} \longrightarrow \mathfrak{M}$ *given by* $\iota(\deg_{\mathrm{M}^{01}}(\mathcal{B})) = \deg_{\mathrm{M}}(\mathcal{B})$ *is an isomorphism. The isomorphism maps closed degrees to closed degrees; the isomorphism, and its inverse, map dense (discrete) degrees to dense (discrete) degrees. In particular* $\mathfrak{M}^{0,1}_{\mathrm{De}} \simeq \mathfrak{M}_{\mathrm{De}}$ *and* $\mathfrak{M}^{0,1}_{\mathrm{Di}} \simeq \mathfrak{M}_{\mathrm{Di}}$*, where* $\simeq$ *denotes lattice-theoretic isomorphism.*

Proof. Notice that if $\mathcal{A} \subseteq \omega^\omega$ then $\mathcal{A} \equiv \mathcal{A}^{01}$, and $\mathcal{A}$ is dense (discrete) if and only if so is $\mathcal{A}^{01}$. Moreover, if $\mathcal{B} \subseteq \mathcal{G}$, then $\mathcal{B} \equiv \mathcal{B}^\omega$ and $\mathcal{B}$ is dense (discrete) if and only if so is $\mathcal{B}^\omega$. Notice also that every closed $\mathcal{B}$ in the Cantor space is also closed in the Baire space. The rest of the proof is a routine check. $\square$

It is worth noticing that if $\mathcal{A}$ is closed in the Baire space, then $\mathcal{A}^{01}$ need not be closed in the Cantor space: for instance, choose a noncomputable f and take

$$\mathcal{A} = \{\langle n \rangle \hat{\ } f : n \in \omega\}.$$

Then the constant function $g(x) = 1$ is in the closure of $\mathcal{A}^{01}$ (since for every n, $\langle 1^n \rangle \hat{\ } \langle 0 \rangle \hat{\ } f^{01} \in \mathcal{A}^{01}$), but not in $\mathcal{A}^{01}$.

Remark 1.8 In the rest of the paper we do not specify whether we work in the Baire space or in the Cantor space, in those cases in which proofs for both $\mathfrak{M}$ and $\mathfrak{M}^{01}$ are, *mutatis mutandis*, virtually the same, or the Cantor case immediately transfers to the Baire space and vice versa, by the isomorphism provided by Theorem 1.7. Of course, mass problems have to be meant as mass problems in the Baire space or in the Cantor space, and Turing functionals must be regarded as Turing functionals on the Baire space or the Cantor space, according to the case. When distinctions between the Cantor space and the Baire space are necessary (for instance when dealing with closed degrees, or with perfect trees) we will treat each case separately.

Finally we observe that the Turing degrees can be viewed as a substructure of the Medvedev lattice. Since $f \leq_{\mathrm{T}} g$ if and only if $\{f\} \leq \{g\}$, it is immediate to observe ([8]) that the Turing degrees embed into the Medvedev lattice, via an embedding preserving least element 0 and join operation $\vee$. Medvedev degrees of singletons are usually called *degrees of solvability.*

2 The closed Medvedev degrees

Since $\mathfrak{M}_{\mathrm{Cl}}$ is a sublattice of $\mathfrak{M}$, it is natural to ask if it is also a Brouwer subalgebra, or at least a Brouwer algebra. We show in fact that neither $\mathfrak{M}_{\mathrm{Cl}}$ nor $\mathfrak{M}^{0,1}_{\mathrm{Cl}}$ is a Brouwer algebra. The proof is based on the following lemma.

Lemma 2.1 *If A, B are sets such that $B \not\leq_{\mathrm{T}} A$, A is 1-generic, and $\mathcal{C}$ is a closed mass problem such that for every $k \in \mathcal{C}$, $k \not\leq_{\mathrm{T}} A$, then there exists a set D such that $k \not\leq_{\mathrm{T}} D$ for all $k \in \mathcal{C}$, and $B \leq_{\mathrm{T}} A \oplus D$.*

Proof. Let A, B and $\mathcal{C}$ be as in the statement of the lemma. In this proof we often identify sets of numbers with their characteristic functions. We recall that a set X is 1-generic if for every c.e. set W of strings in 2^*, there is a $\sigma \subset X$ such that either $\sigma \in W$, or for every $\tau \supseteq \sigma$, $\tau \notin W$.

We will construct D so that $k \not\leq_{\mathrm{T}} D$ for all $k \in \mathcal{C}$, and

$$B \leq_{\mathrm{T}} A \oplus D.$$

We construct D by a finite extension argument. In order that $A \oplus D$ should compute B we use the following coding: the nth bit of B will be $A(m)$ where m is the nth bit on which A and D differ. At stage s we define a finite initial segment τ_s of D.

Stage 0. Let $\tau_0 = \lambda$.

Stage $s+1$. Suppose we have defined τ_s. We want to ensure that $\Psi_s(D)$ is partial or lies outside $\mathcal{C}$, where Ψ_s is the sth Turing functional. Suppose τ_s is of length n and let E be A with the first n bits replaced by τ_s, i.e.

$$E(i) = \begin{cases} \tau_s(i), & \text{if } i < n; \\ A(i), & \text{if } i \geq n. \end{cases}$$

We distinguish the following two cases:

1. $\Psi_s(E)$ is partial. Then there must be some finite initial segment of E which is sufficient to force partiality. In order to see this suppose otherwise, and let m be the least number such that $\Psi_s(E)(m) \uparrow$. Then the c.e. set of strings

   $$V = \{\sigma : \Psi_s(\sigma)(m) \downarrow\}$$

   can be transformed into a c.e. set V' to give a witness that A is not 1-generic – just let Ψ_s be replaced by any Turing functional Φ such that for every function k, $\Phi((A \restriction n)\hat{\ }k) = \Psi_s(\tau_s\hat{\ }k)$, where $A \restriction n$ is the initial segment of A of length n. In this case, then, we can take an initial segment of E sufficient to force partiality. This still codes B correctly, because this initial segment agrees with A on its new bits. Then extend it to a suitable τ_{s+1} so as to code the next bit $B(s)$ of B.

2. $\Psi_s(E)$ is total. In this case $\Psi_s(E) \notin \mathcal{C}$ because $\Psi_s(E) \leq_{\mathrm{T}} A$ and nothing in $\mathcal{C}$ is computable in A. Because $\mathcal{C}$ is closed we can take τ such that $\tau_s \subseteq \tau \subset E$, and nothing extending $\Psi_s(\tau)$ is in $\mathcal{C}$. Finally, choose τ_{s+1} to be some finite extension of τ which codes the next bit $B(s)$ of B.

□

Theorem 2.2 *$\mathfrak{M}_{\mathrm{Cl}}$ and $\mathfrak{M}_{\mathrm{Cl}}^{0,1}$ are not Brouwer algebras.*

Proof. For a given set X, let c_X denote its characteristic function. By the previous lemma, we have that if $B \not\leq_{\mathrm{T}} A$ and A is 1-generic, then for every closed mass problem $\mathcal{C}$ such that $\{c_B\} \leq \{c_A\} \vee \mathcal{C}$ (which implies $k \not\leq_{\mathrm{T}} A$ for every $k \in \mathcal{C}$), one can construct D such that $\mathcal{C} \not\leq \{c_D\}$ but $\{c_B\} \leq \{c_A\} \vee \{c_D\}$, thus showing that there is no least closed $\mathcal{C}$ with the property $\{c_B\} \leq \{c_A\} \vee \mathcal{C}$. As it is irrelevant whether or not $\mathcal{C}$ is a subset of the Cantor space or of the Baire space, this proofs works for both $\mathfrak{M}_{\mathrm{Cl}}$ and $\mathfrak{M}^{0,1}_{\mathrm{Cl}}$. □

Remark 2.3 Recall that *Muchnik reducibility* $\leq_{\mathrm{w}}$ on mass problems (see [9]) is the nonuniform version of Medvedev reducibility: $\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$ if

$$(\forall f \in \mathcal{B})(\exists g \in \mathcal{A})[g \leq_{\mathrm{T}} f].$$

The equivalence classes of mass problems under the equivalence relation $\equiv_{\mathrm{w}}$ generated by $\leq_{\mathrm{w}}$ are called *Muchnik degrees*. It is well known, see [9], that Muchnik reducibility gives rise to a Brouwer algebra on Muchnik degrees, called the *Muchnik lattice*. Notice that for every mass problem $\mathcal{A}$ and function h, $\mathcal{A} \leq \{h\}$ if and only if $\mathcal{A} \leq_{\mathrm{w}} \{h\}$, and if A, B are as in the proof of Theorem 2.2, then $\{c_B\} \leq_{\mathrm{w}} \{c_A\} \vee \mathcal{C}$ still implies that $k \not\leq_{\mathrm{T}} A$ for every $k \in \mathcal{C}$. Therefore the proof of Theorem 2.2 shows that the result stated therein holds of Muchnik reducibility too. Thus the lattice of closed Muchnik degrees is not a Brouwer algebra. See also Simpson's result, [15], stating that the lattice of Muchnik degrees of (lightface) Π^0_1 classes in the Cantor space does not form a Brouwer algebra: for Medvedev reducibility, this is still an open question, originally posed by Terwijn [22, Conjecture 4.3].

3 The dense Medvedev degrees

We now turn our attention to the dense Medvedev degrees, i.e. the Medvedev degrees containing some dense mass problem. It might be worth noticing that the property of M-degrees of not being dense does not coincide at all with the property of being nowhere dense. Indeed, every M-degree is nowhere dense, in the sense that it contains some nowhere dense mass problem: to see this, for every mass problem $\mathcal{A}$ we have that $\mathcal{A} \equiv \mathcal{A} \vee \{f\}$ when f is computable, and $\mathcal{A} \vee \{f\}$ is nowhere dense.

Given a poset $\mathfrak{P} = \langle P, \leq \rangle$, we recall that a subset $B \subseteq P$ is an *automorphism base* for $\mathfrak{P}$, if every automorphism F of $\mathfrak{P}$ which is the identity on B, is also the identity on P. We begin our investigation on the dense degrees with the following result.

Theorem 3.1 *The dense Medvedev degrees form an automorphism base of* $\mathfrak{M}$.

Proof. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are mass problems and $\mathbf{A} \not\leq \mathbf{B}$. We show that there exists a dense $\mathbf{D}$ such that $\mathbf{D} \leq \mathbf{A}$, but $\mathbf{D} \not\leq \mathbf{B}$. (This is enough to show that the dense degrees form an automorphism base of the Medvedev lattice. Indeed, suppose that F is an automorphism which is the identity on the dense degrees, and $F(\mathbf{A}) = \mathbf{B}$ where $\mathbf{A} \not\leq \mathbf{B}$: if $\mathbf{D}$ is dense, $\mathbf{D} \leq \mathbf{A}$, and $\mathbf{D} \not\leq \mathbf{B}$, then $\mathbf{D} = F(\mathbf{D}) \leq F(\mathbf{A}) = \mathbf{B}$, a contradiction.) So, let $\mathcal{A} \in \mathbf{A}$, $\mathcal{B} \in \mathbf{B}$, and construct a dense $\mathcal{D}$ as follows. For every Turing functional Ψ, since $\mathcal{A} \not\leq \mathcal{B}$, choose a function $f = f_\Psi \in \mathcal{B}$ such that $\Psi(f) \uparrow$ or $\Psi(f) \notin \mathcal{A}$, and let

$$\mathcal{D} = \{g : (\forall \Psi)[g \neq \Psi(f_\Psi)]\}.$$

Then $\mathcal{D} \not\leq \mathcal{B}$; moreover $\mathcal{D} \leq \mathcal{A}$ since $\mathcal{A} \subseteq \mathcal{D}$; finally $\mathcal{D}$ is dense since its complement is countable. □

Next, we introduce a class of dense degrees that are particularly useful in examples and applications:

Definition 3.2 For every f, let

$$\mathcal{B}_f = \{g : g \not\leq_{\mathrm{T}} f\},$$

and let $\mathbf{B}_f = \deg_{\mathrm{M}}(\mathcal{B}_f)$.

Mass problems of the form $\mathcal{B}_f$, for some f, and their respective M-degrees have been extensively used in the literature concerning the Medvedev lattice, see e.g. [19]. For instance, it is known, see [19, Lemma 2.2] that each $\mathbf{B}_f$ is join-irreducible and meet-irreducible: in fact, join-irreducibility is a straightforward consequence of the fact that for every noncomputable f,

$$\mathbf{B}_f \wedge \deg_{\mathrm{M}}(\{f\}) = \max \{\mathbf{A} : \mathbf{A} < \mathbf{B}_f\},$$

as follows from the observation that if $\mathcal{B}_f \not\leq \mathcal{A}$ then $\mathcal{A} \not\subseteq \mathcal{B}_f$, and thus there exists some $g \in \mathcal{A}$ with $g \leq_{\mathrm{T}} f$, giving $\mathcal{A} \leq \{f\}$. Moreover:

Theorem 3.3 *The non-zero* M*-degrees of the form* $\mathbf{B}_f$*, with* f *noncomputable, are exactly the* M*-degrees* $\mathbf{X}$ *satisfying* $\mathfrak{M} \models \varphi(\mathbf{X})$*, where* $\varphi(v)$ *is the first order formula in the language of posets,*

$$\varphi(v) := (\exists u < v)(\forall w < v)(w \leq u).$$

As a consequence the M*-degrees of the form* $\mathbf{B}_f$*, for some function* f*, are first-order definable in* $\mathfrak{M}$*.*

They are also characterized by:

$$\mathbf{B}_f = \min \{\mathbf{A} : \mathbf{A} \not\leq \deg_{\mathrm{M}}(\{f\})\}.$$

Since the M*-degrees of the form* $\deg_{\mathrm{M}}(\{f\})$ *(called the degrees of solvability) are first-order definable ([4]), this provides another first-order definition of the* $\mathbf{B}_f$*'s.*

Proof. By the remark preceding the Theorem, it suffices to show that if $\mathbf{B}$ is an M-degree satisfying $\varphi(v)$, then $\mathbf{B} = \mathbf{B}_f$, for some f. Let $\mathbf{A}$ be the greatest M-degree strictly below $\mathbf{B}$. By Dyment's characterization of empty intervals in [4], there exists an f such that $\mathbf{A} = \mathbf{B} \wedge \deg_{\mathrm{M}}(\{f\})$, and $\mathbf{B} \not\leq \deg_{\mathrm{M}}(\{f\})$. Let $\mathcal{B} \in \mathbf{B}$. It follows that $\mathcal{B} \not\leq \{f\}$ hence $\mathcal{B} \subseteq \mathcal{B}_f$, and thus $\mathcal{B}_f \leq \mathcal{B}$. On the other hand if $\mathcal{B}_f < \mathcal{B}$, then $\mathcal{B}_f \leq \mathcal{B} \wedge \{f\}$ (since $\mathbf{A}$ is the greatest degree strictly below $\mathcal{B}_f$), giving $\mathcal{B}_f \leq \{f\}$, a contradiction. Hence $\mathcal{B} \equiv \mathcal{B}_f$. □

Question 3.4 *Is $\mathfrak{M}_{\mathrm{De}}$ a Brouwer subalgebra of $\mathfrak{M}$, or at least a Brouwer algebra?*

4 Ideals and filters

We recall some basic notions and definitions of lattice theory. For more details see for instance [1].

Definition 4.1 In a lattice $\mathfrak{L} = \langle L, \vee, \wedge, \leq \rangle$ a nonempty subset $I \subseteq L$ is said to be an *ideal* if for all $x, y \in L$,

$$y \in I \text{ and } x \leq y \Rightarrow x \in I,$$
$$x, y \in I \Rightarrow x \vee y \in I.$$

Let I be an ideal of $\mathfrak{L}$. I is *proper* if $I \neq L$. A proper ideal is *prime* if for all $x, y \in L$,

$$x \wedge y \in I \Rightarrow x \in I \text{ or } y \in I.$$

If $\mathfrak{L}$ is a lattice then the ideal I generated by a nonempty subset $X \subseteq L$ (i.e. the smallest ideal under inclusion, containing X) is given by

$$I = \left\{ a : (\exists F \subseteq X)[F \text{ finite, } F \neq \emptyset \text{ and } a \leq \bigvee F] \right\}.$$

The dual notion of an ideal is that of a *filter*. It is well known (see e.g. [1]) that I is a prime ideal if and only if the complement I^c is a prime filter.

Definition 4.2 ([4]) Let $\mathfrak{I}$ be the ideal generated by the dense degrees.

The following is trivial by Lemma 1.6:

$$\mathfrak{I} = \{\mathbf{A} : (\exists \mathbf{D} \in \mathfrak{M}_{\mathrm{De}} - \{\mathbf{1}\}) \, [\mathbf{A} \leq \mathbf{D}]\}.$$

It is shown in [18] that $\mathfrak{I}$ is not principal. We will show in Theorem 5.5 that the dense degrees themselves do not form an ideal. Moreover:

Theorem 4.3 ([18]) $\mathfrak{I}$ *is prime.*

Proof. We give the proof for later reference. Suppose that $\mathcal{A} \wedge \mathcal{B} \leq \mathcal{D}$ where $\mathcal{D}$ is dense, and let Ψ be a Turing functional providing the reduction. Suppose that there exists an initial segment α such that $\Psi(\alpha)(0) = 0$. Then consider

$$\mathcal{D}^- = \{f : \alpha\hat{\ }f \in \mathcal{D}\}\,.$$

Clearly $\mathcal{D}^-$ is dense, and the Turing functional $\Phi(f) = \Psi(\alpha\hat{\ }f)$ reduces $\mathcal{D}^-$ to $\mathcal{A}$, showing that $\deg_{\mathrm{M}}(\mathcal{A}) \in \mathfrak{I}$. A similar argument shows that $\deg_{\mathrm{M}}(\mathcal{B}) \in \mathfrak{I}$ if we assume that there exists an initial segment α such that $\Psi(\alpha)(0) = 1$. Since one of these cases must hold, we have proved that either $\deg_{\mathrm{M}}(\mathcal{A}) \in \mathfrak{I}$ or $\deg_{\mathrm{M}}(\mathcal{B}) \in \mathfrak{I}$. □

Corollary 4.4 ([2]) *The complement of $\mathfrak{I}$ is a prime filter, denoted by $\mathfrak{F}$.*

It is shown in [2] that $\mathfrak{F}$ is not principal.

Definition 4.5 ([2]) Let $\mathfrak{F}_{\mathrm{Cl}}$ be the filter generated by the nonzero closed Medvedev degrees.

The following is trivial by Lemma 1.5:

$$\mathfrak{F}_{\mathrm{Cl}} = \{\mathbf{A} : (\exists \mathbf{C} \in \mathfrak{M}_{\mathrm{Cl}} - \{\mathbf{0}\})\,[\mathbf{A} \geq \mathbf{C}]\}\,.$$

It is shown in [18] that $\mathfrak{F}_{\mathrm{Cl}}$ is not principal. Recently Shafer has proved [14] that $\mathfrak{F}_{\mathrm{Cl}}$ is not prime. We will see in Theorem 6.5 that the nonzero closed degrees themselves do not form a filter.

Lemma 4.6 *If $\mathcal{A}$ and $\mathcal{B}$ are mass problems such that $\mathcal{B}$ is closed, and $\mathcal{B} \not\leq \mathcal{A}$ then, for every $\mathcal{C}$,*

$$\mathcal{B} \leq \mathcal{A} \vee \mathcal{C} \Rightarrow \mathcal{C} \textit{ nowhere dense.}$$

Proof. Let $\mathcal{B} \not\leq \mathcal{A}$, with $\mathcal{B}$ closed. The claim follows from the proof of Theorem 2.8 in [20], which shows that if $\mathcal{B}$ is closed and $\mathcal{D}$ is dense then $\mathcal{D} \to \mathcal{B} \equiv \mathcal{B}$. Therefore it suffices to observe that if $\mathcal{B} \leq \mathcal{A} \vee \mathcal{C}$ and $\mathcal{C}$ is dense in some interval then there exists a dense $\mathcal{D}$ such that $\mathcal{B} \leq \mathcal{A} \vee \mathcal{D}$ (if $\mathcal{C}$ is dense in, say, $[[\alpha]]$, then $\mathcal{D} = \{f : \alpha\hat{\ }f \in \mathcal{C}\}$ is dense and $\mathcal{C} \leq \mathcal{D}$), hence $\mathcal{B} \equiv \mathcal{D} \to \mathcal{B} \leq \mathcal{A}$, giving $\mathcal{B} \leq \mathcal{A}$, a contradiction.

For the ease of the reader we include, however, a direct proof of the claim. Suppose that $\mathcal{B} \leq \mathcal{A} \vee \mathcal{C}$ via a Turing functional Ψ, where $\mathcal{B}$ is closed and $\mathcal{C}$ is dense in the basic neighborhood $[[\alpha]]$. Consider the following Turing functional Φ: On input f, compute $\Phi(f)(x)$ by recursion as follows: suppose that $\Phi(f)(x-1)\downarrow$ (if $x > 0$) and we have already computed relatively to f a finite string $\alpha_{x-1} \supseteq \alpha$ (with $\alpha_{-1} = \alpha$). Then look for the first pair β, α_x of finite strings (of the same length) such that $\beta \subset f$, $\alpha_{x-1} \subseteq \alpha_x$ and $\Psi(\beta \oplus \alpha_x)(x)\downarrow$. Define $\Phi(f)(x) = \Psi(\beta \oplus \alpha_x)(x)$; otherwise $\Phi(f)(x)$ is undefined. Let now $f \in \mathcal{A}$. Notice that by density of $\mathcal{C}$ in

$[[\alpha]]$, for every x there is a function $g \in \mathcal{C}$ extending α_{x-1} such that $\Psi(f \oplus g)(x) \downarrow$. Thus for every x we can find β, α_x as above, hence $\Phi(f)$ is total. On the other hand suppose that for every x we pick $g_x \supset \alpha_x$ with $g_x \in \mathcal{C}$, and let $h_x = \Psi(f \oplus g_x)$: then each h_x is a point in the Baire space, and in fact $h_x \in \mathcal{B}$, $\lim_x h_x = \Phi(f)$ (where the limit $\lim_x h_x$ is taken with respect to the Baire topology), and thus $\Phi(f) \in \mathcal{B}$ since $\mathcal{B}$ is closed. This shows that $\mathcal{B} \leq \mathcal{A}$ via Φ, a contradiction. □

Corollary 4.7 ([2]) $\mathfrak{F}_{\mathrm{Cl}} \subseteq \mathfrak{F}$.

Proof. The inclusion $\mathfrak{F}_{\mathrm{Cl}} \subseteq \mathfrak{F}$ follows by the previous lemma since we cannot have $\mathbf{C} \leq \mathbf{D}$ if $\mathbf{D}$ is dense and $\mathbf{C} \neq \mathbf{0}$ is closed. □

That strict inclusion holds in Corollary 4.7 has been shown in [2]. The proof given there uses another interesting filter of the Medvedev lattice, namely the filter generated by the nonzero discrete M-degrees. This filter will be denoted in this paper by $\mathfrak{F}_{\mathrm{Di}}$. By Lemma 1.6, we have

$$\mathfrak{F}_{\mathrm{Di}} = \{\mathbf{A} : (\exists \mathbf{C} \in \mathfrak{M}_{\mathrm{Di}} - \{\mathbf{0}\})\,[\mathbf{A} \geq \mathbf{C}]\}\,.$$

Dyment [4] shows that if $\mathcal{A}$ does not contain computable functions and is discrete, then, for every $\mathcal{B}$,

$$\mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{B} \text{ nowhere dense.}$$

Hence, $\mathfrak{F}_{\mathrm{Di}} \subseteq \mathfrak{F}$.

On the other hand, Bianchini and Sorbi [2] show:

Theorem 4.8 ([2]) $\mathfrak{F}_{\mathrm{Di}} \not\subseteq \mathfrak{F}_{\mathrm{Cl}}$. *Hence* $\mathfrak{F} \not\subseteq \mathfrak{F}_{\mathrm{Cl}}$.

We will show in Corollary 8.4 that $\mathfrak{F}_{\mathrm{Cl}} \not\subseteq \mathfrak{F}_{\mathrm{Di}}$. It is shown in [18] that $\mathfrak{F}_{\mathrm{Di}}$ is not prime and the corresponding quotient lattice has cardinality $2^{2^{\aleph_0}}$. Dyment [4] shows that $\mathfrak{F}_{\mathrm{Di}}$ is not principal. We will show in Theorem 8.6 that the nonzero discrete degrees do not form a filter.

Remark 4.9 The ideals and filters introduced in this section have, of course, corresponding versions in $\mathfrak{M}^{01}$. By virtually the same proofs, all the results and properties shown in this section remain unaltered if one replaces $\mathfrak{M}$ with $\mathfrak{M}^{01}$.

5 The ideal $\mathfrak{I}$ generated by the dense degrees

We define a *perfect tree in the Baire space* to be a function $T : 2^* \longrightarrow \omega^*$ such that for every $\sigma, \tau \in 2^*$,

1. $\sigma \subseteq \tau \Rightarrow T(\sigma) \subseteq T(\tau)$;

2. $T(\sigma\hat{}\langle 0\rangle)|T(\sigma\hat{}\langle 1\rangle)$.

A perfect tree is *computable* if it is computable as a function. If T is a perfect tree, then we denote by $[T]$ the set of infinite paths through T. The set $[T]$ is a *perfect* (i.e. closed and without isolated points) subset of the Baire space. We also say that a string τ *is on* T, if $\tau \subset f$ for some $f \in [T]$.

By abusing notation, we can also regard a tree T in the Baire space as a total mapping $T : 2^\omega \longrightarrow \omega^\omega$, by letting

$$T(f) = \bigcup_{\sigma \subset f} T(\sigma).$$

Then T is a homeomorphism of 2^ω onto $[T]$. Moreover if T is computable then the associated homeomorphism of 2^ω with $[T]$, and its inverse T^{-1}, are Turing functionals. Notice that here $T(f) \equiv_{\mathrm{T}} f$.

Definition 5.1 A mass problem $\mathcal{A}$ is *dense in a perfect tree* T if it is dense in $[T]$, i.e. $[T]$ is contained in $\overline{\mathcal{A}}$, the closure of $\mathcal{A}$ in the Baire space.

Notice that in the above definition we do not require that $\mathcal{A} \subseteq [T]$. If T is computable, this additional property implies that the M-degree of $\mathcal{A}$ is dense: see Theorem 5.4.

If we require that the tree be a function $T : 2^* \longrightarrow 2^*$, then we get the definition of a *perfect tree in the Cantor space*. For perfect trees in the Cantor space, one can make observations similar to those made for trees in the Baire space. In particular, the set $[T]$ of infinite paths is a perfect subset of 2^ω.

The following lemma relates trees in the Cantor space to trees in the Baire space. Given a string $\sigma \in \omega^*$, we define $\sigma^{01} \in 2^*$: if $\sigma = \langle x_0 \cdots x_{n-1}\rangle$ then $\sigma^{01} = \langle 1^{x_0}01^{x_1}0 \cdots 1^{x_{n-1}}0\rangle$. Moreover, if $\sigma \in 2^*$ is a string whose final bit is 0, $\sigma = \langle 1^{x_0}01^{x_1}0 \cdots 1^{x_{n-1}}0\rangle$, then we define $\sigma^\omega \in \omega^*$, $\sigma^\omega = \langle x_0 \cdots x_{n-1}\rangle$. Finally, if T is a perfect tree in the Baire space, then define $T^{01} : 2^* \longrightarrow 2^*$, by $T^{01}(\sigma) = (T(\sigma))^{01}$, and if T is a perfect tree in the Cantor space such that for every nonempty σ, $T(\sigma)$ ends with the bit 0, then define $T^\omega : 2^* \longrightarrow \omega^*$, by $T^\omega(\sigma) = (T(\sigma))^\omega$.

Lemma 5.2 *Under the assumptions specified above, T^{01} is a perfect tree in the Cantor space, and T^ω is a perfect tree in the Baire space.*

Proof. Obvious. □

The given proof of the following theorem works, modulo obvious modifications, both for $\mathfrak{M}$ and $\mathfrak{M}^{01}$. In addition to the usual caveats, here one has also to distinguish between trees in the Cantor space, or trees in the Baire space, depending on whether one is looking for a proof suitable to $\mathfrak{M}^{01}$ or to $\mathfrak{M}$.

Theorem 5.3 *For every mass problem $\mathcal{A}$ of non-zero* M*-degree, if* $\deg_{\mathrm{M}}(\mathcal{A}) \in \mathfrak{I}$ *then $\mathcal{A}$ is dense in some computable perfect tree.*

Proof. Suppose that $\mathcal{A} \leq \mathcal{D}$ via the Turing functional Φ, and $\mathcal{D}$ is dense; we are given that $\mathcal{A}$ does not contain computable functions. Define a computable perfect tree T as follows:

1. $T(\lambda) = \lambda$;

2. Suppose we have defined $T(\sigma)$; find the least quintuple $\tau_0, \tau_1, \rho_0, \rho_1, s$ where $T(\sigma) \subseteq \tau_i$, $\rho_i \subseteq \Phi(\tau_i)[s]$, and $\rho_0 | \rho_1$. (Here the appendix $[s]$ denotes that we perform at most $s-1$ steps in the computations). Then define $T(\sigma\hat{\ }\langle i\rangle) = \tau_i$. Such a quintuple exists, since otherwise, by density of $\mathcal{D}$ and by the fact that $\mathcal{D} \subseteq \mathrm{domain}(\Phi)$, we would have that for every $f \supset T(\sigma)$ and $f \in \mathcal{D}$,

$$\Phi(f) = \bigcup_{T(\sigma) \subseteq \tau} \Phi(\tau):$$

   a contradiction since Φ would map all functions in $\mathcal{D} \cap [[T(\sigma)]]$ to computable functions.

Moreover, we can think of $\Phi \circ T$ as a computable perfect tree too, by letting $\Phi \circ T(\lambda) = \lambda$, and $\Phi \circ T(\sigma\hat{\ }\langle i\rangle) = \rho_i$, where ρ_i and s are as in the definition of $T(\sigma\hat{\ }\langle i\rangle)$ above. We now conclude the proof of our claim, by showing that $\mathcal{A}$ is dense in $\Phi \circ T$. Let $g \in [\Phi \circ T]$, and assume that $\tau \subset g$. Let σ be the least string such that $\tau \subseteq \Phi \circ T(\sigma)$. By density, let $h \in \mathcal{D}$, $h \supset T(\sigma)$. Then $\Phi(h) \supset \tau$, and $\Phi(h) \in \mathcal{A}$. □

The tree T built in the above proof is a Φ-*splitting tree*, i.e. for every σ there exists some x such that $\Phi(T(\sigma\hat{\ }\langle 0\rangle))(x) \downarrow \neq \Phi(T(\sigma\hat{\ }\langle 1\rangle))(x) \downarrow$. A pair (σ_0, σ_1) of strings such that $\Phi(\sigma_0)(x) \downarrow \neq \Phi(\sigma_1)(x) \downarrow$ is called a Φ-*splitting on* x. (In the context of the Cantor space, see for instance [7] for details on Φ-splitting trees, and Φ-splittings.)

Theorem 5.4 *If T is a computable perfect tree, $\mathcal{A} \subseteq [T]$ and $\mathcal{A}$ is dense in T then* $\deg_{\mathrm{M}}(\mathcal{A})$ *is dense.*

Proof. Suppose that $\mathcal{A}$ is dense in some computable perfect tree T, with $\mathcal{A} \subseteq [T]$. Then $\mathcal{A} \leq T^{-1}(\mathcal{A})$ (viewing T and T^{-1} as Turing functionals), and $T^{-1}(\mathcal{A})$ is dense: if σ is any interval, then $\mathcal{A} \cap [[T(\sigma)]] \neq \emptyset$, hence $T^{-1}(\mathcal{A}) \cap [[\sigma]] \neq \emptyset$. □

Theorem 8.5 will show that there are M-degrees not in $\mathfrak{I}$, but containing mass problems that are dense in some computable perfect tree.

One should not believe that all degrees in $\mathfrak{I}$ are dense. In fact:

Theorem 5.5 *There exists an $\mathbf{A} \in \mathfrak{I}$ such that $\mathbf{A}$ is not dense. So the dense degrees are not an ideal.*

Proof. (*First proof.*) The result easily follows from Corollary 4.3 and Lemma 4.6. Indeed consider any mass problem $\mathcal{A}$ with $\deg_{\mathrm{M}}(\mathcal{A}) \in \mathfrak{I}$ and any mass problem $\mathcal{C}$ with $\deg_{\mathrm{M}}(\mathcal{C}) \in \mathfrak{F}$, such that $\mathcal{A} \not\leq \mathcal{C}$, hence $\mathcal{A} \not\leq \mathcal{A} \wedge \mathcal{C}$: for instance take $\mathcal{A} = \mathcal{B}_f = \{g : g \not\leq_{\mathrm{T}} f\}$, assume that f is not computable, and take $\mathcal{C} = \{f\}$. Then $\deg_{\mathrm{M}}(\mathcal{A} \wedge \mathcal{C}) \in \mathfrak{I}$, as $\mathcal{A} \wedge \mathcal{C} \leq \mathcal{A}$, but there is no dense $\mathcal{D}$ such that $\mathcal{A} \wedge \mathcal{C} \equiv \mathcal{D}$, since for such a $\mathcal{D}$ we would have $\mathcal{A} \not\leq \mathcal{D}$ because of $\mathcal{A} \not\leq \mathcal{A} \wedge \mathcal{C}$, hence $\mathcal{C} \leq \mathcal{D}^-$ for some dense $\mathcal{D}^-$ by the proof of Corollary 4.3, giving $\deg_{\mathrm{M}}(\mathcal{C}) \in \mathfrak{I}$.

(*Second Proof.*) For a different, but perhaps more interesting and useful proof, we present a direct construction of mass problems $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}$ is dense, $\mathcal{A} \leq \mathcal{B}$, but $\mathcal{A} \not\equiv \mathcal{D}$ for any dense $\mathcal{D}$. In fact we construct $\mathcal{A}$ to be dense in the interval $\langle 0 \rangle$, but not equivalent to any dense mass problem, so that the mass problems $\mathcal{A}$ and $\mathcal{B} = \{f : \langle 0 \rangle\hat{\ } f \in \mathcal{A}\}$ satisfy the claim. Notice that we do not specify below whether we work in the Baire space or in the Cantor space, as the proofs for both cases are virtually identical.

The construction of $\mathcal{A}$ is by steps. At step n we define a finite mass problem $\mathcal{A}_n$, and a closed mass problem $\mathcal{P}_n$ (with $[[\langle 1 \rangle]] - \mathcal{P}_n \neq \emptyset$) of *prohibited* functions for which we want to guarantee that $\mathcal{A} \cap \mathcal{P}_n = \emptyset$ for the final $\mathcal{A}$. We guarantee that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$, $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$. At the end of the construction we take $\mathcal{A} = \bigcup_n \mathcal{A}_n$. At each step, each parameter keeps the same value as at the preceding step if not explicitly modified. Also, let $\{\sigma_i^0\}_{i\in\omega}$ be a listing of the neighborhoods extending $\langle 0 \rangle$. Finally, we fix some listing $\{(\Psi_n, \Phi_n)\}_{n\in\omega}$ of all pairs of Turing functionals.

Step 0: Let $\mathcal{A}_0 = \emptyset$, and $\mathcal{P}_0 = \emptyset$.

Step $n+1$: Let $(\Psi, \Phi) = (\Psi_n, \Phi_n)$. The idea is to diagonalize against possible reductions $\mathcal{D} \leq \mathcal{D}$ via Φ, and $\mathcal{D} \leq \mathcal{A}$ via Ψ, with $\mathcal{D}$ dense. Assume by induction that $\mathcal{P}_n$ is closed and $[[\langle 1 \rangle]] - \mathcal{P}_n \neq \emptyset$.

We distinguish the following cases:

1. There exists a noncomputable $f \notin \mathcal{P}_n$ such that $\Psi(\Phi(f)) \uparrow$, or $\Psi(\Phi(f)) \downarrow$ and $\Psi(\Phi(f)) \neq f$ and $\Psi(\Phi(f) \notin \mathcal{A}_n$. In this case, choose such an f, let $f \in \mathcal{A}_{n+1}$; if $\Psi(\Phi(f)) \downarrow$ then also let $\Psi(\Phi(f)) \in \mathcal{P}_{n+1}$.

2. Otherwise (i.e. for all noncomputable $f \notin \mathcal{P}_n$, $\Psi(\Phi(f)) \downarrow$, and $\Psi(\Phi(f)) = f$ or $\Psi(\Phi(f)) \in \mathcal{A}_n$), let ρ be such that $[[\rho]] \subseteq [[\langle 1 \rangle]] - \mathcal{P}_n$ (use that $[[\langle 1 \rangle]] - \mathcal{P}_n$ is a nonempty open set), and let $\sigma = \rho\hat{\ }\langle 0 \rangle$. We distinguish two cases:

    (a) There exists $\tau \supseteq \sigma$ with no $\Psi \circ \Phi$-splittings above τ. In this case take a noncomputable $f \supset \tau$, and let $f \in \mathcal{A}_{n+1}$. Then the pair (Ψ, Φ) cannot be used in any equivalence $\mathcal{D} \equiv \mathcal{A}$, since by construction eventually $\mathcal{A}$ will not contain any computable function, whereas by the absence of splittings, on the neighborhood $[[\tau]]$ the Turing functional $\Psi \circ \Phi$, when defined, outputs computable functions, and thus $\Psi(\Phi(f))$ is computable.

(b) Otherwise, prohibit all functions in $[[\sigma]]$, by letting $\mathcal{P}_{n+1} = \mathcal{P}_n \cup [[\sigma]]$. (Notice that by choice of σ, $[[\langle 1\rangle]] - \mathcal{P}_{n+1} \neq \emptyset$, as $\rho\hat{}\langle 1\rangle \subseteq [[\langle 1\rangle]] - \mathcal{P}_{n+1}$.) In this case, we use the density of $\mathcal{D}$ to show that (Ψ, Φ) cannot witness an equivalence $\mathcal{D} \equiv \mathcal{A}$. Indeed, using $\Psi \circ \Phi$-splittings above τ we can build a computable perfect tree T, with $T(\lambda) = \sigma$, which is $\Psi \circ \Phi$-splitting. For all noncomputable $f \in [T]$ we have that either $\Psi(\Phi(f)) \in \mathcal{A}_n$ or $\Psi(\Phi(f)) = f$. Thus by cardinality, in $[T]$ there are noncomputable functions f such that $\Psi(\Phi(f)) = f$. Pick such an f, and let $\tau \subset f$ be such that $\Psi(\Phi(\tau)) \supseteq \sigma$ (without loss of generality we may assume that $\Phi(\tau)$ is in fact a string). By the density of $\mathcal{D}$ there exists a $g \in \mathcal{D}$ such that $g \supset \Phi(\tau)$, but then $\Psi(g) \supseteq \sigma$, thus $\Psi(g) \notin \mathcal{A}$, since all functions in $[[\sigma]]$ are prohibited.

At the end of Step $n+1$, take a noncomputable function $f \supset \sigma_n^0$ which has not been prohibited so far (only finitely many functions extending $\langle 0\rangle$ have been prohibited so far), and let $f \in \mathcal{A}_{n+1}$.

The observations made throughout the construction should make it clear that the mass problems $\mathcal{A}$ and $\mathcal{B} = \{f : \langle 0\rangle\hat{}f \in \mathcal{A}\}$ have the required properties. □

Remark 5.6 The first proof of the previous theorem shows in fact that if $\mathbf{A} \in \mathfrak{I}$, $\mathbf{C} \in \mathfrak{F}$, and $\mathbf{A} \not\leq \mathbf{C}$ then

$$(\forall \mathbf{D})[\mathbf{D} \text{ dense and } \mathbf{A} \wedge \mathbf{C} < \mathbf{D} \Rightarrow \mathbf{A} \leq \mathbf{D}];$$

in particular, if $\mathbf{A}$ is dense then,

$$\mathbf{A} = \min \{\mathbf{D} : \mathbf{D} \text{ dense and } \mathbf{A} \wedge \mathbf{C} < \mathbf{D}\}.$$

Remark 5.7 How does one transfer proofs from the Cantor space to the Baire space, and conversely? Often, proving either case is enough to get the remaining case as an immediate consequence of the isomorphism given in Theorem 1.7. As observed in Remark 1.8 a somewhat careful analysis should be made when closed mass problems and perfect trees are concerned. For instance one can argue that proving Theorem 5.3 for the Baire space immediately yields the result for the Cantor space since T^{01} is a computable perfect tree in the Cantor space if T is a computable perfect tree in the Baire space. To get the result for the Baire space as a consequence of the result for the Cantor space, one has to slightly modify the proof for the Cantor space, and build a computable perfect tree T such that for every nonempty σ, $T(\sigma)$ ends with the bit 0. But this can be easily done, by defining $T(\lambda) = \lambda$, and in the inductive step, $T(\sigma\hat{}\langle i\rangle) = T(\sigma)\hat{}\tau_i\hat{}\langle 0\rangle$, where τ_i is as in the proof of Theorem 5.3. This guarantees that T^ω is a computable perfect tree in the Baire space. The argument then goes like this: if $\mathcal{A} \leq \mathcal{D}$ and $\mathcal{D}$ is dense in the Baire space, then $\mathcal{A}^{01} \leq \mathcal{D}^{01}$ and $\mathcal{D}^{01}$ is dense in the Cantor space. Therefore there exists a tree T in the Cantor space such that for every nonempty string σ, $T(\sigma)$ ends with the bit 0, and $\mathcal{A}^{01}$ is dense in T; it is then easy to conclude that $\mathcal{A}$ is dense in T^ω.

Definition 5.8 A mass problem $\mathcal{X}$ is said to be $\equiv_{\mathrm{T}}$-*closed* if

$$f \in \mathcal{X} \text{ and } f \equiv_{\mathrm{T}} g \Rightarrow g \in \mathcal{X}.$$

Examples of $\equiv_{\mathrm{T}}$-closed mass problems are mass problems that are upwards $\leq_T$-closed (whose M-degrees are called *Muchnik degrees*), and the Turing degrees (regarded as mass problems).

Theorem 5.9 *If $\mathcal{A}$ and $\mathcal{B}$ are mass problems that are $\equiv_{\mathrm{T}}$-closed and $\mathcal{A}$ is of nonzero* M-*degree, then*

$$\mathcal{A} \leq \mathcal{B} \Leftrightarrow \mathcal{B} \subseteq \mathcal{A}.$$

Proof. Suppose first that we work in the Cantor space, and $\mathcal{A}, \mathcal{B} \subseteq 2^\omega$ are mass problems that are $\equiv_{\mathrm{T}}$-closed, $\mathcal{A}$ does not contain computable functions, and $\mathcal{A} \leq \mathcal{B}$, via the Turing functional Φ. Let T be a computable perfect tree associated with $\mathcal{B}$ (which is dense) and Φ as in the proof of Theorem 5.3. Let $f \in \mathcal{B}$: then $T(f) \equiv_{\mathrm{T}} f$, i.e. $T(f) \in \mathcal{B}$, and since T is Φ-splitting, we have that $\Phi(T(f)) \equiv_{\mathrm{T}} T(f)$ (see for instance [7]), and thus $f \in \mathcal{A}$ since $\Phi(T(f)) \in \mathcal{A}$ and $\mathcal{A}$ is $\equiv_{\mathrm{T}}$-closed.

Next, we give the proof for the Medvedev lattice on the Baire space. Suppose that $\mathcal{A}, \mathcal{B}$ are as above, but $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$. Thus $\mathcal{A}^{01} \leq \mathcal{B}^{01}$, hence $\mathcal{B}^{01} \subseteq \mathcal{A}^{01}$ by T-closure in the Cantor space. If now $f \in \mathcal{B}$ then $f^{01} \in \mathcal{B}^{01}$, hence $f^{01} \in \mathcal{A}^{01}$ and thus $f^{01} \in \mathcal{A}$ since $\mathcal{A}^{01} \subseteq \mathcal{A}$ by T-closure, but $f \equiv_{\mathrm{T}} f^{01}$, then again by T-closure, $f \in \mathcal{A}$. This shows that $\mathcal{B} \subseteq \mathcal{A}$. □

Remark 5.10 Notice that in the Cantor space, $T(f) \equiv_{\mathrm{tt}} f$ and $\Phi(T(f)) \equiv_{\mathrm{tt}} T(f)$ in the above proof: this shows that in the Cantor space the claim is true also for mass problems that are $\equiv_{\mathrm{tt}}$-closed, i.e. if $\mathcal{A}, \mathcal{B} \subseteq 2^\omega$ are $\equiv_{\mathrm{tt}}$-closed, then

$$\mathcal{A} \leq \mathcal{B} \Leftrightarrow \mathcal{B} \subseteq \mathcal{A}.$$

Dyment [4, 3.10] states that there exist function $g \leq_{\mathrm{T}} h$ such that the mass problem $\{f : f \equiv_{\mathrm{T}} g\}$ is not comparable with $\{f : f \equiv_{\mathrm{T}} h\}$. In fact:

Corollary 5.11 *For all pair of non computable functions g, h such that $g \not\equiv_{\mathrm{T}} h$, we have that $\{f : f \equiv_{\mathrm{T}} g\}$ is not Medvedev comparable with $\{f : f \equiv_{\mathrm{T}} h\}$.*

Proof. Immediate since by Theorem 5.9, on these mass problems reduciblity is equivalent to inclusion. □

6 The filter $\mathfrak{F}$

An obvious consequence of Theorem 5.3 is:

Theorem 6.1 *If* $\mathbf{A}$ *is non-zero and contains some mass problem that is not dense in any computable perfect tree then* $\mathbf{A} \in \mathfrak{F}$.

Also, by Lemma 4.6,

Corollary 6.2 *For every mass problem* $\mathcal{A}$*, if* $\deg_{\mathrm{M}}(\mathcal{A}) \in \mathfrak{F}_{\mathrm{Cl}}$ *then* $\mathcal{A}$ *is nowhere dense.*

The following result uses the condition on the elements of $\mathfrak{F}$ given in Theorem 6.1 to construct a degree that is in $\mathfrak{F}$ but not in $\mathfrak{F}_{\mathrm{Cl}}$. For a different construction see [2, Corollary 3.7].

Theorem 6.3 *There exists a countable mass problem* $\mathcal{A}$*, of non-zero degree, such that* $\mathcal{A}$ *is not dense in any computable perfect tree, and* $\mathcal{A} \not\geq \mathcal{C}$ *for any closed* $\mathcal{C}$ *of nonzero degree.*

Proof. Let $\{T_e\}_{e\in\omega}$ be a listing of all computable perfect trees. We build a mass problem $\mathcal{A}$ of noncomputable functions, satisfying, for all e, the requirements:

$$\begin{aligned} P_e: &\quad \mathcal{A} \subseteq \mathrm{domain}(\Psi_e) \Rightarrow \overline{\Psi_e(\mathcal{A})} \text{ contains a computable function;} \\ N_e: &\quad \mathcal{A} \text{ not dense in } [T_e]. \end{aligned}$$

(We recall that $\overline{\mathcal{X}}$ denotes the closure of a given mass problem $\mathcal{X}$). Notice that satisfaction of all requirements P_e ensures that there is no closed $\mathcal{C}$ of nonzero degree such that $\mathcal{C} \leq \mathcal{A}$, otherwise there would be some e such that $\mathcal{A} \subseteq \mathrm{domain}(\Psi_e)$ and $\Psi_e(\mathcal{A}) \subseteq \mathcal{C}$, but then $\overline{\Psi_e(\mathcal{A})} \subseteq \mathcal{C}$, giving that $\mathcal{C}$ contains a computable function.

At stage $s > 0$ we define an approximation $\mathcal{A}_s$ to $\mathcal{A}$, such that the closure $\overline{\mathcal{A}_s}$ is countable; moreover we define a neighborhood τ to be *available* if $[[\tau]] \cap [[\rho]] = \emptyset$ for every neighborhood ρ that has been declared prohibited so far, and $[[\tau]] \cap \mathcal{A}_{s-1} = \emptyset$. $\mathcal{A}_s$ may contain only function from available neighborhoods. At the end of each stage we have defined finitely many prohibited neighborhoods, and we guarantee that there remain available neighborhoods.

Stage $s = 0$: do nothing.

Stage $2e + 1$: Take the least available σ. If there exists a string $\tau \supseteq \sigma$ such that for some noncomputable $f \supset \tau$, $\Psi_e(f)\uparrow$, then let $f \in \mathcal{A}_{2e+1}$ for some such $f \supset \tau$.

Otherwise, define by recursion two sequences τ_n, σ_n of strings, and a sequence f_n of functions as follows:

1. let τ_0 be the first string $\tau \supseteq \sigma$ for which $\Psi_e(\tau)(0)\downarrow$, let $\sigma_0 = \tau_0\hat{}\langle 1\rangle$ and choose a noncomputable function $f_0 \supset \tau_0\hat{}\langle 0\rangle$;

2. let τ_{n+1} be the first string $\tau \supseteq \sigma_n$ for which $\Psi_e(\tau)(n)\downarrow$, let $\sigma_{n+1} = \tau_{n+1}{}^\frown\langle 1\rangle$ and choose a noncomputable function $f_{n+1} \supset \tau_{n+1}{}^\frown\langle 0\rangle$;

Let $f_n \in \mathcal{A}_{2e+1}$ for all n. Notice that $\mathcal{A}_{2e+1}$ has countable closure if $\mathcal{A}_{2e}$ has.

Stage $2e+2$: Assume that we have already defined $\mathcal{A}_{2e+1}$, with countable closure, and finitely many prohibited neighborhoods. If there is a prohibited neighborhood ρ such that ρ is on T_e, then do noting: in this case we already have that the final $\mathcal{A}$ is not dense in $[T_e]$ since $\mathcal{A} \cap [[\rho]] = \emptyset$. Otherwise, there exists a string σ such that $[[T_e(\sigma)]] \cap [[\rho]] = \emptyset$ for all prohibited ρ (in fact, any σ such that $T_e(\sigma)$ is longer than all prohibited strings works here), choose such a σ and pick a neighborhood τ such that τ is on T_e, $T_e(\sigma) \subseteq \tau$ and $[[\tau]] \cap \mathcal{A}_{2e+1} = \emptyset$: such a neighborhood exists since by cardinality we cannot have $([T_e] \cap [[T_e(\sigma)]] \subseteq \overline{\mathcal{A}_{2e+1}}$. Suppose that σ' is the least string such that $\tau \subseteq T_e(\sigma')$. Declare *prohibited* the neighborhood $T_e(\sigma'{}^\frown\langle 0\rangle)$. Notice that at next stage there will be available neighborhoods, e.g. those extending $T_e(\sigma'{}^\frown\langle 1\rangle)$. Notice also that $\mathcal{A}_{2e+2} = \mathcal{A}_{2e+1}$.

Finally take $\mathcal{A} = \bigcup_s \mathcal{A}_s$. This ends the construction. We now check that the construction works.

To see that P_e is satisfied, consider the least available string σ at stage $2e+1$, and assume that Ψ_e is defined on all noncomputable functions $f \supset \sigma$. Then we define the sequence $f_n \supset \sigma$, with $f_n \in \mathrm{domain}(\Psi_e)$ whose limit is $\bigcup_n \sigma_n \in \mathrm{domain}(\Psi_e)$. But then $\lim_n \Psi_e(f_n)$ converges to the computable function $\Psi_e(\bigcup_n \sigma_n) \in \overline{\Psi_e(\mathcal{A})}$.

To show that N_e is satisfied, we observe that our action at stage $2e+2$ picks a τ on T_e, and guarantees that $\mathcal{A} \cap [[\tau]] = \emptyset$. □

The above theorem provides also another proof, in addition to Theorem 4.8, that $\mathfrak{F}_{\mathrm{Cl}}$ is strictly included in $\mathfrak{F}$.

Finally, Theorem 6.5 below shows that the nonzero closed degrees do not form a filter, in other words the nonzero closed degrees are properly contained in $\mathfrak{F}_{\mathrm{Cl}}$. We first need the following lemma:

Lemma 6.4 *If $\mathcal{B} \equiv \mathcal{C}$ and $\mathcal{C}$ is closed then $\mathcal{B}$ is either of countable degree or contains a perfect set.*

Proof. Suppose that $\mathcal{C} \subseteq \mathrm{domain}(\Phi)$ and $\Phi(\mathcal{C}) \subseteq \mathcal{B}$. If $\Phi(\mathcal{C})$ is countable then $\mathcal{C} \geq \Phi(\mathcal{C}) \geq \mathcal{B} \geq \mathcal{C}$ and thus $\mathcal{B}$ is of countable degree. If not then, as $\Phi(\mathcal{C}) = \{g : (\exists f)[f \in \mathcal{C} \text{ and } \Phi(f) = g]\}$ is analytic and uncountable, by a classical theorem due to Suslin (see for instance [6]) $\Phi(\mathcal{C})$ contains a perfect set, and thus so does $\mathcal{B} \supseteq \Phi(\mathcal{C})$. □

Theorem 6.5 *There is a closed $\mathcal{A}$ of nonzero degree, and a $\mathcal{B} \geq \mathcal{A}$ with $\mathcal{B}$ not of closed degree.*

Proof. By the Lemma, it suffices to construct a closed $\mathcal{A}$ not containing computable functions, and a subset $\mathcal{B} \subseteq \mathcal{A}$ which is not of countable degree and contains no perfect subset. Suppose $\mathcal{A}$ is perfect and every member is of minimal Turing degree (see for instance [7]). We build $\mathcal{B}$ in $2^{\aleph_0}$ many stages (at stage α we define $\mathcal{B}_\alpha$) with the condition that by the end of each stage we have put fewer than $2^{\aleph_0}$ many elements of $\mathcal{A}$ into $\mathcal{B}$, and have guaranteed that fewer than $2^{\aleph_0}$ many (prohibited) specified elements of $\mathcal{A}$ will never go into $\mathcal{B}$.

As there are only $2^{\aleph_0}$ many perfect sets $\mathcal{P}_\alpha$ we may at stage $\alpha < 2^{\aleph_0}$ guarantee that $\mathcal{P}_\alpha \nsubseteq \mathcal{B}$. If $\mathcal{P}_\alpha \nsubseteq \mathcal{A}$ then there is nothing to worry about. Otherwise, there is an element f of $\mathcal{P}_\alpha$ not yet in $\mathcal{B}_\alpha$ and we satisfy the requirement by prohibiting f from ever entering $\mathcal{B}$, guaranteeing that $f \notin \mathcal{B}$.

Similarly there are only $2^{\aleph_0}$ many countable sets. Let $\{(\mathcal{C}_\alpha, \Phi_\alpha)\}_{\alpha<2^{\aleph_0}}$ be an enumeration of all pairs consisting of a countable set, and a Turing functional. Without loss of generality we may assume that the $\mathcal{C}_\alpha$ do not contain any computable function. We act at stage α to guarantee that one can not have $\mathcal{C}_\alpha \leq \mathcal{B}$ via Φ. If $\Phi_\alpha(\mathcal{A}) = \{g : (\exists f)[f \in \mathcal{A} \text{ and } \Phi_\alpha(f) = g]\}$ is not countable, by Suslin's theorem it contains a perfect subset and so there are $2^{\aleph_0}$ many elements of $\mathcal{A}$ on which Φ_α is one-one. We may choose one of these f (that have not been prohibited so far from entering $\mathcal{B}$) with $\Phi_\alpha(f) \notin \mathcal{C}_\alpha$, such that we can put f into $\mathcal{B}$. This guarantees that $\Phi_\alpha(\mathcal{B}) \nsubseteq \mathcal{C}_\alpha$. On the other hand, if $\Phi_\alpha(\mathcal{A})$ is countable, then there are $2^{\aleph_0}$ many elements f_β of $\mathcal{A}$ such that $\Phi_\alpha(f_\beta)$ is partial for all β or there is an f such that $\Phi_\alpha(f_\beta) = f$ for all β. In the second case, as the f_β are of minimal degree, f is computable. If we now put one of these f_β into $\mathcal{B}$ then we guarantee that $\mathcal{B} \nsubseteq \text{domain}(\Phi_\alpha)$, or $\Phi_\alpha(\mathcal{B}) \nsubseteq \mathcal{C}_\alpha$. This shows that $\mathcal{B} \ngeq \mathcal{C}$ for every countable $\mathcal{C}$ not containing computable functions, thus $\mathcal{B}$ is not of countable degree. □

7 Quotient lattices

Let $\mathfrak{M}_{/\mathfrak{I}}$ denote the quotient lattice obtained dividing $\mathfrak{M}$ by the ideal $\mathfrak{I}$. The equivalence class of a degree $\mathbf{A}$ will be denoted by $\mathbf{A}_{/\mathfrak{I}}$. We recall (see e.g. [1]) that the partial ordering relation in $\mathfrak{M}_{/\mathfrak{I}}$, denoted by $\leq_{/\mathfrak{I}}$, is given by

$$\mathbf{A}_{/\mathfrak{I}} \leq_{/\mathfrak{I}} \mathbf{B}_{/\mathfrak{I}} \Leftrightarrow (\exists \mathbf{D} \in \mathfrak{I})[\mathbf{A} \leq \mathbf{B} \vee \mathbf{D}].$$

The following is an immediate consequence of Lemma 4.6:

Theorem 7.1 $\mathfrak{M}_{\mathrm{Cl}}$ *embeds into* $\mathfrak{M}_{/\mathfrak{I}}$ *as a sublattice.*

The next theorem shows that the congruence provided by $\mathfrak{I}$ is not the identity on the elements of $\mathfrak{F}$.

Theorem 7.2 *There exist* M*-degrees* $\mathbf{A}$ $\mathbf{B}$ *in* $\mathfrak{F}$ *such that* $\mathbf{B} \nleq \mathbf{A}$ *but in the quotient* $\mathfrak{M}_{/\mathfrak{I}}$*,* $\mathbf{B}_{/\mathfrak{I}} \leq_{/\mathfrak{I}} \mathbf{A}_{/\mathfrak{I}}$*.*

Proof. We show that there exist mass problems $\mathcal{B}$, $\mathcal{D}$ and perfect tree T such that $\mathcal{B}$ does not contain any computable function and is not dense in any computable perfect tree (hence its M-degree is in $\mathfrak{F}$), $\mathcal{D}$ is dense, $[T]$ is not dense in any computable perfect tree, $\mathcal{B} \not\leq \{T\}$, and

$$\mathcal{B} \leq \{T\} \vee \mathcal{D}.$$

In writing $\{T\}$ we identify T with its characteristic function, via computable identification of strings with natural numbers: in fact, T will be regarded at times as a tree, or as as its characteristic function, or as a mapping from the Cantor space to the Baire space. We define the desired perfect tree T by induction on the length of the input string as follows, where we refer to some listing $\{T_e\}_{e\in\omega}$ of the computable perfect trees.

1. $T(\lambda) = \lambda$:
2. Suppose that we have defined $T(\sigma)$ for all strings σ such that $|\sigma| = e$. Suppose also that all $T(\sigma)$ have the same length, say k. We define $T(\sigma\hat{}\langle i\rangle)$: pick a function $f \in [T_e]$, with say $f(k) = j$; then define $T(\sigma\hat{}\langle i\rangle) = T(\sigma)\hat{}\langle u_j\rangle\hat{}\langle i\rangle$, where $u_j = 0$ if $j \neq 0$, and $u_j = 1$ if $j = 0$.

Notice that $[T]$ (which is in fact a tree in the Cantor space, as $T : 2^* \longrightarrow 2^*$) is not dense in any computable perfect tree. Indeed, if T_e is a computable perfect tree then there is an $f \in [T_e]$ (the one we pick in the construction at the relevant stage) such that $f \upharpoonright e+1$ is incomparable with all $T(\tau)$ with $|\tau| = e+1$. Now take $\mathcal{D} = \{f \in 2^\omega : f \equiv_{\mathrm{T}} T'\}$ (where T' is the jump of T), and let $\mathcal{B} = T(\mathcal{D})$: indeed it suffices to take any dense $\mathcal{D} \subseteq 2^\omega$ whose elements are not $\leq_{\mathrm{T}} T$. We have that $\mathcal{D}$ is dense, $\mathcal{B}$ is dense in no computable perfect tree, and $\mathcal{B} \not\leq \{T\}$: the last claim follows from the observation that $f \leq_{\mathrm{T}} T \oplus T(f)$, and thus $T(f) \leq_{\mathrm{T}} T$ for no $f \in \mathcal{D}$. Finally $\mathcal{B} \leq \{T\} \vee \mathcal{D}$, as desired. □

Let now $\mathfrak{M}_{/\mathfrak{F}}$ denote the quotient lattice obtained dividing $\mathfrak{M}$ by the filter $\mathfrak{F}$. It is known ([2]) that the cardinality of $\mathfrak{M}_{/\mathfrak{F}}$ is $2^{2^{\aleph_0}}$.

The equivalence class of a degree $\mathbf{A}$ will be denoted by $\mathbf{A}_{/\mathfrak{F}}$. We recall that the partial ordering relation in $\mathfrak{M}_{/\mathfrak{F}}$, denoted by $\leq_{/\mathfrak{F}}$, is given by

$$\mathbf{A}_{/\mathfrak{F}} \leq_{/\mathfrak{F}} \mathbf{B}_{/\mathfrak{F}} \Leftrightarrow (\exists \mathbf{C} \in \mathfrak{F})[\mathbf{A} \wedge \mathbf{C} \leq \mathbf{B}].$$

In the following theorem we show that the congruence associated with the filter is not the identity on the elements of $\mathfrak{I}$.

Theorem 7.3 *There exist* M*-degrees* $\mathbf{A}$ *and* $\mathbf{B}$ *in* $\mathfrak{I}$ *such that* $\mathbf{B} \not\leq \mathbf{A}$ *but in the quotient* $\mathfrak{M}_{/\mathfrak{F}}$, $\mathbf{B}_{/\mathfrak{F}} \leq_{/\mathfrak{F}} \mathbf{A}_{/\mathfrak{F}}$.

Proof. Take $\mathbf{B} = \mathbf{B}_f$ and $\mathbf{A} = \mathbf{B}_f \wedge \deg_{\mathrm{M}}(\{f\})$, where f is not computable. Then $\mathbf{B} \wedge \deg_{\mathrm{M}}(\{f\}) \leq \mathbf{A}$, and $\deg_{\mathrm{M}}(\{f\}) \in \mathfrak{F}_{\mathrm{Cl}} \subseteq \mathfrak{F}$. □

Question 7.4 *What is the cardinality of* $\mathfrak{M}_{/\mathfrak{I}}$*?*

8 A few remarks on degrees of enumerability

If A is a set then let

$$\mathcal{E}_A = \{f : \text{range}(f) = A\},$$

and let $\mathbf{E}_A = \deg_{\mathrm{M}}(\mathcal{E}_A)$. If $A \neq \emptyset$, then $\mathcal{E}_A$ is called the *problem of enumerability of* A, and $\mathbf{E}_A$ is called the *degree of enumerability of* A. As already observed by Medvedev, [8], if $A, B \neq \emptyset$ then $A \leq_{\mathrm{e}} B$ if and only if $\mathbf{E}_A \leq \mathbf{E}_B$, thus the degrees of enumerability are isomorphic with the enumeration degrees (preserving 0 and $\vee$).

A set A is of *quasiminimal* e-degree if A is not c.e. and for every total function f, if $f \leq_{\mathrm{e}} A$ then f is computable. Moreover, A is of *nontotal* e-degree if there is no total function f such that $f \equiv_{\mathrm{e}} A$.

Lemma 8.1 ([4]) *If $\mathcal{A} \leq \mathcal{E}_A$ and $\mathcal{A}$ is countable then there exists $f \in \mathcal{A}$ such that $f \leq_{\mathrm{e}} A$. Thus if A has nontotal* e-*degree then $\mathbf{E}_A$ does not contain any countable mass problem.*

Theorem 8.2 ([2]) *If A is immune then $\mathbf{E}_A \in \mathfrak{F}_{\mathrm{Cl}}$.*

Proof. If A is immune then

$$\mathcal{E}_A \geq \{f : f \text{ 1-1 and } \text{range}(f) \subseteq A\},$$

while the latter mass problem is clearly closed and of non-zero degree. □

There are quite natural classes of dense M-degrees that are bounded above by degrees of enumerability. For instance, given any A let $\mathcal{E}_A^{\geq_{\mathrm{e}}} = \{f : A \leq_{\mathrm{e}} f\}$: clearly $\mathcal{E}_A^{\geq_{\mathrm{e}}}$ is dense and $\mathcal{E}_A^{\geq_{\mathrm{e}}} \leq \mathcal{E}_A$. It is known ([12]) that $\mathcal{E}_A^{\geq_{\mathrm{e}}} \leq \mathcal{E}_B^{\geq_{\mathrm{e}}}$ if and only if $A \leq_{\mathrm{e}} B$. For another example of dense degrees below degrees of enumerability, define $\mathcal{E}_A^* = \{f : \text{range}(f) =^* A\}$, where $=^*$ denotes equality modulo a finite set.

Theorem 8.3 *For every nonempty A, $\mathcal{E}_A^*$ is dense and if A is not c.e. then $\mathcal{E}_A^* < \mathcal{E}_A$.*

Proof. The only nontrivial part is to show that $\mathcal{E}_A \not\leq \mathcal{E}_A^*$ if A is not c.e.: if $\mathcal{E}_A \leq \mathcal{E}_A^*$ via the Turing functional Ψ then $A = \bigcup_{\sigma \in \omega^*} \Psi(\sigma)$, giving that A is c.e.. □

The degrees of enumerability turn out to be useful to make observations on $\mathfrak{I}$ and $\mathfrak{F}$. We have already recalled in Theorem 4.8 that $\mathfrak{F}_{\mathrm{Di}} \not\subseteq \mathfrak{F}_{\mathrm{Cl}}$. The converse inclusion does not hold either:

Corollary 8.4 $\mathfrak{F}_{\mathrm{Cl}} \not\subseteq \mathfrak{F}_{\mathrm{Di}}$

Proof. Let A be an immune set of quasiminimal e-degree (see for instance [13]). Then $\mathbf{E}_A \in \mathfrak{F}_{\mathrm{Cl}}$ by Theorem 8.2. On the other hand by Lemma 8.1, there cannot be any nonzero countable (hence any nonzero discrete) M-degree $\mathbf{C}$ such that $\mathbf{C} \leq \mathbf{E}_A$. □

Further to our anticipation following Theorem 5.4, we are now in a position to prove that there are mass problems that are dense in some computable perfect tree, but with M-degree not in $\mathfrak{I}$, which shows that the converse of Theorem 5.3 is false.

Theorem 8.5 *There exists a mass problem $\mathcal{A}$ which is dense in some computable perfect tree, but such that* $\deg_{\mathrm{M}}(\mathcal{A}) \notin \mathfrak{I}$.

Proof. Let A be an immune set. Let $x, y \in A$, with $x \neq y$. We are going to show that $\mathcal{E}_A$ is dense in some computable perfect tree T, but on the other hand $\deg_{\mathrm{M}}(\mathcal{E}_A) \notin \mathfrak{I}$ since by Theorem 8.2 and Corollary 4.7, $\mathbf{E}_A \in \mathfrak{F}$.

We define T as follows:

1. $T(\lambda) = \lambda$;
2. Suppose that we have defined $T(\sigma)$. Then define:

$$T(\sigma\hat{\ }\langle 0\rangle) = T(\sigma)\hat{\ }\langle x\rangle,$$
$$T(\sigma\hat{\ }\langle 1\rangle) = T(\sigma)\hat{\ }\langle y\rangle.$$

Then $T(\sigma\hat{\ }\langle 0\rangle)|T(\sigma\hat{\ }\langle 1\rangle)$ for every σ, and for every $g \in [T]$, $\mathrm{range}(g) \subseteq A$, thus $[T] \subseteq \overline{\mathcal{E}_A}$. □

Finally:

Theorem 8.6 *There exist degrees* $\mathbf{A}$ *and* $\mathbf{B}$ *such that* $\mathbf{A} \neq \mathbf{0}$, $\mathbf{A} < \mathbf{B}$, *and* $\mathbf{A}$ *is discrete, but* $\mathbf{B}$ *is not discrete. Thus the nonzero discrete degrees are properly contained in* $\mathfrak{F}_{\mathrm{Di}}$.

Proof. Let f be a noncomputable total function, and let A be such that $f <_{\mathrm{e}} A$, but $\deg_{\mathrm{e}}(A)$ is not total. (In fact for every f there is such a set A, by relativizing the construction of a quasiminimal e-degree.) Then $\{f\} < \mathcal{E}_A$, but by Lemma 8.1, we have that $\deg(\mathcal{E}_A)$ does not contain any countable mass problem. □

References

[1] R. Balbes and P. Dwinger. *Distributive Lattices.* University of Missouri Press, Columbia, 1974.

[2] C. Bianchini and A. Sorbi. A note on closed degrees of difficulty of the Medvedev lattice. *Math. Logic Quarterly*, 42:127–133, 1996.

[3] S. B. Cooper. *Computability Theory.* Chapman & Hall/CRC Mathematics, Boca Raton, London, New York, Washington, DC, 2003.

[4] E. Z. Dyment. Certain properties of the Medvedev lattice. *Mathematics of the USSR Sbornik*, 30:321–340, 1976. English Translation.

[5] Yu. L. Ershov. Note on a problem of Rogers. *Algebra and Logic*, 8:285, 1969.

[6] A. S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics.* Springer–Verlag, Heidelberg, 1995.

[7] M. Lerman. *Degrees of Unsolvability.* Perspectives in Mathematical Logic. Springer–Verlag, Heidelberg, 1983.

[8] Y. T. Medevdev. Degrees of difficulty of the mass problems. *Dokl. Nauk. SSSR*, 104(4):501–504, 1955.

[9] A. A. Muchnik. On strong and weak reducibility of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. Russian.

[10] R. A. Platek. A note on the cardinality of the Medvedev lattice. *Proc. Amer. Math. Soc.*, 25:917, 1970.

[11] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.

[12] M. Rozinas. The semilattice of e-degrees. In *Recursive Functions*, pages 71–84. Ivanov. Gos. Univ., Ivanovo, 1978. (Russian) MR 82i:03057.

[13] M. G. Rozinas. Partial degrees of immune and hyperimmune sets. *Siberian Math. J.*, 19:613–616, 1978.

[14] P. Shafer. Characterizing the join-irreducible Medvedev degrees. To appear in *Notre Dame J. Formal Logic.*

[15] S. G. Simpson. Mass problems and intuitionism. *Notre Dame J. Formal Logic*, 49(2):127–136, 2008.

[16] E. Z. Skvortsova. Faithful interpretation of the intitionistic prpositional calculus by an initial segment of the Medvedev lattice. *Sibirsk. Mat. Zh.*, 29(1):171–178, 1988. Russian.

[17] R. I. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987.

[18] A. Sorbi. On some filters and ideals of the Medvedev lattice. *Arch. Math. Logic*, 30:29–48, 1990.

[19] A. Sorbi. Embedding Brouwer algebras in the Medvedev lattice. *Notre Dame J. Formal Logic*, 32(2):266–275, 1991.

[20] A. Sorbi. Some quotient lattices of the Medvedev lattice. *Z. Math. Logik Grundlag. Math.*, 37:167–182, 1991.

[21] A. Sorbi and S. Terwijn. Intermediate logics and factors of the Medvedev lattice. *Ann. Pure Appl. Logic*, 155(2):69–86, 2008.

[22] S. A. Terwijn. The Medvedev lattice of computably closed sets. *Arch. Math. Logic*, 45:179–190, 2006.

[23] S. A. Terwijn. On the structure of the Medvedev lattice. *J. Symbolic Logic*, 73(2):543–558, 2008.