Every Incomplete Computably Enumerable Truth-Table Degree Is Branching*

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Abstract

If r is a reducibility between sets of numbers, a natural question to ask about the structure $\mathcal{C}_r$ of the r-degrees containing computably enumerable sets is whether every element not equal to the greatest one is branching (i.e., the meet of two elements strictly above it). For the commonly studied reducibilities, the answer to this question is known except for the case of truth-table (tt) reducibility. In this paper, we answer the question in the tt case by showing that every tt-incomplete computably enumerable truth-table degree $\mathbf{a}$ is branching in $\mathcal{C}_{tt}$. The fact that every Turing-incomplete computably enumerable truth-table degree is branching has been known for some time. This fact can be shown using a technique of Ambos-Spies and, as noticed by Nies, also follows from a relativization of a result of Degtev. We give a proof here using the Ambos-Spies technique because it has not yet appeared in the literature. The proof uses an infinite injury argument. Our main result is the proof when $\mathbf{a}$ is Turing-complete but tt-incomplete. Here we are able to exploit the Turing-completeness of $\mathbf{a}$ in a novel way to give a finite injury proof.

1 Introduction

If r is one of the reducibilities between sets of natural numbers studied in computability theory, one can form the structures $\mathcal{D}_r$, $\mathcal{D}_r(\leq \mathbf{0}'_r)$ and $\mathcal{C}_r$ consisting of all the r-degrees, the r-degrees of sets r-reducible to the 1-complete computably enumerable set $K = \{e|\{e\}(e)\downarrow\}$, and the r-degrees containing computably enumerable sets, respectively. (We write $\mathbf{0}'_r$ for the r-degree of K.) These structures are always partially ordered sets and are usually upper semi-lattices. Among

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the many algebraic questions one might ask about these structures, a basic one is whether every element not equal to the greatest element is branching, where we call a degree *branching* if it is meet-reducible, i.e., it is the meet of two degrees strictly above it. In the case of $\mathcal{C}_r$ and the most commonly studied reducibilities r (namely, many-one (m), truth-table (tt), weak truth-table (wtt) and Turing (T)) we have the following results. Every incomplete c.e. m-degree is branching. This can be seen as follows. Given a c.e. m-degree, $\mathbf{c} < \mathbf{0}'_m$, by a result of Denisov [2], there is a c.e. m-degree $\mathbf{a}$ with $\mathbf{c} < \mathbf{a} < \mathbf{0}'_m$. Now, by a result of Ershov and Lavrov [3], there is a strong minimal cover $\mathbf{b}$ of $\mathbf{c}$ in the c.e. m-degrees (that is, $\mathbf{c} < \mathbf{b}$ and for all c.e. m-degrees $\mathbf{d}$, if $\mathbf{d} < \mathbf{b}$, then $\mathbf{d} \leq \mathbf{c}$) such that $\mathbf{b}$ is incomparable with $\mathbf{a}$. The two degrees $\mathbf{a}$ and $\mathbf{b}$ now have meet $\mathbf{c}$. (See Odifreddi [8], Exercise X.5.5 for a direct proof of the fact that every c.e. m-degree is branching.) Every incomplete c.e. wtt-degree is branching by a result of Cohen [1]. (See Odifreddi [8], Exercise X.6.8.) In the case of Turing reducibility, the situation is a bit more complicated. There are nonzero branching c.e. T-degrees and $\mathbf{0}$ is also branching, but there are also incomplete c.e. T-degrees that are not branching (Lachlan [6]). In fact, both the branching (Slaman [9]) and nonbranching (Fejer [4]) c.e. T-degrees are dense in $\mathcal{C}_T$. Thus, among these problems, only the truth-table case has been unresolved. In this paper, we show that every incomplete c.e. truth-table degree is branching.

We call a c.e. truth-table degree *Turing-complete* if its members belong to $deg_T(K)$. The fact that every Turing-incomplete c.e. truth-table degree is branching was first shown by an unpublished argument using a technique due to Ambos-Spies. Later, Nies showed that this fact also follows from a relativization of a result of Degtev. (Nies' argument is given in the hint to Exercise X.7.14.c of Odifreddi [8].) Ambos-Spies' technique was used by Nies and Shore [7] in showing a related result. To make this paper more complete, and because the technique is interesting in its own right, we begin by showing how the result in [7] can be strengthened slightly in order to get a result that has as a corollary the fact that every Turing-incomplete c.e. truth-table degree is branching. Then, in the main part of the paper, we show (Theorem 4) that every Turing-complete, tt-incomplete truth-table degree is branching. In combination with the result on Turing-incomplete c.e. degrees, this gives the result of the title of this paper. We conclude the paper by mentioning some related open problems.

Our notation is for the most part standard. (See for example Soare [10].) For D a set and e a number, the e-section of D, denoted $D^{[e]}$, is the set $\{\langle e, x\rangle | \langle e, x\rangle \in D\}$, where $\langle \cdot, \cdot \rangle$ is a standard pairing function. We use $D^{[<e]}$ to denote $\bigcup\{D^{[e']} | e' < e\}$, and similarly for notations such as $D^{[\geq e]}$. Our notation for truth-table reductions is from [5]. In particular, if $\{e\}(x)\downarrow$, then $[e](x)$ denotes the truth-table with Gödel number $\{e\}(x)$ and $|[e](x)|$ denotes the length of this truth table. (Here, as in [5], we take this length to be one more than the largest number asked about in the truth table.) For any set A, $[e]^A(x)$ is 1 or 0 according to whether or not A satisfies $[e](x)$. We then have that for any two sets A and B, $A \leq_{tt} B$ if and only if there is an e with $\{e\}$ total such that for all x, $[e]^B(x) = A(x)$. Also, we let F_k denote the finite set with canonical index k.

2 Turing Incomplete Truth-Table Degrees

In this section, we present a slight strengthening of a result from [7] which will allow us to conclude that every Turing incomplete c.e. truth-table degree is branching. We first give a slightly modified version of Lemma 3.3 of [7].

Lemma 1 *Let $(D_n)_{n\in\omega}$ be a uniformly c.e. sequence of sets and let*

$$D = \{\langle x, n\rangle | x \in D_n\}.$$

Suppose that, for each e, the set $D^{[\leq e]}$ is T-incomplete. Then, there exist c.e. sets B, C such that

$$(\forall n)[D_n \leq_{tt} B, C] \tag{1}$$

$$(\forall n)[B, C \not\leq_{tt} D_n] \tag{2}$$

$$(\forall Z)[Z \leq_{tt} B, C \Rightarrow (\exists e)[Z \leq_{tt} D^{[\leq e]}]] \tag{3}$$

We say that B, C form an exact pair for the sequence (D_n).

Proof: The difference between this result and Lemma 3.3 of [7] is the addition of (2). To make (1) hold, we meet the requirements

$$P_n: \quad D^{[n]} =^* B^{[n]} =^* C^{[n]}$$

for $n \in \omega$ (where $X =^* Y$ means that the symmetric difference of X and Y is finite). To make (3) hold, we have requirements

$$Q_e: \quad Z = [e]^B = [e]^C \to Z \leq_{tt} D^{[\leq e]}$$

for all $e \in \omega$. (These requirements are as in [7].) To make (2) hold, we introduce requirements

$$R^X_{e,n}: \quad X \neq [e]^{D^{[n]}} \quad (X = B, C)$$

for $e, n \in \omega$.

The strategies for the P_n and Q_e requirements are as in [7]. We briefly review these strategies here. The strategy for the P_n requirements is the obvious purely positive one: at stage s, each number in $D^{[n]}$ that is greater than the restraint on P_n is put into both B and C. For this strategy to succeed, it is necessary that the liminf of the restraint on P_n be finite and also that only finitely many numbers be put into $B^{[n]}$ and $C^{[n]}$ by requirements other than P_n.

To meet Q_e, we try to make the antecedent $[e]^B = [e]^C$ false, that is, we look for an x and a finite set F such that

$$[e]^{B\cup F}(x) \neq [e]^C(x) \text{ and } F \cap \omega^{[\leq e]} = \emptyset.$$

Given such x and F, we enumerate F into B and attempt to preserve B and C on the use of the computations. (The restriction that $F \cap \omega^{[\leq e]} = \emptyset$ is needed to prevent action for Q_e from injuring higher priority P_n requirements.) If, in spite of these efforts, $[e]^B = [e]^C = Z$, then for every x, $Z(x) = [e]^C(x) = [e]^{B\cup\omega^{[>e]}}(x)$. (If not, then we can begin a successful attack on Q_e through x after B and C have settled down on the use.) Thus, $Z \leq_{tt} B^{[\leq e]}$. Assuming that higher priority P_n requirements have been met, $B^{[\leq e]} \equiv_{tt} D^{[\leq e]}$, so Q_e is met. Because the P_n requirements are infinitary, it is possible that a given Q_e makes infinitely many attacks and that each attack is later cancelled due to the action of a higher priority P_n. Thus the Q_e requirements can be infinitary and this causes problems because action of the Q_e's can injure lower priority P_n and $Q_{e'}$ requirements. The solution is to use the T-incompleteness of each $D^{[\leq e]}$ in a way suggested by Ambos-Spies. Requirement Q_e appoints and cancels multiple followers x^e_m ($m \geq 0$), each with an associated F^e_m. When the follower is appointed, restraint is put on, and if the restraint is ever violated, the follower is cancelled. An attack can be made through follower x^e_m only if $m \in K$. We can now use the T-incompleteness of $D^{[\leq e]}$ to show that for some m, x^e_m is undefined at the end of infinitely many stages. (The least such m is called the outcome of requirement Q_e.) Indeed, suppose Q_e has no outcome. Then it must be that $[e]^B = [e]^C$, since otherwise we would stop appointing followers for Q_e. Assuming that $D^{[n]} =^* B^{[n]} =^* C^{[n]}$ for each $n \leq e$, we argue that $K \leq_T D^{[\leq e]}$, a contradiction. Given m, recursively in $D^{[\leq e]}$, compute a stage t such that x^e_m has a value x at the end of stage t and the sets $B^{[\leq e]}$ and $C^{[\leq e]}$ have settled down on the use of $[e](x)$ by the end of stage t. Then, m is in K if and only if $m \in K_t$, because if m appears in K after stage t, then a successful diagonalization would take place and no more followers would be appointed.

The fact that each Q_e has an outcome means that the modified strategy for Q_e still works: if $[e]^B = [e]^C = Z$ and m is the outcome of Q_e and z is the final value of x^e_{m-1}, then for all $x > z$, $Z(x) = [e]^C(x) = [e]^{B\cup\omega^{[>e]}}(x)$, since otherwise x^e_m would get a permanent final value. Although the action taken for a given Q_e can still be infinitary, the action taken for $x^e_{m'}$ with m' less than the outcome of Q_e is finitary. Thus, we put the construction on the tree $T = \omega^{<\omega}$. For each $\gamma \in T$, P_γ and Q_γ are versions of the requirements P_i and Q_i, where $i = |\gamma|$. If $e < i$ and β is γ restricted to e, then P_γ, Q_γ guess that the outcome of Q_β is $\gamma(e)$. When Q_β appoints a follower x^β_n, the associated set F^β_n is not allowed to interfere with any P_γ, Q_γ with $\gamma <_L \beta * n$. This ensures that the construction is finitary along the true path.

This completes our review of the strategies in [7]. To handle the $R^X_{e,n}$ requirements, we let $(R_i)_{i\in\omega}$ be an effective listing of these requirements. For each

γ in the tree $\omega^{<\omega}$, there is a strategy R_γ for R_i, where $i = |\gamma|$. The strategy for R_γ is Sacks coding; that is, if $R_{|\gamma|}$ is $R^X_{e,n}$, then R_γ codes K into X until a difference between X and $[e]^{D^{[n]}}$ occurs. (This must happen since $D^{[n]}$ is tt-incomplete.) Because the reduction being diagonalized against is a truth-table reduction, the action of R_γ is finite.

Now we will describe the construction in detail. We give only the differences from the construction in [7], so the reader will need to refer to that paper. Strategies P_γ and Q_γ work as in [7], except that P_γ tries to code $D^{[i]}$ into $X^{[2n(\gamma)]}$ for $X = B, C$, instead of into $X^{[n(\gamma)]}$, where $n(\gamma)$ is the code number assigned to γ. Strategy R_γ does its coding into $X^{[2n(\gamma)+1]}$.

To measure the length of agreement for the R_i requirements, we define

$$l'(i,s) = \max\{x|(\forall y < x)[X(y) = [e]^{D^{[n]}}(y)[s]]\},$$

where $R_i = R^X_{e,n}$. The construction is the same as in [7] except for the following changes. 1) In Step 1, we consider numbers $x = \langle z, 2n(\beta)\rangle$ instead of $x = \langle z, n(\beta)\rangle$. 2) We add a Step 1.5 (where R_β codes K) given by

> For each $x < s$, if $l'(e,s) \geq x \geq R(\beta,[s,e])$ and $x = \langle z, 2n(\beta)+1\rangle$ for $z \in K_s$, then enumerate x into X, where R_e is $R^X_{i,n}$.

3) In Step 3, replace equation (5.9) by

$$F \cap (\omega^{[2n(\gamma)]} \cup \omega^{[2n(\gamma)+1]}) = \emptyset \text{ for each } \gamma < \beta * m.$$

In the verification, Lemma 5.1 of [7] is modified by changing $X^{[\gamma]}$ and $\langle z, n(\gamma)\rangle$ in Part (iv) to $X^{[2n(\gamma)]}$ and $\langle z, 2n(\gamma)\rangle$, respectively, and by adding a Part (v) which states

(v) If $\gamma \subseteq \alpha$, $i = |\gamma|$, and R_i is $R^X_{e,n}$, then $X \neq [e]^{D^{[n]}}$ and $X^{[2n(\gamma)+1]}$ is finite.

The proof of the new Part (v) of the lemma goes as follows. By the inductive hypothesis, it is only necessary to prove (v) for $\gamma = \alpha$. Let R_{e+1} be $R^X_{i,n}$ and let s_α be as in the statement of the lemma. By the argument for Part (iv) of the lemma given in [7], no Q_β puts a number into $X^{[2n(\alpha)+1]}$ after stage s_α. Suppose that $X = [i]^{D^{[n]}}$. Then, $\lim_s l'(e+1,s) = \infty$. We claim that

$$X^{[2n(\alpha)+1]} = X^{[2n(\alpha)+1]}_{s_\alpha} \cup \{\langle z, 2n(\alpha)+1\rangle | \langle z, 2n(\alpha)+1\rangle \geq R(\alpha) \wedge z \in K\}. \quad (4)$$

To see the right-to-left inclusion of (4), suppose that $x = \langle z, 2n(\alpha)+1\rangle \geq R(\alpha)$ and $z \in K$. Take $s \geq s_\alpha$ such that s is an α-stage, $z \in K_s$, $s > x$ and $l'(e+1,s) \geq x$. Since $R(\alpha,[s,e+1]) = R(\alpha)$, x is enumerated into X at stage $[s,e+2]$. For the other inclusion, suppose that $x \in X^{[2n(\alpha)+1]} - X^{[2n(\alpha)+1]}_{s_\alpha}$. Then, x did not enter X through the action of any Q_β, so x entered X at Step 1.5

of Stage $[s, e+2]$, where s is an α-stage $> s_\alpha$. Since $R(\alpha, [s, e+1]) = R(\alpha)$, $x \in \{\langle z, 2n(\alpha)+1\rangle | \langle z, 2n(\alpha)+1\rangle \geq R(\alpha) \wedge z \in K\}$.

From (4), we now get $K \leq_{tt} X^{[2n(\alpha)+1]} \leq_{tt} X \leq_{tt} D^{[n]}$, contradicting assumption that $D^{[n]}$ is Turing-incomplete.

Thus, $X \neq [i]^{D^{[n]}}$, so $l'(e+1, s)$ is bounded as $s \to \infty$. This implies that the set of elements put into $X^{[2n(\alpha)+1]}$ by R_α is finite. As already mentioned, the set of elements put into $X^{[2n(\alpha)+1]}$ by Q_β requirements is finite. Thus, $X^{[2n(\alpha)+1]}$ is finite. This completes the proof of Part (v).

There are a few other modifications to the proof of Lemma 3.3 of [7] which are necessary to prove this lemma, but they are routine and are left to the reader. ∎

An ideal $\mathbf{I}$ of $\mathcal{C}_{tt}$ is called Σ^0_3 if $\{i | deg_{tt}(W_i) \in \mathbf{I}\}$ is Σ^0_3.

Theorem 2 *Let* $\mathbf{I}$ *be a* Σ^0_3*-ideal of* $\mathcal{C}_{tt}$ *consisting of only Turing-incomplete truth-table degrees. Then,* $\mathbf{I}$ *has an exact pair* $\mathbf{b}, \mathbf{c}$ *such that neither of* $\mathbf{b}$ *and* $\mathbf{c}$ *belongs to* $\mathbf{I}$.

Proof: This theorem follows from Lemma 1 as Theorem 3.2 follows from Lemma 3.3 in [7]. Namely, by a theorem of Yates (see [10], page 253) there exists a uniformly c.e. sequence of sets $(D_n)_{n\in\omega}$ such that $\{W_e | deg_{tt}(W_e) \in \mathbf{I}\} = \{D_n | n \in \omega\}$. Apply Lemma 1 and let $\mathbf{b} = deg_{tt}(B)$ and $\mathbf{c} = deg_{tt}(C)$. Then, $\mathbf{b}, \mathbf{c}$ form an exact pair for $\mathbf{I}$ and do not belong to $\mathbf{I}$. ∎

Theorem 3 *If* $\mathbf{a}$ *is a Turing-incomplete c.e. tt-degree, then* $\mathbf{a}$ *is branching.*

Proof: Let $\mathbf{a}$ be a Turing-incomplete c.e. truth-table degree and define $\mathbf{I} = \{\mathbf{d} \in \mathcal{C}_{tt} | \mathbf{d} \leq \mathbf{a}\}$. Fix a c.e. set $A \in \mathbf{a}$. We have

$$deg_{tt}(W_i) \in \mathbf{I} \Leftrightarrow (\exists e)(\forall x)(\forall t)(\exists s \geq t)[\{e\}_s(x)\downarrow \wedge W_{i,s}(x) = [e]^{A_s}(x)],$$

so $\mathbf{I}$ is a Σ^0_3-ideal. All the members of $\mathbf{I}$ are Turing-incomplete. Thus, by Theorem 2, there is an exact pair $\mathbf{b}, \mathbf{c}$ for $\mathbf{I}$ such that neither $\mathbf{b}$ nor $\mathbf{c}$ belongs to $\mathbf{I}$. It follows that $\mathbf{a} < \mathbf{b}, \mathbf{c}$ and that $\mathbf{a}$ is the meet of $\mathbf{b}$ and $\mathbf{c}$ in $\mathcal{C}_{tt}$, so $\mathbf{a}$ is branching. ∎

3 Turing Complete Truth-Table Degrees

In this section, we present the main result of the paper, namely, that every Turing-complete tt-incomplete c.e. degree is branching.

Theorem 4 *If* $\mathbf{b}$ *is a Turing-complete, tt-incomplete c.e. tt-degree, then* $\mathbf{b}$ *is branching in* $\mathcal{C}_{tt}$.

Proof: Let B be a c.e. set in $\mathbf{b}$. We want to construct c.e. sets A_0, A_1 such that $A_i \not\leq_{tt} B$ for $i = 0, 1$ and such that $deg_{tt}(A_0 \oplus B)$ and $deg_{tt}(A_1 \oplus B)$ have meet $\mathbf{b}$ in $\mathcal{C}_{tt}$. We will actually meet the following requirements $R_{e,i}$ for $e \in \omega$ and $i = 0, 1$:

$$R_{e,i} : [e]^{A_i \oplus B} \neq A_{1-i}.$$

These requirements certainly imply $A_i \not\leq_{tt} B$ for $i = 0, 1$ and the meet requirements will hold automatically due to the way we achieve the $R_{e,i}$ requirements.

An intuitive and oversimplified description of how we meet requirement $R_{e,i}$ is the following. Because $K \not\leq_{tt} B$, we know that if we make A_i look like ω (above higher priority restraint) and we make A_{1-i} look like K (on a row), then there must be an x such that $[e]^{A_i \oplus B}(x) \neq A_{1-i}(x)$. Not only is there such an x, but K can find it; that is, there is a function $f \leq_T K$ such that if A_i looks like ω up to $f(e)$ (above higher priority restraint) and A_{1-i} looks like K on a specified row up to $f(e)$, then $[e]^{A_i \oplus B} \neq A_{1-i}$. Since $K \leq_T B$, there is a j_0 with $\{j_0\}^B = f$. Then, for all x, $\{j_0\}_s^{B_s}(x) \rightarrow f(x)$ as $s \rightarrow \infty$. At stage s of the construction, we will make A_i look like ω up through $\{j_0\}_s^{B_s}(e)$ and A_{1-i} look like K_s through the same number (on a row). In fact, if we get a new computation $\{j_0\}_s^{B_s}(e)$ at stage s, we make A_i look like ω and A_{1-i} look like K up through s. Because of the usual convention that $\{j_0\}_s^{B_s}(e) \leq s$, this suffices to meet $R_{e,i}$. The reason we put potentially many more numbers than necessary to meet $R_{e,i}$ into A_i is to ensure that meet requirements hold.

In outline, the argument that $deg_{tt}(A_0 \oplus B)$ and $deg_{tt}(A_1 \oplus B)$ have meet $deg_{tt}(B) = \mathbf{b}$ in $\mathcal{C}_{tt}$ is as follows. It suffices to show that, given e_0, e_1 with $\{e_0\}, \{e_1\}$ total, there is an e' with $\{e'\}$ total such that for all x, $[e']^B(x)$ is either $[e_0]^{A_0 \oplus B}(x)$ or $[e_1]^{A_1 \oplus B}(x)$. Given x, $[e']$ finds $M = \max\{|[e_0](x)|, |[e_1](x)|\}$ and then at stage M finds the highest priority $R_{e,i}$ such that $\{j_0\}_M^{B_M}(e)$ converges by an incorrect computation. Then, at some later stage, every number greater than the current restraint on $R_{e,i}$ and less than M will be dumped into A_i and no number smaller than the current restraint will be put in A_i. This allows $[e']$ to correctly compute $[e_i]^{A_i \oplus B}(x)$. We will see that this is in fact a truth-table reduction.

To begin the formal construction, we define

$$f(\langle e, m, k \rangle) = \mu y[\{e\}(\langle y, m \rangle)\uparrow \vee [e]^{(\omega^{[\geq m]} \cup F_k) \oplus B}(\langle y, m \rangle)\downarrow \neq K(y)].$$

Since $\omega^{[\geq m]} \cup F_k$ is recursive and $K \not\leq_{tt} B$, f is total. Since B is Turing complete, $f \leq_T B$, say $f = \{j_0\}^B$. The extra parameters in the definition of f, compared to the intuitive discussion just given, are to account for the effect of higher priority requirements. If the requirements of higher priority than $R_{e,i}$ control the first m rows and F_k is the set of numbers that the higher priority requirements put into $\omega^{[<m]}$, then $\omega^{[\geq m]} \cup F_k$ is the value A_i would have if $R_{e,i}$ puts every element it can into A_i.

We make the following assumptions:

$$\text{if } x = \langle e,m,k\rangle \text{ and } \{e\}(\langle f(x),m\rangle)\downarrow, \text{ then } u(B;j_0,x) > \\ |[e](\langle f(x),m\rangle)| \text{ and } K(f(x)) = K_{u(B;j_0,x)}(f(x)); \tag{5}$$

$$\{j_0\}_0^D(y)\uparrow \text{ for all } D, y; \tag{6}$$

$$\text{if } \{j_0\}_s^{B_s}(x)\downarrow \text{ and } B_s \upharpoonright u(B_s;j_0,x,s) \neq B_{s+1}\upharpoonright u(B_s;j_0,x,s), \\ \text{then } \{j_0\}_{s+1}^{B_{s+1}}(x)\uparrow. \tag{7}$$

The last of these assumptions is the usual "hat trick". With these assumptions, we have the following lemma, which summarizes how we will meet the $R_{e,i}$ requirements.

Lemma 5 *Let $\{e\}$ be total and let s be such that $\{j_0\}_s^{B_s}(\langle e,m,k\rangle)\downarrow$ and*

$$B_s\upharpoonright u(B_s;j_0,\langle e,m,k\rangle,s) = B\upharpoonright u(B_s;j_0,\langle e,m,k\rangle,s).$$

If

$$A_i^{[\geq m]}\upharpoonright s = \omega^{[\geq m]}\upharpoonright s, \tag{8}$$

$$A_i^{[<m]} = F_k, \text{ and} \tag{9}$$

$$(\forall y \leq s)(\langle y,m\rangle \in A_{1-i} \leftrightarrow y \in K_s) \tag{10}$$

then $R_{e,i}$ is met.

Proof: Let $x = \langle e,m,k\rangle$ and $u = u(B_s;j_0,x,s)$. By the usual convention, $s \geq u, f(x)$ and by the assumption of the lemma, $u = u(B;j_0,x)$. Because $\{e\}$ is total, we have, by definition of f, that $[e]^{(\omega^{[\geq m]}\cup F_k)\oplus B}(\langle f(x),m\rangle)\downarrow \neq K(f(x))$. Thus, the lemma will be proven if we can show that $[e]^{A_i\oplus B}(\langle f(x),m\rangle) = [e]^{(\omega^{[\geq m]}\cup F_k)\oplus B}(\langle f(x),m\rangle)$ and that $K(f(x)) = A_{1-i}(\langle f(x),m\rangle)$. Towards establishing the first of these equalities, note that by (5), $s \geq u > |[e](\langle f(x),m\rangle)|$. Thus, (8) and (9) give us the desired equality, since A_i equals $\omega^{[\geq m]}\cup F_k$ up to the length of the appropriate truth table. For the second equality, we have, by (10), $A_{1-i}(\langle f(x),m\rangle) = K_s(f(x))$. Since $s \geq u$, (5) gives $K_s(f(x)) = K(f(x))$, completing the proof. ∎

Let $(R_n)_{n\in\omega}$ be an effective ordering of the $R_{e,i}$'s. The construction will be a finite injury one. From time to time during the construction, an attack may be begun on a requirement R_n. The attack may later be canceled. At most one attack on a given requirement will be active at a time. When an attack is made on R_n, two parameters m_n and x_n will be defined. If the attack is later canceled, then these values become undefined.

$R_n = R_{e,i}$ *requires attention* at stage $s+1$ if either

$$R_n \text{ is not under attack at the end of stage } s \tag{11}$$

or

$$\text{there is an attack on } R_n \text{ at the end of stage } s,\ \{j_0\}_s^{B_s}(x_{n,s})\downarrow \text{ and } \{j_0\}_{s-1}^{B_{s-1}}(x_{n,s})\uparrow. \tag{12}$$

Construction:

Stage 0: Do nothing.

Stage s+1: Find the highest priority requirement R_n that requires attention. Let $R_n = R_{e,i}$. Proceed according to which case in the definition of requires attention holds.

(11) holds: Begin an attack on R_n. Set $m_n = s$ and $x_n = \langle e, s, k\rangle$, where $F_k = A_{i,s}^{[<s]}$. Put $\langle y, m_n\rangle$ into A_i for all $y \leq s$. For all $y \in K_s$, put $\langle y, m_n\rangle$ into A_{1-i}.

(12) holds: Put $\langle y, z\rangle$ into A_i for all y, z with $m_n \leq z \leq s$ and $y \leq s$. For all $y \in K_s$, put $\langle y, m_n\rangle$ into A_{1-i}.

In either case, cancel all attacks on $R_{n'}$ with $n' > n$.

End of Construction

Lemma 6 *For each n, R_n requires attention only finitely often and an uncanceled attack is made on R_n.*

Proof: Suppose that the result is true for all $n' < n$. Take s_0 such that no $R_{n'}$ with $n' < n$ requires attention at a stage $> s_0$. If no attack on R_n is in progress at the end of stage s_0, one will be made at stage $s_0 + 1$, so there is an attack in progress at the end of stage $s_0 + 1$. This attack will never be canceled and R_n never requires attention via (11) after stage $s_0 + 1$. Let x_n, m_n be the values associated with the final attack and take $s_1 \geq s_0 + 1$ such that $\{j_0\}_{s_1}^{B_{s_1}}(x_n)\downarrow$ and $B_{s_1} \restriction u(B_{s_1}; j_0, x_n, s_1) = B \restriction u(B_{s_1}; j_0, x_n, s_1)$. Then, R_n does not require attention via (12) after stage $s_1 + 1$, so R_n requires attention only finitely often. ∎

Lemma 7 *Each R_n is met.*

Proof: Let $R_n = R_{e,i}$. We may assume that $\{e\}$ is total, since otherwise the result is immediate. Suppose that the final attack on R_n is begun at stage $s_0 + 1$. Then, the final values of m_n and x_n are s_0 and $\langle e, s_0, k\rangle = \langle e, m_n, k\rangle$, respectively, where $F_k = A_{i,s_0}^{[<s_0]}$. Let $s + 1 \geq s_0 + 1$ be the last stage at which R_n requires attention. We will use Lemma 5 applied with $m = m_n$ to show that R_n is met. First, we want to show that $\{j_0\}_s^{B_s}(x_n)\downarrow$ and $B_s \restriction u(B_s; j_0, x_n, s) = B \restriction u(B_s; j_0, x_n, s)$. Suppose not. Let $t > s$ be the first stage at which $\{j_0\}_t^{B_t}(x_n)\downarrow$ by a B-correct computation. Then, by the "hat trick" (7), $\{j_0\}_{t-1}^{B_{t-1}}(x_n)\uparrow$, so R_n requires attention at stage $t + 1 > s + 1$, contradicting our assumption.

To show (8), note that if $\langle y, z\rangle \in \omega^{[\geq m_n]} \restriction s$, then $m_n \leq z \leq s$ and $y \leq s$. We may have either (11) or (12) holding at stage $s + 1$ (depending on whether

$s = s_0$ or $s > s_0$), but in either case, $\langle y, z\rangle$ is put into A_i at stage $s+1$. Thus, $A_i^{[\geq m_n]}\restriction s = \omega^{[\geq m_n]}\restriction s$.

Towards showing the last two facts we need in order to apply Lemma 5, note that by construction, if some $R_{n'}$ puts a number y into $A_0 \cup A_1$ at stage $s'+1$, then $y \in \omega^{[\geq m_{n',s'}]} \cap \omega^{[\leq s']}$. Also, each $R_{n'}$ with $n' > n$ is canceled at stage s_0+1 and any value assigned to $m_{n'}$ at a later stage will be $> s_0$. Furthermore, no $R_{n'}$ with $n' < n$ receives attention at a stage $\geq s_0+1$. It follows from these facts that $F_k = A_{i,s_0}^{[<s_0]} = A_i^{[<s_0]} = A_i^{[<m_n]}$, so (9) holds.

Finally, to establish (10), suppose that $y \in K_s$. Then, $\langle y, m_n\rangle \in A_{1-i,s} \subseteq A_{1-i}$ because R_n receives attention at stage $s+1$. Conversely, if $\langle y, m_n\rangle \in A_{1-i}$, then by the considerations of the previous paragraph, $\langle y, m_n\rangle$ was put into A_{1-i} by R_n. Since R_n last acts at stage $s+1$, if $\langle y, m_n\rangle \in A_{1-i}$, then $y \in K_s$. Thus, (10) holds and by Lemma 5, R_n is met. ∎

Lemma 8 *If $\{e_0\}$ and $\{e_1\}$ are both total, then there is an e' such that $\{e'\}$ is total and for all x, either $[e']^B(x) = [e_0]^{A_0\oplus B}(x)$ or $[e']^B(x) = [e_1]^{A_1\oplus B}(x)$. (So, if $[e_0]^{A_0\oplus B} = [e_1]^{A_1\oplus B} = g$, then $[e']^B = g$.)*

Proof: Given x, we compute the truth table $[e'](x)$ as follows: Find the truth tables $[e_0](x)$ and $[e_1](x)$ and let $M = \max\{|[e_0](x)|, |[e_1](x)|\}$. Carry out the construction through the end of stage $s_0 = M+1$. Suppose that at the end of stage s_0 there are attacks in progress on requirements $R_0, \ldots, R_{n_0}$. (Note that by construction, if there is an attack in progress on R_n at the end of a stage s and $n' < n$, then there is an attack in progress on $R_{n'}$ at the end of stage s.) The truth table $[e'](x)$ will have length M and will query all the numbers $0, \ldots, M-1$. Given $\sigma \in 2^M$, find the minimum $n \leq n_0$, if any, such that either $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\uparrow$ or $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\downarrow$ and $B_{s_0-1}\restriction u(B_{s_0-1}; j_0, x_{n,s_0}, s_0-1) \neq \sigma\restriction u(B_{s_0-1}; j_0, x_{n,s_0}, s_0-1)$. (Note that if $n \leq n_0$ and $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\downarrow$, then $u(B_{s_0-1}; j_0, x_{n,s_0}, s_0-1) < s_0-1 = M = |\sigma|$.) If no such n exists, then $[e']^\sigma(x)$ is $[e_0]^{A_{0,s_0}\oplus\sigma}(x)$. If such a minimal n exists, then let R_n be $R_{e,i}$. We define $[e']^\sigma(x)$ to be $[e_i]^{(A_{i,s_0}^{[<m_{n,s_0}]}\cup\omega^{[\geq m_{n,s_0}]})\oplus\sigma}(x)$. Then, e' is certainly a truth table reduction. We have to show that $[e']^{B\restriction M}(x)$ is either $[e_0]^{A_0\oplus B}(x)$ or $[e_1]^{A_1\oplus B}(x)$.

Suppose first that there is no $n \leq n_0$ such that either $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\uparrow$ or $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\downarrow$ and, letting $u = u(B_{s_0-1}; j_0, x_{n,s_0}, s_0-1)$, $B_{s_0-1}\restriction u \neq (B\restriction M)\restriction u = B\restriction u$. Then, by induction on $s \geq s_0+1$, it follows that at no such stage s does any R_n with $n \leq n_0$ receive attention. Thus, $A_{0,s_0}^{[<s_0]} = A_{0,s_0}^{[\leq M]} = A_0^{[\leq M]}$. It follows that $A_{0,s_0}\restriction M = A_0\restriction M$ and hence $[e']^{B\restriction M}(x) = [e_0]^{A_{0,s_0}\oplus(B\restriction M)}(x) = [e_0]^{A_0\oplus B}(x)$, as desired.

Now suppose that n is minimal such that $n \leq n_0$ and either $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\uparrow$ or $\{j_0\}_{s_0-1}^{B_{s_0-1}}(x_{n,s_0})\downarrow$ and, letting $u = u(B_{s_0-1}; j_0, x_{n,s_0}, s_0-1)$, $B_{s_0-1}\restriction u \neq (B\restriction M)\restriction u = B\restriction u$. Then, no $R_{n'}$ with $n' < n$ requires attention at any stage

$s \geq s_0 + 1$, so $A_{j,s_0}^{[<m_{n,s_0}]} = A_j^{[<m_{n,s_0}]}$, for $j = 0, 1$. Let R_n be $R_{e,i}$. At some stage $t \geq s_0 + 1$, R_n will receive attention. At stage t, all numbers $\langle y, z \rangle$ with $y < t$ and $m_{n,s_0} \leq z < t$ are put into A_i. Since $t \geq s_0 + 1 > M$, $A_i^{[\geq m_{n,s_0}]} \restriction M = \omega^{[\geq m_{n,s_0}]} \restriction M$. Thus,

$$\begin{aligned} & A_i \restriction M = A_i^{[<m_{n,s_0}]} \restriction M \cup A_i^{[\geq m_{n,s_0}]} \restriction M \\ & = A_{i,s_0}^{[<m_{n,s_0}]} \restriction M \cup \omega^{[\geq m_{n,s_0}]} \restriction M = (A_{i,s_0}^{[<m_{n,s_0}]} \cup \omega^{[\geq m_{n,s_0}]}) \restriction M. \end{aligned}$$

Hence, $[e']^{B \restriction M}(x) = [e_i]^{(A_{i,s_0}^{[<m_{n,s_0}]} \cup \omega^{[\geq m_{n,s_0}]}) \oplus (B \restriction M)}(x) = [e_i]^{A_i \oplus B}(x)$, as desired. ∎

This completes the proof of Theorem 4.

4 Open Problems

We conclude the paper by listing three problems related to the topic of the paper that we are unable to solve using the methods developed here.

Q1) Does every Σ_3^0-ideal $\mathbf{I}$ of $\mathcal{C}_{tt}$, different from $\mathcal{C}_{tt}$, have an exact pair $\mathbf{b}, \mathbf{c}$ such that neither of $\mathbf{b}$ and $\mathbf{c}$ belongs to $\mathbf{I}$?

(The restriction in (Q1) that $\mathbf{b}, \mathbf{c}$ do not belong to $\mathbf{I}$ is relevant only when $\mathbf{I}$ is a principal ideal. If $\mathbf{I}$ is not principal, then neither half of an exact pair for $\mathbf{I}$ can belong to $\mathbf{I}$.) Theorem 2 gives an affirmative answer to (Q1) when $\mathbf{I}$ does not contain any Turing-complete degrees. Theorem 4 gives an affirmative answer to (Q1) when $\mathbf{I}$ is a principal ideal and contains Turing complete degrees. An affirmative answer to (Q1) for arbitrary $\mathbf{I}$ would make it possible to obtain our result that every *tt*-incomplete c.e. truth-table degree is branching in $\mathcal{C}_{tt}$ from a more general uniformity of $\mathcal{C}_{tt}$. The method of Section 3 does not appear to give an affirmative answer to (Q1) in the remaining open case, namely, when $\mathbf{I}$ is nonprincipal and contains Turing-complete degrees.

A set A is said to be *bounded truth-table (btt) reducible* to a set B if there are numbers e and k such that $A = [e]^B$ and for all x, the truth table $[e](x)$ queries at most k numbers.

Q2) Is every *btt*-incomplete c.e. bounded truth-table degree branching in $\mathcal{C}_{btt}$?

The method of Section 2, unchanged, shows that every Turing-incomplete c.e. bounded truth-table degree is branching in $\mathcal{C}_{btt}$. However, the proof in Section 3 does not show that every Turing-complete btt-incomplete bounded truth-table degree is branching in $\mathcal{C}_{btt}$. This is because in Lemma 8, the truth-table reduction $[e']$ is not bounded, even if the reductions $[e_0]$ and $[e_1]$ are bounded.

So far, when a structural property of $\mathcal{C}_{tt}$ has been shown, it has turned out to be possible to show the same result, by a more difficult proof, for $\mathcal{D}_{tt}(\leq \mathbf{0}'_{tt})$ and then for the structure $\mathcal{D}_{wtt}(\leq \mathbf{0}'_{wtt})$, where *wtt* stand for weak truth-table reducibility (i.e., Turing reducibility with a recursive bound on the use function). In fact, no elementary difference is known between these three structures. In view of the results in this paper, it is natural to ask

Q3) Is every degree in $\mathcal{D}_{tt}(\leq \mathbf{0}'_{tt})$, other than $\mathbf{0}'_{tt}$, branching in $\mathcal{D}_{tt}(\leq \mathbf{0}'_{tt})$?

A negative answer to this question would be the most interesting, but would be surprising. We believe that the method from Section 3, with additional complications, can be used to give an affirmative answer to (Q3) when the degree is Turing-complete, but the situation for Turing-incomplete degrees is unknown – the exact pair result from Section 2 is not easily adapted to this case.

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