The $\forall\exists$-Theory of $\mathcal{R}(\leq, \vee, \wedge)$ is Undecidable

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May 15, 2003

Abstract

The three quantifier theory of $(\mathcal{R}, \leq_T)$, the recursively enumerable degrees under Turing reducibility, was proven undecidable by Lempp, Nies and Slaman [1998]. The two quantifier theory includes the lattice embedding problem and its decidability is a long standing open question. A negative solution to this problem seems out of reach of the standard methods of interpretation of theories because the language is relational. We prove the undecidability of a fragment of the theory of $\mathcal{R}$ that lies between the two and three quantifier theories with $\leq_T$ but includes function symbols.

Theorem: ***The two quantifier theory of*** $(\mathcal{R}, \leq, \vee, \wedge)$***, the r.e. degrees with Turing reducibility, supremum and infimum (taken to be any total function extending the infimum relation on*** $\mathcal{R}$***) is undecidable.***

The same result holds for various lattices of ideals of $\mathcal{R}$ which are natural extensions of $\mathcal{R}$ preserving join and infimum when it exits.

The first author was partially supported by a VIGRE Postdoctoral Fellowship under NSF grant number 9983660 to Cornell University. The third author was partially supported by NSF Grant DMS-0100035. The authors offer their thanks to the referee for several helpful comments.

1 Introduction

A major theme in the study of degree structures of all types has been the question of the decidability or undecidability of their theories. This is a natural and fundamental question that is an important goal in the analysis of these structures. It also serves as a guide and organizational principle for the development of construction techniques and algebraic information about the structures. A decision procedure implies (and requires) a full understanding and control of the first order properties of a structure. Undecidability results typically require and imply some measure of complexity and coding in the structure. Once a structure has been proven undecidable, it is natural to try to determine both the extent and source of the complexity. One the one hand, one wants to determine the degree of the theory. On the other hand, one strives to find the dividing line between decidability and undecidability in terms of fragments of the theory. The first has frequently brought with it considerable information about second order properties such as definability and automorphisms. The second requires the most algebraic information and development of construction techniques.

Our interest here is in $\mathcal{R}$, the r.e. degrees under Turing reducibility, and some natural extensions of this structure, but, for the sake of comparison, we also discuss $\mathcal{D}$ and $\mathcal{D}(\leq \mathbf{0}')$, the Turing degrees of all sets and the ones below $\mathbf{0}'$. For $\mathcal{D}$; $\mathcal{D}(\leq \mathbf{0}')$ the results came fairly early. The first paper on the structure $\mathcal{D}$ of the Turing degrees as a whole, Kleene-Post [1954], developed the finite extension method (essentially Cohen forcing for one quantifier formulas of arithmetic) and proved that all finite partial orderings can be embedded in both $\mathcal{D}$ and $\mathcal{D}(\leq \mathbf{0}')$. As these structures are partial orderings, this suffices to show that the one quantifier ($\exists$) theories are decidable. (An existential sentence is true in either structure if and only if it is consistent with the theory of partial orders, or equivalently, if there is a partial order with a domain of size the number of variables in the formula.)

Once the embedding problem is settled, the next level of algebraic questions about the structures concern extension of embeddings. The first example here is density (or, from the other side minimal covers). A long development of construction techniques building on Spector's original construction [1956] of a minimal degree essentially by forcing with recursive trees lead to Lachlan's [1968] result that every countable distributive lattice is isomorphic to an initial segment of $\mathcal{D}$. This coding of distributive lattices is sufficient to get the undecidability of the theory as Lachlan [1968] notes. Combining these initial segment techniques with the finite extension method, Simpson [1977] showed that the theory of $\mathcal{D}$ is recursively isomorphic to $Th^2(\mathbb{N})$, true second order arithmetic.

Finding the dividing line between decidability and undecidability required Lerman's [1971] result that every finite lattice (not just the distributive ones) is isomorphic to an initial segment of $\mathcal{D}$. On one hand, combining this with the finite extension method solved the extension of embedding problem in such a way that it gave the decidability of the two quantifier ($\forall\exists$) theory of $\mathcal{D}$ (Shore [1978] and Lerman [see 1983, Appendix A]). (By the extension of embedding problem we mean determining for which partial orders

$\mathcal{X} \subseteq \mathcal{Y}$ does every embedding of $\mathcal{X}$ into $\mathcal{D}$ have an extension to one of $\mathcal{Y}$.) The ability to code all finite lattices also sufficed for Schmerl (see Lerman [1983, Appendix A]) to prove that the three quantifier ($\forall\exists\forall$) theory of $\mathcal{D}$ is undecidable.

A similar analysis of $\mathcal{D}(\leq 0')$ was then carried out first by a significant elaboration of the construction techniques to get enough initial segments results below $0'$ to give undecidability (Epstein [1979] and Lerman). Lerman then proved the full analog that every finite (even recursive) lattice is isomorphic to an initial segment of $\mathcal{D}(\leq 0')$ (Lerman [1983, ch. XII]). This immediately gives the undecidability of the three quantifier theory. Then these results were extended and combined with extension of embedding results below an arbitrary r.e. degree (Lerman and Shore [1988]) to get the decidability of the two quantifier theory. They were also used to show (Shore [1981]) that the theory of $\mathcal{D}(\leq 0')$ is recursively isomorphic to true first order arithmetic.

The road has been much longer for the analysis of the r.e. degrees, $\mathcal{R}$. It began with the finite injury (or $0'$) priority method of Friedberg [1957] and Muchnik [1956] that produced incomparable r.e. degrees and so an embedding of the simplest nontrivial Boolean algebra. The method sufficed to embed all finite (even countable) partial orderings (Sacks [1963]) and so decide the one quantifier theory of $\mathcal{R}$ in the same way that Kleene and Post's work decided that of $\mathcal{D}$ and $\mathcal{D}(\leq 0')$. As the r.e. degrees are dense (by the infinite injury (or $0''$) methods of Sacks [1964]), the next steps in the analysis could not follow the path laid out for $\mathcal{D}$. Many years of development of construction techniques and algebraic information ensued. Lachlan's monster (or $0'''$ injury) methods were eventually used by Harrington and Shelah [1982] to prove that $\mathcal{R}$ is undecidable. The degree of its theory, as by now one should expect, is also that of true first order arithmetic (Harrington and Slaman; Slaman and Woodin; Nies, Shore and Slaman [1998]).

The situation for these three degree structures is summarized in the following table:

	$\mathcal{R}$	$\mathcal{D}$	$\mathcal{D}(\leq \mathbf{0}')$
$\exists$	Dec	Dec	Dec
$\forall\exists$	?	Dec	Dec
$\forall\exists\forall$	Undec	Undec	Undec
Th	$Th(N)$	$Th^2(N)$	$Th(N)$

This leaves us with determining the boundary line between decidability and undecidability for $\mathcal{R}$. Once again, a long hiatus and much work on other developments led to the undecidability of the three quantifier theory by Lempp, Nies and Slaman [1998]. The extension of embedding problem was solved by Slaman and Soare [2001] but the question of the decidability of the two quantifier theory of $\mathcal{R}$ remains open. A major obstacle is the lattice embedding problem of determining which finite lattices can be embedded in $\mathcal{R}$. Despite some forty years of effort by many researchers on both embedding and nonembedding results, this question is still unsolved. The best result to date is Lerman [2000] which shows that the question for an important class of lattices is of degree at most

$0''$. Even if the lattice embedding problem is shown to be decidable, there are further difficulties related to Lachlan's [1966] nondiamond result that there is no embedding of the four element Boolean algebra into $\mathcal{R}$ that preserves both 0 and 1.

Thus we remain a long way from the decidability of the two quantifier theory of $\mathcal{R}$. On the other hand, the methods used to prove undecidability of other degree structures, interpretation of theories with simple fragments known to be undecidable, cannot work for the two quantifier theory of $\mathcal{R}$ with just $\leq_T$, or even any extension by relation symbols, since the most we can code into this fragment is the validity (perhaps in all finite models) of an $\forall\exists$ sentence in a finite relational language but this problem is always decidable. (The point here is that, since the language is relational, any such sentence with n variables is satisfiable if and only if it is satisfiable in some structure of size at most n. As there are only finitely many such structures, this question is decidable. The basic result is classical (Bernays and Schönfikel [1928] and Ramsey [1930]). Its application to interpretations in structures such as $\mathcal{R}$ is pointed out in Shore [1999, p. 179].)

The only hope for an undecidability result at the two quantifier level then is to add function symbols. One would then try to interpret some theory with function symbols or, more directly, to code register machines. (The coding of register machines is at the basis of much of the work on undecidability of various severely restricted quantification classes of formulas as in Börger, Grädel and Gurevich [1997].) The natural function on $\mathcal{R}$ is the join operator $\vee$. However, it is uniformly locally finite, i.e. the closure of any finite set is finite with size bounded by a fixed recursive function of the cardinality of the starting set and so cannot, on its own, be used to generate the infinite (or at least unbounded) structures need for coding even register machines. The next thing to try in terms of the known structural work on $\mathcal{R}$ is the infimum operator $\wedge$. This has the advantage that finitely generated substructures can be infinite (Lerman, Shore and Soare [1984]). The obvious problem with this approach is that not every pair of r.e. degrees has an infimum and so $\wedge$ is not a total function on $\mathcal{R}$ as is required. We can, of course, consider total extensions of the partial infimum relation but would not want the undecidability to be an artifact of our (perhaps perverse) choice of extension. Our solution is to prove undecidability in a sufficiently uniform way so that the proof is independent of the choice of extension. This we do for our main result.

Theorem 1.1 *For any total extension $\wedge$ of the partial infimum relation on $\mathcal{R}$, the two quantifier ($\forall\exists$) theory of $\mathcal{R}(\leq,\vee,\wedge)$ is undecidable.*

Now it is routine to eliminate $\vee$ by replacing it with its definition (as least upper bound) at the expense of adding one quantifier. Thus, for example, $\forall x,y\exists z(x\vee y\geq z)$ is equivalent to $\forall x,y\exists z\forall w(x,y\leq w\rightarrow w\geq z)$. This translation shows that the $\forall\exists$-theory of $\mathcal{R}(\leq,\vee)$ is reducible to the $\forall\exists\forall$-theory of $\mathcal{R}(\leq)$. The same would be true of $\wedge$ (as greatest lower bound) were it a total infimum function, i.e. $\forall x,y\exists z(x\wedge y\leq z)$ would be equivalent to $\forall x,y\exists z\forall w(x,y\geq w\rightarrow w\leq z)$. This needn't be true for arbitrary extensions of the partial infimum relation on $\mathcal{R}$ but the care that we take with our coding to guarantee that it works for all extensions allows us to argue that the $\forall\exists$ sentences of $\mathcal{R}(\leq, \vee, \wedge)$ that witness undecidability can, in fact, be replaced uniformly by equivalent $\forall\exists\forall$ sentences of $\mathcal{R}(\leq)$ so that the previous best result on undecidability is also a consequence of our proof.

Corollary 3.1 *(Lempp, Nies and Slaman [1998]): The three quantifier ($\forall\exists\forall$) theory of $\mathcal{R}(\leq)$ is undecidable.*

We will give the details of the proof in Section 3, once we explain the specific coding of register machines that we employ. As essentially similar codings of register machines can easily be carried out in lattices, the usual interpretation of lattices in $\mathcal{D}$ as initial segments shows that our main result also holds for the degrees as a whole and those below $0'$.

Corollary 3.2 *For any total extension $\wedge$ of the partial infimum relation on $\mathcal{D}$ ($\mathcal{D}(\leq 0')$), the two quantifier ($\forall\exists$) theory of $\mathcal{D}$ ($\mathcal{D}(\leq 0')$) with $\leq$, $\vee$ and $\wedge$ is undecidable.*

As for $\mathcal{R}$, the arguments here also imply the previous result that the $\forall\exists\forall$ theories of these structures are undecidable. At least in case of $\mathcal{D}$, the boundary here is very fine as Jockusch and Slaman [1993] have proven that the $\forall\exists$ theory of $\mathcal{D}(\leq, \vee)$ is decidable.

A new corollary of our proof of undecidability is one for the (lattice) structure $\mathcal{I}(\mathcal{R})$ of ideals of $\mathcal{R}$ with $\vee$ and $\wedge$. Here both operations are naturally total on the structure. $I \vee J$ is the ideal generated by $I \cup J$, i.e. the downward closure of $\{a \vee b \mid a \in I \ \&\ b \in J\}$, and $I \wedge J$ is the ideal $I \cap J$. This structure is an interesting one that has been studied, for example, by Calhoun [1993], Lerman and Calhoun [2001] and Nies [2003]. Also of interest are the structures $\mathcal{I}_n(\mathcal{R})$ for $n \geq 4$ of the Σ^0_n ideals of $\mathcal{R}$ (those ideals I such that $\{e \mid \deg(W_e) \in I\}$ is Σ^0_n) which are each lattices with the same operations as $\mathcal{I}(\mathcal{R})$. (Note that by standard index set results $\mathcal{I}_n(\mathcal{R})$ is trivial for $n = 1, 2$: If an ideal I of $\mathcal{R}$ does not contain $\mathbf{0}'$ (but does contain $\mathbf{0}$) then by Yates' representation theorem [1966] (see Soare [1987 XII, 1.3) applied to K, the complete Π^0_4 set is of the form $\{k \mid \forall e(W^{[e]}_{f(k)} \in I)\}$ for some recursive f and so I must be at least Σ^0_3. On the other hand, the class of Σ^0_3 ideals is not closed under $\vee$, as can be seen by considering a high degree $\mathbf{h}$ and a splitting of $\mathbf{h}$ into two low degrees $\mathbf{a}, \mathbf{b}$. The principal ideals generated by $\mathbf{a}$ and $\mathbf{b}$ are Σ^0_3 but their join is the one generated by $\mathbf{h}$ which is Σ^0_4 but not Σ^0_3.) Each of these lattices ($\mathcal{I}(\mathcal{R})$ and $\mathcal{I}_n(\mathcal{R})$ for $n \geq 4$) is a natural extension of $\mathcal{R}$ in the sense that the natural embedding taking a degree in $\mathcal{R}$ to the principal ideal it generates is an embedding that preserves order and join as well as infimum when it is defined.

Corollary 3.3 *The two quantifier ($\forall\exists$) theory of $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$ for $n \geq 4$), the lattice of (Σ_n) ideals of $\mathcal{R}$, with $\subseteq$, $\vee$ and $\wedge$ is undecidable.*

Once again, after we have the details of the coding in place, an algebraic analysis shows that the principal ideals generated by the degrees doing the coding in $\mathcal{R}$ perform the same job in $\mathcal{I}(\mathcal{R})$. As both $\vee$ and $\wedge$ are total functions on $\mathcal{I}(\mathcal{R})$, their routine elimination as described above gives the undecidability of the three quantifier theory.

Corollary 3.4 *The three quantifier ($\forall\exists\forall$) theory of $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$ for $n \geq 4$), the lattice of (Σ_n) ideals of $\mathcal{R}$, with just $\subseteq$ is undecidable.*

We also remark that similar algebraic observations show that we can characterize the degrees of the theories of these ideal structures. Indeed the ideas of Nies, Shore and Slaman [1998] would have sufficed as well.

Corollary 3.5 *The theory of $\mathcal{I}(\mathcal{R})$ is recursively isomorphic to that of true second order arithmetic and that of $\mathcal{I}_n(\mathcal{R})$ to that of true first order arithmetic for each $n \geq 4$.*

2 Coding Register Machines

In this section we will explain the algebraic aspects of our codings and derive the main theorem, assuming these codings can be interpreted in $\mathcal{R}$. The next section will provide the proofs of the corollaries about other degree structures. The final section will supply the recursion theoretic arguments to show that the structures described here can be realized in the r.e. degrees.

We begin with a standard description of the k-register machines of Shepherdson and Sturgis [1963] and Minsky [1961] and their representation in predicate logic as in Nerode and Shore [1997, III.8] or Börger, Grädel and Gurevich [1997, 2.1]).

A *k-register machine* consists of k many storage locations called registers. Each register contains a natural number. There are only two types of operations that these machines can perform in implementing a program. First, they can increase the content of any register by one and then proceed to the next instruction. Second, they can check if any given register contains the number 0 or not. If so, they go on to the next instruction. If not, they decrease the given register by one and can be told to proceed to any instruction in the program. Formally, we define register machine programs and their execution as follows:

A *k-register machine program I* is a finite sequence $I_1, \ldots, I_t, I_{t+1}$ of instructions operating on a sequence of numbers $x_1, \ldots, x_k$, where each instruction I_m, for $m \leq t$, is of one of the following two forms:

(i) $x_i := x_i + 1$ (replace x_i by $x_i + 1$)

(ii) If $x_i \neq 0$, then $x_i := x_i - 1$ and go to j. (If $x_i \neq 0$, replace it by $x_i - 1$ and proceed to instruction I_j.)

It is assumed that after executing some instruction I_m, the execution proceeds to I_{m+1}, the next instruction on the list, unless I_m directs otherwise. The execution of such a program proceeds in the obvious way on any input of values for $x_1, \ldots, x_k$ (the initial content of the registers) to change the values of the x_i and progress through the list of instructions. The final instruction, I_{t+1}, is always a halt instruction. Thus, if I_{t+1} is ever reached, the execution terminates with the current values of the x_i. In general, we denote the assertion that an execution of the program I is at instruction I_m with values $n_1, \ldots, n_k$ of the variables by $I_m(n_1, \ldots, n_k)$.

The standard translation of a register machine M describes the action of M by a system of universal axioms in the language of one unary function s thought of as the successor function on $\mathbb{N}$. For technical reasons peculiar to our later coding in $\mathcal{R}$, we want to use distinct domains D_i with least elements 0_i and successor functions s_i for each register. In our application, these sets and operations will be defined from parameters in $\mathcal{R}$. For now, we describe the axioms needed in predicate logic with additional k-ary relations P_m corresponding to the instructions I_m.

For each instruction I_m, $1 \leq m \leq t$, include an axiom of the appropriate form:

(i) $P_m(x_1, \ldots, x_k) \to P_{m+1}(x_1, \ldots, x_{i-1}, s_i(x_i), x_{i+1}, \ldots, x_k)$.

(ii) $P_m(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k) \to P_{m+1}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_k)$
$\wedge\ P_m(x_1, \ldots, x_{i-1}, s_i(y), x_{i+1}, \ldots x_k) \to P_j(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$.

(Note that being a successor is equivalent to being nonzero.)

Let $P(I)$ be the finite set of universal axioms corresponding in this translation to register program I. It is easy to prove that, program I halts on input $(n_1, \ldots, n_k)$ if and only if the sentence $F_k(n_1, \ldots, n_k) \equiv P_1(s^{n_1}(0), \ldots s^{n_k}(0)) \to \exists x_1, \ldots, \exists x_k[P_{t+1}(x_1, \ldots, x_k)]$ is a logical consequence of $P(I)$. More specifically for our purposes, the machine halts if and only if $F_k(n_1, \ldots, n_k)$ is true in every model of $P(I)$ in any class of structures that contains ones isomorphic to the standard model (where each s_i on D_i is isomorphic to the standard successor function s on $\mathbb{N}$) with all possible recursively enumerable k-ary relations P_m on $D_1 \times \cdots \times D_k$. (Validity implies truth in all the structures in our class and if I fails to halt, the standard interpretation of the predicates as the r.e. relations $I_m(n_1, \ldots, n_k)$ gives a structure of the required form in which $F(n_1, \ldots, n_k)$ is false.) As it is a classical fact (Shepherdson and Sturgis [1963]; Minsky [1961]) that the halting problem for 2-register machine programs is r.e. complete, it suffices to code all such standard models with binary predicates to get undecidability.

As usual for interpretations, we now want to provide formulas $\Delta_i(\vec{q}, x)$, $\Pi_m(\vec{q}, x, y)$ and terms $\sigma_i(\vec{q}, x)$ of $\mathcal{R}(\leq, \vee, \wedge)$ defining, for each choice of parameters $\vec{q}$, sets D_i ($i = 1, 2$), binary relations P_m on $D_1 \times D_2$ ($1 \leq m \leq t+1$) and unary functions s_i on D_i ($i = 1, 2$). We take q_1 and q_2 to be the interpretations of 0 in D_1 and D_2 respectively. We now interpret our formulas $P(I) \to F(n_1, \ldots, n_k)$ in the usual way. We relativize the quantifiers to the appropriate domain, i.e. $\exists x_i(\ldots)$ becomes $\exists x_i(\Delta_i(\vec{q}, x) \wedge \ldots)$ and

$\forall x_i(\dots)$ becomes $\forall x_i(\Delta_i(\vec{q},x) \to \dots)$. We then replace occurrences of $s_i(x_i)$ by $\sigma_i(\vec{q},x_i)$ and ones of $P_m(x_1,x_2)$ by $\Pi_m(\vec{q},x_1,x_2)$. We indicate this translation by $*$. We also need a correctness condition Θ that says that $q_i \in D_i$ and the σ_i define functions on the D_i: $\Delta_1(\vec{q},q_1) \wedge \Delta_2(\vec{q},q_2) \wedge \ \forall x_1(\Delta_1(\vec{q},x_1) \to \Delta_1(\vec{q},\sigma_1(x_1)) \wedge \forall x_2(\Delta_2(\vec{q},x_2) \to \Delta_2(\vec{q},\sigma_2(x_2))$. The class of sentences of $\mathcal{R}(\leq,\vee,\wedge)$ that we want will then be those of the form $\forall\vec{q}[\Theta \to (P(I)^* \to F_2^*)]$ where I ranges over programs for 2-register machines.

As long as the class of structures given by all choices of parameters $\vec{q}$ includes ones isomorphic to the standard model with all possible r.e. relations as the P_m, truth in $\mathcal{R}$ for this class of sentences will be undecidable. It is clear that to get these sentences to be $\forall\exists$ ones it is sufficient to get quantifier free definitions (Δ_i and Π_m) of the domains and relations (and the worst that would work would be equivalent Σ_1 and Π_1 definitions). As long as there are realizations of the D_i as a uniformly low independent set of degrees in $\mathcal{R}$, we can define arbitrary r.e. relations on them from parameters in a quantifier free form by using the following special case of Lemma 7.1 of Nies, Shore and Slaman [1998]:

Lemma 2.1 *(Nies, Shore and Slaman [1998]) If $\langle \mathbf{a}_j \rangle$ is a uniformly r.e. independent set with $\oplus\mathbf{a}_i$ low and S is any r.e. set then there are $\mathbf{u},\mathbf{v}$ such that $S = \{i : \mathbf{u} \leq \mathbf{a}_i \vee \mathbf{v}\}$.*

If we assume, for example, that $D_i = \{\mathbf{g}_{2j+i} : j \in \omega\}$ (identified with $\mathbb{N}$ in the obvious way) for some independent set of degrees $\mathbf{g}_l$ with $\oplus\mathbf{g}_l$ low, then we can apply the lemma to the set of degrees $\{\mathbf{g}_{2j} \vee \mathbf{g}_{2k+1} : j,k \in \omega\}$ with $S_m = \{\langle j,k\rangle : P_m(j,k)\}$ for any r.e. relation P_m to provide parameters $\mathbf{u}_m,\mathbf{v}_m$ such that the formula $\mathbf{u}_m \leq x_1 \vee x_2 \vee \mathbf{v}_m$ defines the isomorphic copy of P_m on $D_1 \times D_2$ and can be taken as the desired quantifier free Π_m. Thus the source of all our concerns is providing a quantifier free definition from parameters of a uniformly r.e. independent set $\langle \mathbf{g}_l \rangle$ with a term of $\mathcal{R}$ that gives the successor relation on them. (Once we have such a set we can pick out the even and odd parts using the same lemma (or exact pairs) and then take the successor functions on each of these two disjoint sets to be simply the two-fold iteration of the original successor function.)

The two known methods for constructing independent sets definable from parameters are essentially those of Harrington and Shelah [1982] and Slaman and Woodin (see Nies, Shore and Slaman [1998]). The sets they defined from parameters are as follows:

- $\mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c}) = \{\mathbf{g} \leq \mathbf{r} : \mathbf{g}$ is maximal s.t. $\mathbf{g} \vee \mathbf{b} \not\geq \mathbf{c}\}$
- $\mathbf{SW}(\mathbf{r},\mathbf{p},\mathbf{q}) = \{\mathbf{g} \leq \mathbf{r} : \mathbf{g}$ is minimal s.t. $\mathbf{g} \vee \mathbf{p} \geq \mathbf{q}\}$.

Here the elements $\mathbf{g}_i$ of the sets typically constructed are uniformly r.e. and independent while $\mathbf{r}$ is taken to be their effective sum and can be made low. Thus the only problem is that the definitions of these sets requires a universal quantifier. We could reduce this to a quantifier free definition by requiring that they define the same set $\mathbf{G}$, for then

- $\mathbf{G}(\mathbf{r},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q}) = \{\mathbf{g} \leq \mathbf{r} : \mathbf{g} \vee \mathbf{p} \geq \mathbf{q} \,\&\, \mathbf{g} \vee \mathbf{b} \not\geq \mathbf{c}\}$.

As a technical convenience that simplifies the construction we note that if we have an HS set $\langle \mathbf{g}_i \rangle$ defined from parameters $\mathbf{r}, \mathbf{b}$ and $\mathbf{c}$ then we can weaken the conditions corresponding to the definition of the SW set to require only that, for each i, $\mathbf{g}_i \vee \mathbf{p} \geq \mathbf{q}$ and, for any $\mathbf{w} \leq \mathbf{g}_i$, if $\mathbf{w} \vee \mathbf{p} \geq \mathbf{q}$ then $\mathbf{w} = \mathbf{g}_i$. This clearly suffices to show that $\mathbf{G}(\mathbf{r},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q}) = \mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c})$. (That $\mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c}) \subseteq \mathbf{G}(\mathbf{r},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q})$ follows from the condition that $\mathbf{g}_i \vee \mathbf{p} \geq \mathbf{q}$ for each $\mathbf{g}_i \in \mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c})$. To see that $\mathbf{G}(\mathbf{r},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q}) \subseteq \mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c})$, consider any $\mathbf{w} \in \mathbf{G}(\mathbf{r},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q})$. By the maximality condition on $\mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c})$ there is a $\mathbf{g}_i \in \mathbf{HS}(\mathbf{r},\mathbf{b},\mathbf{c})$ such that $\mathbf{w} \leq \mathbf{g}_i$. Now our weakened requirements guarantee that $\mathbf{w} = \mathbf{g}_i$ as required.) Thus we wish to show that there are parameters $\mathbf{r}, \mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q}$ such that the set $\mathbf{G} = \langle \mathbf{g}_i : i \in \omega \rangle$ they define is uniformly r.e. and independent with a low sum.

In addition, we want to define a successor function on these degrees. Actually, we define one on the $\mathbf{g}_{2j}$ taking $\mathbf{g}_{2j}$ to $\mathbf{g}_{2j+2}$ and so our required domain will be these degrees. (Again they can be defined from parameters using the lemma as described above but in fact our construction will also build a degree $\mathbf{f}_1$ such that $\mathbf{D} = \{\mathbf{g}_{2i}\}_{i\in\omega} = \{\mathbf{x} : \mathbf{x} \in \mathbf{G} \,\&\, \mathbf{x} \leq \mathbf{f}_1 \ \vee \mathbf{x} = \mathbf{g}_0\}$.) We use the effective successor structure from Shore [1981] employed in Nies, Shore and Slaman [1998]. This calls for the construction of additional parameters $\mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1$ such that, for $i \in \omega$, $(\mathbf{g}_{2i} \vee \mathbf{e}_0) \wedge \mathbf{f}_0 = \mathbf{g}_{2i+1}$ and $(\mathbf{g}_{2i+1} \vee \mathbf{e}_1) \wedge \mathbf{f}_1 = \mathbf{g}_{2i+2}$. We can then define the desired successor function on $\mathbf{D}$ by $s(\mathbf{g}_{2i}) = (((\mathbf{g}_{2i} \vee \mathbf{e}_0) \wedge \mathbf{f}_0) \vee \mathbf{e}_1) \wedge \mathbf{f}_1 = \mathbf{g}_{2i+2}$. The required result is then the following:

Theorem 2.2 *There are r.e. sets $R, B, C, P, Q, E_0, E_1, F_0, F_1, \langle H_i : i \in \omega \rangle$ and $\langle G_i : i \in \omega \rangle$ with $R = \oplus G_i$; $H_i = \oplus_{k \neq i} G_k$; $F_0 = \oplus G_{2k+1}$ and $F_1 = \oplus G_{2k+2}$ such that (for all i and W)*

1. $G_i \not\leq_T H_i$.
2. R is low.
3. If $W \leq_T G_i$ and $Q \leq_T W \oplus P$ then $G_i \leq_T W$.
4. If $W \leq_T R$ and $C \not\leq_T W \vee B$ then $(\exists k) W \leq_T G_k$.
5. $C \not\leq_T G_i \vee B$.
6. $Q \leq_T G_i \vee P$.
7. $\deg(G_{2i+1}) = \deg(G_{2i} \oplus E_0) \wedge \deg(F_0)$.
8. $\deg(G_{2i+2}) = \deg(G_{2i+1} \oplus E_1) \wedge \deg(F_1)$.

The proof of this theorem is given in the final section.

Note that as the structures required for the undecidability are coded by parameters such that all the infima needed to define σ_i exist in $\mathcal{R}$, the structures coded in $\mathcal{R}(\leq, \vee, \wedge)$ include all the ones needed for the undecidability for any (total) extension $\wedge$ of the partial infimum relation on $\mathcal{R}$. Thus the construction of r.e. degrees $\mathbf{r}, \mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1$ as described above suffices to prove our main result (Theorem 1.1) that the two quantifier theory of $\mathcal{R}(\leq, \vee, \wedge)$ is undecidable for any total extension $\wedge$ of the infimum relation.

We now turn to establishing the corollaries mentioned in the Introduction.

3 Applications to Other Structures

We first show that our codings provide a new proof of Lempp, Nies and Slaman's result that the $\forall\exists\forall$ theory of $\mathcal{R}(\leq)$ is undecidable. We need to find a translation of the sentences $\forall\vec{q}[\Theta \to (P(I)^* \to F_2^*)]$ of $\mathcal{R}(\leq, \vee, \wedge)$ into $\forall\exists\forall$ ones of $\mathcal{R}(\leq)$ which preserve truth in $\mathcal{R}$. Note first that the definitions Δ_i and Π_m of the domains D_i and predicates P_m use $\leq$ and $\vee$ but not $\wedge$. Our only use of the infimum operation is in the definitions σ_i of the terms representing the successor operations on the D_i. In our translation these terms are defined by composition from the successor function s on $\mathbf{D} = \{\mathbf{g}_{2i}\} = \{\mathbf{x} : \mathbf{x} \in \mathbf{G} \,\&\, \mathbf{x} \leq \mathbf{f}_1 \vee \mathbf{x} = \mathbf{g}_0\}$ given by $s(\mathbf{g}_{2i}) = (((\mathbf{g}_{2i} \vee \mathbf{e}_0) \wedge \mathbf{f}_0) \vee \mathbf{e}_1) \wedge \mathbf{f}_1 = \mathbf{g}_{2i+2}$. Our primary task then is to eliminate the uses of s in our formulas.

We begin with the correctness condition Θ which for s says that $\forall x \in \mathbf{D}(s(x) \in \mathbf{D})$, i.e. $(((\mathbf{g}_{2i} \vee \mathbf{e}_0) \wedge \mathbf{f}_0) \vee \mathbf{e}_1) \wedge \mathbf{f}_1 \in \mathbf{D}$. We use the set $\bar{\mathbf{D}} = \{\mathbf{g}_{2i+1}\} = \{\mathbf{x} : \mathbf{x} \in \mathbf{G} \,\&\, \mathbf{x} \leq \mathbf{f_0}\}$ as well and break up the condition into the conjunction of two similar assertions: $\forall \mathbf{x} \in \mathbf{D}((\mathbf{x} \vee \mathbf{e}_0) \wedge \mathbf{f}_0 \in \bar{\mathbf{D}})$ and $\forall \mathbf{x} \in \bar{\mathbf{D}}((\mathbf{x} \vee \mathbf{e}_1) \wedge \mathbf{f}_1 \in \mathbf{D})$. The first is replaced by $(\forall \mathbf{x} \in \mathbf{D})(\exists \mathbf{y} \in \bar{\mathbf{D}})(\mathbf{y} \leq \mathbf{x} \vee \mathbf{e}_0, \mathbf{f})_\mathbf{0} \& (\forall \mathbf{x} \in \mathbf{D})(\forall \mathbf{y}, \mathbf{z} \in \bar{\mathbf{D}})(\mathbf{y}, \mathbf{z} \leq \mathbf{x} \vee \mathbf{e}_0, \mathbf{f}_0 \to \mathbf{y} = \mathbf{z})$ and the second by the analogous statement switching $\mathbf{D}$ with $\bar{\mathbf{D}}$ and $\mathbf{e}_0, \mathbf{f}_0$ with $\mathbf{e}_1, \mathbf{f}_1$. We can now eliminate $\vee$ from this sentence at the expense of one additional quantifier in the usual way to get our Π_2 correctness condition $\hat{\Theta}$ in the language of $\mathcal{R}(\leq)$. The first one becomes $(\forall \mathbf{x} \in \mathbf{D})(\forall \mathbf{u})(\exists \mathbf{y} \in \bar{\mathbf{D}})[\forall \mathbf{v}(\mathbf{v} \geq \mathbf{x}, \mathbf{e}_0 \to \mathbf{v} \geq \mathbf{u}) \to \mathbf{y} \leq \mathbf{u}, \mathbf{f}] \& (\forall \mathbf{x} \in \mathbf{D})(\forall \mathbf{y}, \mathbf{z} \in \bar{\mathbf{D}})(\forall \mathbf{u})[\forall \mathbf{v}(\mathbf{v} \geq \mathbf{x}, \mathbf{e}_0 \to \mathbf{v} \geq \mathbf{u}) \to ((\mathbf{y}, \mathbf{z} \leq \mathbf{u}, \mathbf{f}_0) \to \mathbf{y} = \mathbf{z}))]$ and the second is analogous.

Our typical sentence on the list of ones showing undecidability now looks like $\forall\vec{q}[\hat{\Theta} \to (P(I)^* \to F_2^*)]$. Our next task is to eliminate the uses of s (and so $\wedge$) in these formulas. We might as well view $P(I)^* \to F_2^*$ as a single Σ_1 sentence in $\leq, \vee, s$. Our correctness condition $\hat{\Theta}$ says that for each $\mathbf{r} \in \mathbf{D}$ there is a unique $\mathbf{v} \in \bar{\mathbf{D}}$ and $\mathbf{w} \in \mathbf{D}$ such that $(\mathbf{v} \leq \mathbf{r} \vee \mathbf{e}_0, \mathbf{f}_0) \,\&\, (\mathbf{w} \leq \mathbf{v} \vee \mathbf{e}_1, \mathbf{f}_1)$. We can use this property to replace each instance of an application of s. We proceed by an induction on the complexity of terms. Suppose our formula is of the form $\exists \vec{\mathbf{r}} \varphi(\vec{\mathbf{r}}, s(\mathbf{r}_0))$. (Note that φ necessarily includes a clause $\mathbf{r}_0 \in \mathbf{D}$.) We replace this with the sentence $\exists \vec{\mathbf{r}}(\exists \mathbf{v}_0 \in \bar{\mathbf{D}})(\exists \mathbf{w}_0 \in \mathbf{D})[(\mathbf{v}_0 \leq \mathbf{r} \vee \mathbf{e}_0, \mathbf{f}_0) \,\&\, (\mathbf{w}_0 \leq \mathbf{v}_0 \vee \mathbf{e}_1, \mathbf{f}_1) \,\&\, \varphi(\vec{\mathbf{r}}, \mathbf{w}_0/s(\mathbf{r}_0))]$. (We use the notation $\mathbf{w}_0/s(\mathbf{r}_0)$

to indicate that we have substituted $\mathbf{w}_0$ for the term $s(\mathbf{r}_0)$ in the ambient formula.) Assuming the correctness condition $\hat{\Theta}$, this is clearly equivalent to the original $\exists\vec{\mathbf{r}}\varphi(\vec{\mathbf{r}}, s(\mathbf{r}_0))$. We can now proceed inductively to eliminate all occurrences of s and produce a Σ_1 formula in $\leq, \vee$ equivalent under the assumption $\hat{\Theta}$ to our original $\exists\vec{\mathbf{r}}\varphi(\vec{\mathbf{r}}, s(\mathbf{r}_0))$. We can now apply the dual procedure to the one used to eliminate $\vee$ from Π_1 formulas in $\leq, \vee$ to get a Σ_2 formula $\Psi(I)$ in just $\leq$ equivalent to $(P(I)^* \rightarrow F_2^*)$. We then have our new family of formulas $\forall\vec{q}[\hat{\Theta} \rightarrow \Psi(I)]$ which are $\forall\exists\forall$ and whose validity in $\mathcal{R}$ is undecidable as required to prove Corollary 3.1.

Corollary 3.1 *(Lempp, Nies and Slaman [1998]): The three quantifier* ($\forall\exists\forall$) *theory of* $\mathcal{R}(\leq)$ *is undecidable.*

Next, we consider $\mathcal{D}$ and $\mathcal{D}(\leq 0')$. First note that it is straightforward to construct Δ_2^0 lattices with top r, individual elements $b, c, p, q, e_0, e_1, f_0, f_1$ and a family of independent (even minimal) elements g_i satisfying all the algebraic facts required in Theorem 2.2 and additional elements u and v defining any fixed r.e. subset of the g_i as in Lemma 2.1. We can now use the standard embedding theorems from Lerman [1983] to realize these lattices as initial segments of $\mathcal{D}$ or $\mathcal{D}(\leq 0')$. Our arguments for undecidability now work just as well in these structures and so we have the analogous results.

Corollary 3.2 *For any total extension* $\wedge$ *of the partial infimum relation on* $\mathcal{D}$ ($\mathcal{D}(\leq 0')$)*, the two quantifier* ($\forall\exists$) *theory of* $\mathcal{D}$ ($\mathcal{D}(\leq 0')$) *with* $\leq$, $\vee$ *and* $\wedge$ *is undecidable.*

Finally, we turn our attention to the lattices $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$, $n \geq 4$) of (Σ_n) ideals of $\mathcal{R}$ with $\vee$ and $\wedge$. Recall that the operations $\vee$ and $\wedge$ are defined in the usual way for structures of ideals ($\mathbf{I} \vee \mathbf{J}$ is the ideal generated by $\mathbf{I} \cup \mathbf{J}$ and $\mathbf{I} \wedge \mathbf{J}$ is the ideal $\mathbf{I} \cap \mathbf{J}$) and are both total operators on $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}_n(\mathcal{R})$ for $n \geq 4$.

Corollary 3.3 *The two quantifier* ($\forall\exists$) *theory of* $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$, $n \geq 4$)*, the lattice of* (Σ_n) *ideals of* $\mathcal{R}$*, with* $\vee$ *and* $\wedge$ *is undecidable.*

Proof. We claim that the principal ideals generated by the degrees constructed to satisfy Theorem 2.2 and Lemma 2.1 have all the required properties in $\mathcal{I}(\mathcal{R})$ that the degrees themselves had in $\mathcal{R}$. The crucial fact is the quantifier free definability of the set $\{\mathbf{g}_i\}$ as $\mathbf{G}(\mathbf{r}, \mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q})$. We denote the principal ideal generated by a degree $\mathbf{x}$ by $(\mathbf{x})$ and want to establish the corresponding facts in $\mathcal{I}(\mathcal{R})$. Consider any ideal $\mathbf{I} \subseteq (\mathbf{r})$ such that $(\mathbf{q}) \subseteq \mathbf{I} \vee (\mathbf{p})$ and $(\mathbf{c}) \nsubseteq \mathbf{I} \vee (\mathbf{b})$. The first assumption tells us that there is an $\mathbf{e} \in \mathbf{I}$ such that $\mathbf{q} \leq \mathbf{e} \vee \mathbf{p}$ while the second guarantees that $\mathbf{c} \nleq \mathbf{e} \vee \mathbf{b}$. Thus $\mathbf{e}$ is one of the $\mathbf{g}_i$ and so $(\mathbf{g}_i) \subseteq \mathbf{I}$. On the other hand, if $\mathbf{h} \in \mathbf{I}$ then (by our second assumption again) $\mathbf{c} \nleq \mathbf{h} \vee \mathbf{g}_i \vee \mathbf{b}$. The **HS** maximality property of $\mathbf{g}_i$ then guarantees that $\mathbf{h} \vee \mathbf{g}_i \leq \mathbf{g}_i$ and so $\mathbf{I} \subseteq (\mathbf{g}_i)$ as required.

The other facts needed from Theorem 2.2 are that $(\mathbf{g}_{2i+1}) = [(\mathbf{g}_{2i}) \vee (\mathbf{e}_0)] \wedge (\mathbf{f}_0)$ and the analogous one for $(\mathbf{g}_{2i+2})$. These follow immediately from the trivial general facts about $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$, $n \geq 4$) that, for all degrees $\mathbf{x}, \mathbf{y}$, $(\mathbf{x} \vee \mathbf{y}) = (\mathbf{x}) \vee (\mathbf{y})$ and, if $\mathbf{x} \wedge \mathbf{y}$ exists, $(\mathbf{x} \wedge \mathbf{y}) = (\mathbf{x}) \wedge (\mathbf{y})$. The only other algebraic fact needed is that the principal ideals given by the degrees constructed for Lemma 2.1 have the analogous property in $\mathcal{I}(\mathcal{R})$. This too follows immediately from the first trivial fact.

The same arguments work for $\mathcal{I}_n(\mathcal{R})$ for $n \geq 4$. $\square$

As remarked above, when $\vee$ and $\wedge$ are total functions the two quantifier theory with $\leq$, $\vee$ and $\wedge$ is reducible to the three quantifier theory with just $\subseteq$ and so we also have proven Corollary 3.4.

Corollary 3.4 *The three quantifier ($\forall\exists\forall$) theory of $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$, $n \geq 4$), the lattice of (Σ_n) ideals of $\mathcal{R}$, with just $\subseteq$ is undecidable.*

We now explain how similar considerations characterize the theories of these structures of ideals.

Corollary 3.5 *The theory of $\mathcal{I}(\mathcal{R})$ is recursively isomorphic to that of true second order arithmetic and that of $\mathcal{I}_n(\mathcal{R})$ to that of true first order arithmetic for each $n \geq 4$.*

Proof. Consider the effective successor models $\{\mathbf{g}_{2i} : i \in \omega\}$ in $\mathcal{R}$ with the relevant parameters as constructed here. As remarked above the effective successor models defined by the ideals $(\mathbf{g}_{2i}) = \{\mathbf{a} \leq_T \mathbf{g}_{2i}\}$ generated by the relevant degrees are definable in the same way in $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$, $n \geq 4$), using the analogous successor function. We begin by noting that we could add parameters to define additional relations of the form supplied by Lemma 2.1. We want to choose ones that define a structure for arithmetic on one subset of the set $\mathbf{D} = \{(\mathbf{g}_{2i})\}$ of ideals. We let $\mathbf{D}_k = \{(\mathbf{g}_{8i+2k}) : i \in \omega\}$ for $k \leq 3$ and define the required relations for order, addition and multiplication on $\mathbf{D}_0$. We begin with parameters that pick out the relations $S_k = \{\{(\mathbf{g}_{8i}), (\mathbf{g}_{8i+2k})\} : i \in \omega\}$ for $1 \leq k \leq 3$ that identify the corresponding elements of $\mathbf{D}_0$ and $\mathbf{D}_k$. We can then define, for example, the natural ordering on $\mathbf{D}_0$ by parameters that pick out $\{\{(\mathbf{g}_{2i+1}), (\mathbf{g}_{2j+2})\} : i \leq j\}$ and similarly plus and times by picking out $\{\{(\mathbf{g}_{2i+1}), (\mathbf{g}_{2j+2}), (\mathbf{g}_{2k+3})\} : i + j = k\}$ and $\{\{(\mathbf{g}_{2i+1}), (\mathbf{g}_{2j+2}), (\mathbf{g}_{2k+3})\} : i \cdot j = k\}$. One can then say that the structure so defined is a model of arithmetic in the usual way.

The problem now is to find a nonempty definable class $\mathcal{C}$ of structures in $\mathcal{I}(\mathcal{R})$ ($\mathcal{I}_n(\mathcal{R})$), containing the structure defined above, such that every structure in $\mathcal{C}$ is isomorphic to the standard model of arithmetic. $\mathcal{C}$ will be defined by saying that there exist parameters $I_{\mathbf{e}_i}$ and $I_{\mathbf{f}_i}$ ($i = 0, 1$), $I_{\mathbf{b}}$, $I_{\mathbf{c}}$, etc. (corresponding to the parameters $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{b}$, $\mathbf{c}, \ldots$ in our construction on degrees), and an ideal J_0 serving as 0 in the model of arithmetic, satisfying a conjunction of correctness conditions. (Notice that here we cannot assume or require that these parameters be principal ideals.)

We use the effectiveness of the successor function. As is argued in Shore [1981] or Nies, Shore and Slaman [1998], we can generate all representatives of degrees in each element $J_i = s^i(J_0)$ of the structure, uniformly in i, in a way that is effective in $\leq$ and $\vee$ on degrees, beginning with representatives of the degrees in the ideals J_0, $I_{\mathbf{e}_i}$, and $I_{\mathbf{f}_i}$ $(i = 0, 1)$.

$$\begin{aligned}\deg(W_k) \in s(J_0) &\iff \deg(W_k) \in (((J_0 \vee I_{\mathbf{e}_0}) \wedge I_{\mathbf{f}_0}) \vee I_{\mathbf{e}_1}) \wedge I_{\mathbf{f}_1} \\ &\iff (\exists b, c, d)[\deg(W_b) \in J_0 \;\&\; \deg(W_c) \in I_{\mathbf{e}_0} \;\&\; \deg(W_d) \in I_{\mathbf{e}_1} \;\& \\ &\qquad \deg(W_b \oplus W_c) \in I_{\mathbf{f}_0} \;\&\; W_k \equiv_T W_b \oplus W_c \oplus W_d \;\&\; \deg(W_k) \in I_{\mathbf{f}_1}].\end{aligned}$$

Now all statements here are Σ_n (including the Turing equivalence, since $n \geq 4$), because all these ideals are in $\mathcal{I}_n(\mathcal{R})$. By iterating, we derive a Σ_n formulation of "$\deg(W_e) \in s^i(J_0)$" uniformly in i.

The independence of the defined set $\mathbf{G}(I_\mathbf{r}, I_\mathbf{b}, I_\mathbf{c}, I_\mathbf{p}, I_\mathbf{q})$ can also be guaranteed by a correctness condition saying that for each element of the defined set there is something above all the others but not above it. Thus the ideal generated by all the J_i will not contain any elements of $\mathbf{G}(I_\mathbf{r}, I_\mathbf{b}, I_\mathbf{c}, I_\mathbf{p}, I_\mathbf{q})$ other than the J_i. So as usual, if we require of our model that every proper initial segment have a maximal element (all in the ordering defined on the structure) then we have picked out precisely the standard models. Once we have defined this class of standard models of arithmetic, we have guaranteed that each theory is at least as complicated as true first order arithmetic. As each $\mathcal{I}_n(\mathcal{R})$ is arithmetical, this completely characterizes the complexity of their theories. For $\mathcal{I}(\mathcal{R})$ we simply note that the independence of the J_i guarantees that every subset is uniquely determined by the ideal it generates and so quantification over $\mathcal{I}(\mathcal{R})$ codes full second order quantification over each standard model of arithmetic picked out by our definition. As $\mathcal{I}(\mathcal{R})$ is itself defined in second order arithmetic, its theory is equivalent to that of true second order arithmetic as required. $\square$

Corollary 3.6 *For $m > n \geq 4$, the ideal lattices $\mathcal{I}_m(\mathcal{R})$ and $\mathcal{I}_n(\mathcal{R})$ are not elementarily equivalent, nor is any of them elementarily equivalent to $\mathcal{I}(\mathcal{R})$.*

Proof. Let $S \subseteq \omega$ be a Σ_{n+1}-complete set. The sentence ψ which we build to distinguish the lattice $\mathcal{I}_m(\mathcal{R})$ from $\mathcal{I}_n(\mathcal{R})$ will say that there exists a standard model of arithmetic encoded in the lattice by parameters $I_{\mathbf{e}_0}$, $I_{\mathbf{f}_0}$, etc., such that some element of the lattice can use this model to compute S. The existence of a standard model requires only the existence of a set of parameters satisfying the conditions given in the preceding proof. In the rest of the sentence, we say that there exists an element I in the lattice such that for every i,

$$i \in S \iff J_i \subseteq I,$$

where $J_i = s^i(J_0)$ is the ideal corresponding to i in the specified standard model. (Thus I codes the set S in this model.) To say "$i \in S$" in the language of lattices, we use the standard model given by the parameters and the Σ_{n+1} definition of S.

In $\mathcal{I}_m(\mathcal{R})$, by the preceding results, we have a standard model of arithmetic on certain ideals $\{J_i : i \in \omega\}$. Let J_S be the ideal consisting of finite joins $\mathbf{d}_1 \vee \cdots \vee \mathbf{d}_p$ with each $\mathbf{d}_j \in J_{i_j}$ for some $i_j \in S$. We claim that J_S lies in $\mathcal{I}_m(\mathcal{R})$:

$$\{e : (\exists p)(\exists i_0, \ldots i_p \in S)[\deg(W_e) \in \vee_{j=0}^{p} J_{i_j}]\} =$$
$$\{e : (\exists p, i_0, \ldots i_p, k_0, \ldots k_p)(\forall l \leq p)[i_l \in S \ \& \ \deg(W_{k_l}) \in s^{i_l}(J_0) \ \& \ W_e \leq_T \bigoplus_{j=0}^{p} W_{k_j}]\}.$$

As noted above, $s^{i_l}(J_0)$ is a Σ_m-ideal uniformly in i. Thus J_S is indeed a Σ_m-ideal, and $\mathcal{I}_m(\mathcal{R})$ (and $\mathcal{I}(\mathcal{R})$) satisfy ψ.

Now suppose that we have parameters in $\mathcal{I}_n(\mathcal{R})$ defining a standard model of arithmetic on ideals $\{J_i : i \in \omega\}$, and that $I \in \mathcal{I}_n(\mathcal{R})$ is a (not necessarily principal) ideal with $\{i : J_i \subseteq I\} = S$. Then there would be a Σ_n formula θ defining $\{e : \deg(W_e) \in I\}$, so we would have

$$i \in S \iff J_i \subseteq I \iff (\forall \mathbf{a} \in J_i)[\mathbf{a} \in I] \iff (\forall e)[\deg(W_e) \in s^i(J_0) \implies \theta(e)].$$

This is impossible for the Σ_{n+1}-complete set S, since the rightmost formula is Π_{n+1}. Therefore the sentence ψ fails in $\mathcal{I}_n(\mathcal{R})$, whereas it holds in $\mathcal{I}_m(\mathcal{R})$ and in $\mathcal{I}(\mathcal{R})$. ■

4 Construction

To prove our required technical result on degrees, Theorem 2.2, it suffices to construct r.e. sets satisfying the requirements of the following Theorem:

Theorem 4.1 *There exist sets G_i $(i \in \omega)$, P, Q, B, C, E_0, E_1, and $R = \oplus G_i$ satisfying the following requirements for all e, i, j, k, and x in ω and all computable functionals Ω, Φ, Λ, Υ, and Ψ:*

Requirements:

$\mathcal{D}_{i,\Omega}$: $G_i \neq \Omega^{H_i}$, *where* $H_i = \oplus_{k\neq i}\, G_k$

$\mathcal{L}_{\Phi,x}$: $[(\exists^\infty s)\ \Phi^R(x)[s]\downarrow] \implies \Phi^R(x)\downarrow$

$\mathcal{M}_{i,j,\Lambda,\Upsilon}$: $W_j = \Upsilon^{G_i} \implies [\Lambda^{W_j\oplus P} = Q \implies (\exists\Theta)\ G_i = \Theta^{W_j}]$

$\mathcal{N}_{e,\Phi}$: $W_e = \Phi^R \implies [(\exists\Gamma)\ C = \Gamma^{W_e\oplus B}$ *or* $(\exists k)(\exists\Delta)\ W_e = \Delta^{G_k}]$

$\mathcal{P}_{i,\Psi}$: $C \neq \Psi^{G_i\oplus B}$

$\mathcal{R}_k$: $(\exists\Xi) Q = \Xi^{G_k\oplus P}$.

Lattice requirements: (Here $F_0 = \oplus_k\, G_{2k+1}$ *and* $F_1 = \oplus_k\, G_{2k+2}$*.)*

$\mathcal{T}_{2i}$: $G_{2i+1} \leq_T G_{2i} \oplus E_0$

$\mathcal{T}_{2i+1}$: $G_{2i+2} \leq_T G_{2i+1} \oplus E_1$

$\mathcal{U}_{e,2i}$: $\Phi_e^{G_{2i}\oplus E_0} = \Phi_e^{F_0}$ *total* $\implies \Phi_e^{F_0} \leq_T G_{2i+1}$

$\mathcal{U}_{e,2i+1}$: $\Phi_e^{G_{2i+1}\oplus E_1} = \Phi_e^{F_1}$ *total* $\implies \Phi_e^{F_1} \leq_T G_{2i+2}$.

(Here and afterwards, the notation "$[s]$" at the end of a term or equation indicates that we refer to the approximation at stage s of each set, oracle and function used there. Thus, for example, the hypothesis of the requirement $\mathcal{L}_{\Phi,x}$ is that there are infinitely many stages s at which $\Phi_s^{R_s}(x)$ converges.)

As in Nies, Shore and Slaman [1998], we begin by choosing an effective ordering of all the $\mathcal{D}$-, $\mathcal{M}$-, $\mathcal{N}$-, $\mathcal{P}$-, $\mathcal{R}$-, and $\mathcal{U}$-requirements, in order type ω, such that for all i and j:

- $\mathcal{R}_i$ precedes every $\mathcal{D}_{i,\Omega}$ and every $\mathcal{U}_{e,i}$ in the ordering (i.e. $\mathcal{R}_i$ has higher priority than $\mathcal{D}_{i,\Omega}$ and $\mathcal{U}_{e,i}$); and
- both $\mathcal{R}_i$ and $\mathcal{R}_j$ precede every $\mathcal{M}_{i,j,\Lambda,\Upsilon}$.

(The requirements $\mathcal{T}_i$ are global requirements and will not be given a priority rank or placed on the tree. The requirements $\mathcal{L}_{\Phi,x}$ will play a role in priority arguments, as described below, but they also are not placed on the tree.)

This ordering yields a specific priority order on the $\mathcal{N}$-requirements, which we write as $\mathcal{N}_0, \mathcal{N}_1, \ldots$, defining e_i and Φ_i so that $\mathcal{N}_i$ denotes $\mathcal{N}_{e_i,\Phi_i}$. Next we construct a tree

T. Each node on the tree will have a specific requirement assigned to it, will play a particular strategy to attempt to satisfy that requirement, and will have one immediate successor for each possible outcome of the requirement. For brevity, if the requirement $\mathcal{D}_{i,\Omega}$ is assigned to a node α, we will call α a $\mathcal{D}_{i,\Omega}$-node, and also a $\mathcal{D}$-node; similarly with all other requirements. Below, we name the outcomes for each type of node and explain how the construction works to select one of the outcomes and satisfy the node's requirement.

We view the tree T as growing upwards from a root node. The relation $\prec$ will represent higher priority: $\alpha \prec \beta$ if α is to the left of β on T or $\alpha \subsetneq \beta$, i.e. exactly when α has higher priority than β. To define T and determine which requirement is assigned to each node $\rho \in T$, we need the following definition.

Definition 4.2 Let $\rho \in T$. Each requirement is either *active along* ρ (via a single node $\subset \rho$), or *satisfied along* ρ (again via a single node $\subset \rho$), or neither, according to the following inductive definition. (Notice that a requirement cannot be both active and satisfied along the same node.)

If ρ is the empty string, then no requirement is active or satisfied along ρ. Otherwise, let $\eta = \rho^-$, the immediate predecessor of ρ.

If a $\mathcal{D}$-, $\mathcal{M}$-, $\mathcal{N}$-, $\mathcal{R}$-, or $\mathcal{U}$-requirement is assigned to η, then every requirement active or satisfied along η via some β is also active or satisfied (respectively) along ρ via β. Also, the requirement assigned to η is active along ρ via η (if it is an $\mathcal{N}$-requirement and $\rho = \eta\hat{\ }\langle\infty\rangle$) or satisfied along ρ via η (otherwise).

If η is a $\mathcal{P}_{i,\Psi}$-node, then we must consider the successors of η separately.

- If $\rho = \eta\hat{\ }\langle f\rangle$ or $\rho = \eta\hat{\ }\langle w\rangle$, then every requirement active or satisfied along η via some β is also active or satisfied (respectively) along ρ via β, and $\mathcal{P}_{i,\Psi}$ itself is satisfied along ρ via η.
- Otherwise, according to our definition below of the successors of η, $\rho = \eta\hat{\ }\langle a_l\rangle$ for some $l \in \omega$ such that $\mathcal{N}_l$ is active along η via some α. We then define this $\mathcal{N}_l$ to be satisfied along ρ via α, and every $\mathcal{N}$-requirement active or satisfied along α via some $\beta \subset \alpha$ to be active or satisfied (respectively) along ρ via the same β. All other $\mathcal{N}$-requirements are neither active nor satisfied along ρ. (In particular, $\mathcal{N}$-requirements which were active or satisfied along η but not along α will be injured by the action we take at node ρ and hence are neither active nor satisfied along ρ.) Requirements of types other than $\mathcal{N}$ which were active or satisfied along η via any β remain active or satisfied (respectively) along ρ via β, and $\mathcal{P}_{i,\Psi}$ itself is satisfied along ρ via η.

With this definition, we assign to ρ the requirement of highest priority that is neither active nor satisfied along ρ. The immediate successors of ρ depend on the type of requirement assigned. For each possible outcome y of ρ as defined below, we add an immediate successor $\rho\hat{\ }\langle y\rangle$ of ρ to T.

The possible outcomes of each node, and their meanings, are as follows:

- If ρ is a $\mathcal{D}_{i,\Omega}$-node, then the two possible outcomes for ρ, in order, are $f < w$. The node ρ finds a witness element w^i_ρ, as in a Friedberg-Muchnik construction, waits for the witness to be realized, and then attempts to put it into G_i. The outcome w holds while we wait for the witness element to be realized. If it is never realized, then it never enters G_i, so the requirement is satisfied. If it is realized at some stage, then we preserve the convergence of the computation $\Omega^{H_i}(w^i_\rho)\downarrow= 0$ by initializing all nodes $\succ \rho$, and attempt to enumerate the witness element into G_i, by allowing it to enter the pinball machine associated with the satisfaction of the $\mathcal{U}$- and $\mathcal{T}$-requirements, starting at node ρ. Each $\mathcal{U}$- and $\mathcal{P}$-node below ρ periodically allows elements ("balls") to pass its gate, thereby giving those elements its permission to enter their target sets. Other elements may be assigned to the witness as traces and targeted for sets E_0, E_1, or $G_{i-1}, G_{i-2}, \ldots$ to satisfy the $\mathcal{T}$-requirements. Assuming that ρ is on the true path, w^i_ρ will eventually pass every gate below ρ and enter G_i, at which point we switch to the outcome f. This represents a "finite win" for the requirement $\mathcal{D}_{i,\Omega}$, since we have now satisfied $G_i(w^i_\rho) \neq \Omega^{H_i}(w^i_\rho)$.

  $\mathcal{D}$-nodes (and $\mathcal{M}$-nodes, described below) do injure the negative requirements $\mathcal{L}_{\Phi,x}$ by enumerating elements into R. At certain stages a requirement $\mathcal{L}_{\Phi,x}$ may initialize cofinitely many nodes on T in order to preserve the computation $\Phi^R(x)$. Also, each time the node ρ is initialized by another node (as opposed to being initialized by an $\mathcal{L}$-requirement), it loses some priority vis-a-vis the $\mathcal{L}$-requirements. This guarantees that even if ρ is to the right of the true path and enumerates infinitely many elements into R, it will only injure each $\mathcal{L}$-requirement finitely often.

- If ρ is a $\mathcal{U}_{e,i}$-node, then the possible outcomes of ρ are:

  $$p_0 < p_1 < p_2 < \cdots ,$$

  ordered as given. The outcome p_r represents a restraint of length r placed on $G_i \oplus E_0$ and F_0 (or on $G_i \oplus E_1$ and F_1, depending on the parity of i) as in the pinball-style constructions for lattice embeddings in Lerman [1973] and related works. A "ball targeted for G_i" is a number which some $\mathcal{D}$- or $\mathcal{M}$-node α would like to put into G_i, and will be named w^i_α. The node ρ acts as a gate in the pinball machine. For every $\alpha \supset \rho$, every ball w^i_α must wait at gate ρ until the ball can enter G_i without injuring the requirement $\mathcal{U}_{e,i}$. Occasionally such an α may also want to put a ball e^j_α into the set E_j (for $j = 0$ or 1), and again the gate ρ will make that ball wait until the enumeration will not injure $\mathcal{U}_{e,i}$. If the hypothesis of $\mathcal{U}_{e,i}$ is satisfied, then the restraints will drop to 0 infinitely often; if not, then they will converge to a finite limit, and every node above that outcome will have the correct guess about the limit, hence will only use witnesses large enough not to injure that restraint.

- If ρ is an $\mathcal{R}_k$-node, then the only possible outcome is ∞. At each ρ-stage, we extend the functional $\Xi^{G_k \oplus P}$ being built by $\mathcal{R}_k$. The only possible injury to this construction occurs when some $\mathcal{M}_{k,j,\Lambda,\Upsilon}$-node wishes to enumerate an element into Q, for reasons described below. Each $\mathcal{M}$-node enumerates at most one element x after its last initialization, and any $\mathcal{M}$-node $\supset \rho$ will enumerate its x into Q only after ensuring that some change in $G_k \oplus P$ will allow ρ to redefine $\Xi_\rho^{G_k \oplus P}(x) = 1$.

- If ρ is an $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node, then the possible outcomes are

$$f < \infty < w.$$

We have some control over the enumeration of W_j using the hypothesis $W_j = \Upsilon^{G_i}$, and this in turn affects the hypothesis $\Lambda^{W_j \oplus P} = Q$. The outcome w denotes a nonexpansionary stage for the latter hypothesis, meaning that the length of agreement between W_j and Υ^{G_i} is not sufficient for us to guarantee any increase in the length of agreement between $\Lambda^{W_j \oplus P}$ and Q. If there are cofinitely many nonexpansionary stages, then $\mathcal{M}_{i,j,\Lambda,\Upsilon}$ will be satisfied.

On the other hand, if the length of agreement between $W_j[s]$ and $\Upsilon^{G_i}[s]$ has increased sufficiently to enable a longer length of agreement l between $\Lambda^{W_j \oplus P}[s]$ and $Q[s]$ to be computed, we call $s+1$ a ρ-expansionary stage, and we attempt to extend our functional Θ_ρ to compute G_i from W_j on the domain l, setting the use $\theta_\rho^{W_j}(y)[s+1] = \lambda^{W_j \oplus P}(y)[s]$ and preserving the equality of these uses, as described below. If we succeed, the outcome is ∞. If there are infinitely many ρ-expansionary stages and we succeed in extending Θ_ρ at every one, then the true path will contain $\rho\hat{\ }\langle\infty\rangle$.

If, at some ρ-expansionary stage $s+1$, we cannot extend Θ_ρ as above, then we will be able to achieve a finite win, denoted by the outcome f. Some number, which we designate as x_ρ, must have entered G_i since the last ρ-expansionary stage $r+1$, with no change to $W_j \upharpoonright \lambda^{W_j \oplus P}(x_\rho)[r]$ up until stage s (since such a change would allow us to redefine $\Theta_\rho^{W_j}(x_\rho)[s+1]$). We attempt to enumerate x_ρ into Q to make $\Lambda^{W_j \oplus P} \neq Q$, since the absence of any W_j-change ensures that $\Lambda^{W_j \oplus P}(x_\rho)[s]\downarrow = 0$. (Recall that the use θ_ρ matches the use λ.) We will preserve this finite win by initialization, imposing sufficient restraints on P and G_i to prevent W_j from changing, since $W_j = \Upsilon^{G_i}$.

Before enumerating x_ρ into Q, however, we must ensure that this enumeration will not injure any higher-priority $\mathcal{R}_k$-node β, since such a β builds a functional Ξ_β with the intention that $\Xi_{\beta_k}^{G_k \oplus P} = Q$. When we enumerate x_ρ into Q at a later stage $t+1$, therefore, we will want to enumerate $\xi_{\beta_k}^{G_k \oplus P}(x_\rho)[t]$ into P, allowing β_k to redefine its functional at the next β_k-stage. On the other hand, if $\xi_{\beta_k}^{G_k \oplus P}(x_\rho) < \lambda^{W_j \oplus P}(x_\rho)[t]$, then this enumeration would allow $\Lambda^{W_j}(x_\rho)[t+1]$ to change as well, which would destroy our diagonalization. For $k \neq i$, we avoid this problem by first enumerating an element $w_\rho^k = \xi_{\beta_k}^{G_k \oplus P}(x_\rho)[s]$ into the set G_k, which allows β_k to increase the ξ_{β_k}-use without permitting any change in Θ_ρ. (Of course, this takes time, since w_ρ^k must proceed through the pinball machine, starting at ρ, before entering G_k.) When we finally enumerate x_ρ into Q at a stage $t+1$, our P-enumerations at that stage will allow for changes in $\Xi^{W_j}_{\beta_k}(x_\rho)[t+1]$ while maintaining $\Lambda^{W_j \oplus P}(x_\rho)[t+1]{\downarrow}= 0$.

The requirement $\mathcal{R}_i$ is also assigned to some $\beta_i \subset \rho$, and the strategy above would not work for it, since any G_i-enumeration after stage s could allow a change in $W_j = \Upsilon^{G_i}[s]$, hence in $\Lambda^{W_j \oplus P}(x_\rho)[s]$, ruining our diagonalization. (P-enumerations, which could also ruin the diagonalization, are discussed below.) Fortunately, we do not need any G_i-enumeration, because the ball x_ρ was chosen at a stage $s+1$ when it had just entered G_i itself. β_i will have increased the use $\xi^{G_i \oplus P}_{\beta_i}(x_\rho)[s]$ to be large, hence $> \lambda^{W_j \oplus P}(x_\rho)[s]$. (Recall that any change in this λ-use between stages r and s would have allowed us to extend the functional Θ_ρ so that $\Theta^{W_j}_\rho(x_\rho) = G_i(x_\rho)[s+1]$.) Therefore, we make no further G_i-enumerations, but simply enumerate $\xi^{G_i \oplus P}_{\beta_i}(x_\rho)[s]$ into P at the stage $t+1$ when x_ρ enters Q. β_i will then be able to redefine $\Xi^{G_i \oplus P}_{\beta_i}(x_\rho) = 1$ at the next β_i-stage, as required.

It is important in the preceding construction that we keep the use $\theta^{W_j}_\rho(x)$ equal to $\lambda^{W_j \oplus P}(x)$ at each stage. If the λ-use became larger, then a number $> \theta^{W_j}_\rho(x_\rho)[r]$ which entered W_j between stages r and s might leave $\lambda^{W_j \oplus P}(x_\rho) > \xi^{G_i \oplus P}_{\beta_i}(x_\rho)[s]$ so that the P-enumerations would destroy our diagonalization against $\Lambda^{W_j \oplus P} = Q$, without letting us redefine $\Theta^{W_j}_\rho(x_\rho)[s+1] = 1$. So we must ensure that $P \restriction \lambda^{W_j \oplus P}(x)$ is preserved for every x which might eventually play the role of x_ρ, i.e. any ball w^i_γ (with $\rho\hat{\ }\langle\infty\rangle \subseteq \gamma$) targeted for G_i.

To make this happen, we refuse to allow any ball w^k_α to enter the pinball machine (the preliminary step to entering G_i) if its entry into G_i could create a P-enumeration which might upset the strategy for a higher-priority w^i_γ. In particular, if $\rho\hat{\ }\langle\infty\rangle \subseteq \alpha$, then α thinks that the expansionary outcome of ρ holds, and so α refuses to release any ball w^k_α into the pinball machine until $w^k_\alpha \in \mathrm{dom}(\Theta^{W_j}_\rho)$, that is, until the λ-use on all numbers $\leq w^k_\alpha$ has been chosen and guaranteed by agreement between W_j and Υ^{G_i}. (In particular, the λ-use of all balls from higher-priority nodes γ has been chosen by then.) If w^k_α does enter G_k, then the $\mathcal{R}_k$-node $\beta \subset \alpha$ will subsequently choose the use $\xi^{G_k \oplus P}_\beta(w^k_\alpha)$ to be large, hence larger than the λ-use of any ball w^i_γ from any node $\gamma \prec \alpha$. This use $\xi^{G_k \oplus P}_\beta(w^k_\alpha)$ may subsequently be enumerated into P by some other $\mathcal{M}$-node, but it will not change $P \restriction \lambda^{W_j \oplus P}(w^i_\gamma)$ for any $\gamma \prec \alpha$. Thus, it will not injure the strategy of any $\mathcal{M}$-node of higher priority than α.

(The same is true for any trace for w^k_α: the corresponding $\mathcal{R}$-node will choose the Ξ-use of the trace to be large after the trace enters its target set, hence after w^k_α was released, hence after w^i_γ entered $\mathrm{dom}(\Theta^{W_j}_\rho)$. So this Ξ-use will also be greater than $\lambda^{W_j \oplus P}(w^i_\gamma)$ and can safely be enumerated into P. Therefore, we define every trace to be certified automatically. Only witness balls chosen by $\mathcal{D}$- and $\mathcal{M}$-nodes must wait for certification.)

We will say that w^k_α is *certified* when it has entered the domains of all such $\Theta^{W_j}_\rho$-functionals (for all ρ with $\rho\hat{\ }\langle\infty\rangle \subseteq \alpha$), and we require all balls targeted for any G_k (whether from $\mathcal{D}$-nodes, from $\mathcal{M}$-nodes, or traces for other balls) to be certified before moving through the pinball machine. If α is on the true path, then the expansionary outcome $\rho\hat{\ }\langle\infty\rangle$ does hold, and so the domain of $\Theta^{W_j}_\rho$ will eventually grow large enough to include w^k_α. Thus no ball from a node on the true path will be forced to wait forever before entering the pinball machine.

- If ρ is an $\mathcal{N}_{e,\Phi}$-node, then the possible outcomes of ρ are ∞ and w, ordered with $\infty < w$. The outcome w denotes a nonexpansionary stage, i.e. a stage at which the length of agreement between W_e and Φ^R has not increased, so that we wait without taking any action. ∞ represents the outcome of an expansionary stage; if we have infinitely many such stages, then $\mathcal{N}_{e,\Phi}$ goes about the business of trying to build Γ_ρ to compute C from $W_e \oplus B$. As described below, this process can be injured by lower-priority $\mathcal{P}$-nodes, making this a $0'''$-construction.

- If ρ is a $\mathcal{P}$-node, let K be the finite set of higher-priority $\mathcal{N}$-requirements which ρ may injure:

$$K = \{k : \mathcal{N}_k \text{ is active along } \rho \text{ via some } \alpha_k\} = \{k_0 < k_1 < \cdots k_n\}.$$

  Then the set of possible outcomes of ρ is the following, ordered as given:

$$f < a_{k_0} < a_{k_1} < \cdots < a_{k_n} < w.$$

  For $\mathcal{P}$-nodes ρ, we attempt to achieve a finite win by choosing a witness z_ρ, waiting for $\Psi^{G_i \oplus B}(z_\rho)[s]\downarrow = 0$ at some stage s, and then putting z_ρ into C, so as to force $C \neq \Psi^{G_i \oplus B}$. The outcome f denotes our success in doing so, with the construction initializing nodes $\succ \rho$ at stage s to preserve $(G_i \oplus B)\upharpoonright \psi^{G_i \oplus B}(z_\rho)[s]$. The outcome w denotes that we are waiting for this convergence to occur. (If we wait forever, then $\mathcal{P}_{i,\Psi}$ will be satisfied.)

  However, an $\mathcal{N}_{e,\Phi}$-node $\alpha \subset \rho$ may object to letting z_ρ enter C, since this would disrupt its own computation of C from $W_e \oplus B$ via its functional Γ_α. The easiest way around this difficulty is to enumerate the current use $\gamma^{W_e \oplus B}_\alpha(z_\rho)[s]$ into B, thereby allowing α to change the value of $\Gamma^{W_e \oplus B}_\alpha(z_\rho)$ to 1. However, we can only do this if $\gamma^{W_e \oplus B}_\alpha(z_\rho) > \psi^{G_i \oplus B}(z_\rho)[s]$, since otherwise the change in B would ruin the convergence $\Psi^{G_i \oplus B}(z_\rho)[s] = 0$ and leave $\mathcal{P}_{i,\Psi}$ still unsatisfied.

  To handle this issue, we check in turn with each requirement $\mathcal{N}_k = \mathcal{N}_{e_k,\Phi_k}$ active at ρ via one of the α_k, starting with the lowest-priority one $\mathcal{N}_{k_n}$ and working down to the highest-priority one $\mathcal{N}_{k_0}$. If $\gamma^{W_{e_k} \oplus B}_{\alpha_k}(z_\rho) > \psi^{G_i \oplus B}(z_\rho)[s]$, then $\mathcal{N}_k$ does not object to the entry of z_ρ into C, and we continue with the next-higher-priority $\mathcal{N}$-requirement. Otherwise, we wait until the next ρ-stage, offering $W_{e_k}\upharpoonright \gamma^{W_{e_k} \oplus B}_{\alpha_k}(z_\rho)[s]$ the opportunity to change (for technical reasons having to do with Lemma 5.15).

If no such W_{e_k}-change occurs, we then enumerate $\gamma_{\alpha_k}^{W_{e_k}\oplus B}(z_\rho)[s]$ into $B[s+1]$, destroying both of the computations $\Gamma_{\alpha_k}^{W_{e_k}\oplus B}(z_\rho)[s]$ and $\Psi^{G_i\oplus B}(z_\rho)[s]$. In this case we make the outcome a_k eligible at stage s. In doing so, we give up our hope of a finite win for ρ with the current realization of z_ρ, and also disrupt the computations $\Gamma_{\alpha_l}^{W_{e_l}\oplus B}(z_\rho)[s]$ for every $l > k$. This procedure will result in a win for ρ if we repeat it infinitely often, since in that case $\Psi^{G_i\oplus B}(z_\rho)$ must diverge. For the node α_k, each time we make such a B-enumeration, we take a further step in the construction of a functional $\Delta_{\rho,k}$. If $\rho\hat{}\langle a_k\rangle$ lies on the true path, then the functional $\Delta_{\rho,k}$ will receive such attention infinitely often and will compute W_{e_k} from G_i, thereby satisfying $\mathcal{N}_k$. Hence we say that $\mathcal{N}_k$ is satisfied via α along nodes $\supseteq \rho\hat{}\langle a_k\rangle$, meaning that α_k does not actively try to protect its functional Γ_{α_k} at such nodes, since ρ has constructed $\Delta_{\rho,k}$ to satisfy $\mathcal{N}_k$ instead. This outcome is described in more detail on page 28, in Subcase 3 of the construction for $\mathcal{P}$-nodes.

While the outcome a_k does satisfy $\mathcal{N}_k$, it also disrupts the functionals Γ_{α_l} for all $l > k$ in K, without doing anything to build Δ-functionals for the requirements $\mathcal{N}_l$. Those requirements all have lower priority than $\mathcal{N}_k$, and are immediately reassigned, in the same order, to $\rho\hat{}\langle a_k\rangle$, its immediate successors, their immediate successors, and so on until each has been assigned to another node σ on each path through $\rho\hat{}\langle a_k\rangle$. Thus, along every path through $\rho\hat{}\langle a_k\rangle$, each such $\mathcal{N}_l$ now is assigned to a new $\alpha \supset \rho$, with the assurance that the requirement $\mathcal{N}_k$ will never destroy the functional Γ_α the way it destroyed Γ_{α_l}. By induction, therefore, each $\mathcal{N}_l$ will be reassigned to higher nodes only finitely often along any fixed path through the tree.

The $\mathcal{N}_j$-requirements which are active along α_k emerge with their Γ-functionals unscathed by ρ. These nodes all have higher priority than $\mathcal{N}_k$ (i.e. have $j < k$), so will have

$$\gamma_{\alpha_j}^{W_{e_j}\oplus B}(z_\rho) < \gamma_{\alpha_k}^{W_{e_k}\oplus B}(z_\rho)[s].$$

Therefore the numbers enumerated into B by ρ will not injure the functionals Γ_{α_j}.

The outcome a_k leaves $\mathcal{N}_k$ satisfied, not active, and, assuming inductively that all higher-priority $\mathcal{N}$-requirements remain either active forever or satisfied forever along the true path, $\mathcal{N}_k$ will remain satisfied forever there as well.

The $\mathcal{P}$-node also functions as a gate in the pinball machine, temporarily restraining balls targeted for sets G_j. When z_ρ is realized at a stage s, the restraint will keep balls $< \psi^{G_i\oplus B}(z_\rho)[s]$ from entering G_i, thereby protecting the convergence of $\Psi^{G_i\oplus B}(z_\rho)[s]$. (In the finite-win situation, of course, the source nodes for all such balls are initialized when z_ρ enters C. If the true path passes through $\rho\hat{}\langle a_k\rangle$, then this restraint becomes arbitrarily large, but drops back to 0 each time we destroy the convergence of $\Psi^{G_i\oplus B}(z_\rho)$.) At other stages, for the sake of Lemma 5.15, we wish to ensure that the only balls which enter the set $R = \oplus_j G_j$ are balls entering G_i, so we restrain balls targeted for sets G_j with $j \neq i$. Since $\mathcal{N}_k$ assumes that $W_{e_k} = \Phi^R$, this restraint will ensure that any change in W_{e_k} at these stages can be traced to a change in G_i, allowing us to redefine the functional $\Delta_{\rho,k}^{G_i}$ on the element which entered W_{e_k}, as $\mathcal{N}_k$ requires. These retraints are also set to 0 periodically, at stages s when a change in W_{e_k} would allow us to increase the use $\gamma_{\alpha_k}^{W_{e_k}\oplus B}(z_\rho)[s+1]$ to be $> \psi^{G_i\oplus B}(z_\rho)[s]$. Such a change in W_{e_k} would let us move closer to the finite-win situation and would lead to initialization of the node $\rho\hat{\ }\langle a_k\rangle$, so there is no reason to protect W_{e_k} at such stages. Thus no ball will be restrained forever by any $\mathcal{P}$-node on the true path.

This completes our description of the outcomes of nodes on T and the meaning attached to each.

Construction. As in Nies, Shore, and Slaman [1998], each stage $s+1$ of the construction consists of (at most) s substages, along with two steps which are executed at the end of every stage. At each substage $t < s$, only one node $\rho \in T$, of length t, will be eligible to act, and that ρ will then designate at most one of its immediate successors in T to be eligible to act at the following substage. (Alternatively, ρ may refuse to make any of its successors eligible.) The empty node is always eligible to act at substage 0 of any stage. The choice of which nodes are eligible to act corresponds to our approximation at stage s of the *true path* through T, i.e. the path g such that for each $\rho \subset g$, the successor of ρ along g denotes the ultimate outcome of the strategy played by ρ to satisfy the requirement assigned to it. $g(n)$ will be the leftmost node of length n which is eligible to act at infinitely many stages.

To *initialize* a strategy means to make all its parameters undefined and all functionals which it constructs completely undefined. At stage 0, we initialize every node. At each subsequent substage we initialize every node which lies to the right of any node eligible to act at that substage. Occasionally the construction will instruct us to initialize other nodes as well, but each node on the actual true path will only be initialized finitely often.

A number is *large* if it is greater than every other number seen thus far in the construction. By convention, our functionals are built so that, for any fixed oracle, the use function is strictly increasing.

At stage $s+1$ and substage $t < s$, let ρ be the node of T eligible to act at this stage and substage. If we have just completed substage $s-1$, or if an eligible node refuses to make any of its successors eligible to act at the next substage, then we proceed to the final steps of the stage, which describes which balls are allowed to move on the pinball machine at that stage. We then terminate the stage.

Let $s''+1$ be the last stage at which ρ was initialized, and let $s'+1$ be the most recent stage $> s''+1$ at which ρ was eligible to act. (If there has been no such stage since $s''+1$, we take $s' = s''$.) The action of ρ depends on the type of requirement assigned to it.

If ρ is a $\mathcal{D}_{i,\Omega}$-node, we proceed in the style of Friedberg and Muchnik.

1. If no witness element w^i_ρ is currently defined, then pick a large witness element w^i_ρ and target it for G_i. (Thus, for every gate $\alpha \subset \rho$ on the pinball machine, this w^i_ρ will be greater than the restraint currently maintained by that gate. If ρ is on the true path, then w^i_ρ will be large enough that every such α will eventually allow w^i_ρ to pass its gate.) We also choose a large number e^j_ρ, where j is 0 if i is odd and 1 if i is even, and target it for E_j. The ball e^j_ρ serves as a trace for w^i_ρ, for the sake of requirement $\mathcal{T}_{i-1}$. We then initialize every requirement $\supsetneq \rho$ and end this substage, with no node eligible to act at the next substage.

2. If w^i_ρ is currently defined but either $\Omega^{H_i}(w^i_\rho)\uparrow [s]$ or $\Omega^{H_i}(w^i_\rho)\downarrow\neq 0[s]$, then continue with the next substage, making $\rho\hat{\ }\langle w\rangle$ eligible to act at that substage. (Recall that $H_i = \oplus_{k\neq i} G_k$. In this case we say that w^i_ρ has not yet been *realized*.)

3. If w^i_ρ is currently defined and $\Omega^{H_i}(w^i_\rho)\downarrow= 0[s]$, we check whether w^i_ρ is certified at stage s. By definition, w^i_ρ is *certified at stage* t if for every node $\sigma \subset \rho$ such that a requirement $\mathcal{M}_{i',j,\Lambda,\Upsilon}$ is assigned to σ and $\sigma\hat{\ }\langle\infty\rangle \subseteq \rho$, we have $w^i_\rho \in \mathrm{dom}(\Theta^{W_j}_\sigma)[t']$, where $t'+1$ is the greatest σ-expansionary stage $\leq t$.

   (It is important to note that this definition of certified only applies to balls chosen by $\mathcal{D}$- and $\mathcal{M}$-nodes, not to their traces. Every trace, whether targeted for E_j or for a set G_k, is automatically certified and enters the pinball machine immediately upon being chosen by Instruction 4.4.)

   If w^i_ρ is not certified at stage s, then we initialize all nodes $\supsetneq \rho$ (so they will never injure the computation $\Omega^{H_i}(w^i_\rho)[s]$) and terminate this substage, with no node eligible to act at the next substage.

4. If w^i_ρ is currently defined and certified and $\Omega^{H_i}(w^i_\rho)\downarrow= 0[s]$, we let w^i_ρ and e^i_ρ enter the pinball machine, following Instruction 4.3 below. We then initialize all nodes $\supsetneq \rho$ and end this substage, with no node eligible to act at the next substage.

5. If w^i_ρ has entered the pinball machine, but has not yet entered G_i, we end this substage, with no node eligible to act at the next substage.

6. If w^i_ρ has already been enumerated into G_i by stage s, then end this substage, making $\rho\hat{\ }\langle f\rangle$ eligible to act at the next substage.

Instruction 4.3 (Entering the Pinball Machine) *The* gates *of the pinball machine are precisely the $\mathcal{P}$-nodes and the $\mathcal{U}$-nodes. If a ball entering the pinball machine at stage $s+1$ has subscript ρ, we call ρ the* source node *for that ball. ρ will be either a $\mathcal{D}$-node or an $\mathcal{M}$-node.*

- *If there is no gate $\alpha \subset \rho$, then we enumerate e^j_ρ into $E_j[s+1]$ and w^i_ρ into $G_i[s+1]$. (Notice that every ball which enters the pinball machine, either from a $\mathcal{D}$-node or an $\mathcal{M}$-node, is already certified, hence allowed to enter its target set.)*

- *If there is a gate $\alpha \subset \rho$, then we drop w^i_ρ to the greatest such α, and drop e^j_ρ to the greatest gate $\sigma \subseteq \alpha$ to which a requirement $\mathcal{U}_{e,2k+j}$ is assigned. If there is no such σ, then we enumerate e^j_ρ into $E_j[s+1]$ and appoint a new trace or traces for w^i_ρ, following Instruction 4.4.*

Instruction 4.4 (Assigning Traces) *At stage $s+1$, if the ball w^i_β is waiting at a $\mathcal{P}$-gate, we assign a large trace e^0_β (if i is odd) or e^1_β (if i is even). If it is waiting at a $\mathcal{U}_{e,2k}$-gate α, we follow these directions:*

- *If i is even and $i > 0$, then we assign a new trace e^1_β, chosen large and targeted for E_1. This ball starts at gate α.*
- *If i is odd and $i \neq 2k+1$, then we assign a new trace w^{i-1}_β, chosen large and targeted for G_{i-1}, and this trace is assigned its own trace e^1_β targeted for E_1. Each of these two balls starts at gate α. (Since $i-1 \neq 2k$, this will not threaten the restraint imposed by $\mathcal{U}_{e,2k}$.) The trace w^{i-1}_β is immediately considered certified.*
- *If $i = 2k+1$, then the ball w^i_β is not assigned any traces at this gate. Instead, w^i_β passes gate α immediately and drops to the greatest gate $\alpha' \subsetneq \alpha$ to which either a $\mathcal{P}$-requirement or a requirement $\mathcal{U}_{e',k'}$ with $k'+1 \neq i$ is assigned. We then follow these same instructions with α' in place of α. If there is no such node α', then w^i_β enters $G_i[s+1]$. In this case we check whether w^i_β was a trace for another ball. If so, then we follow these same instructions for that ball at the gate at which it is currently waiting.*
- *If $i = 0$, then the ball w^i_β waits at gate α but is not assigned any traces.*

To create traces for a ball w^i_β waiting at a $\mathcal{U}_{e,2k+1}$-gate α, we follow the analogous directions, with the special case occurring when $i = 2k+2$. The ball w^i_β and the one or two traces defined above together constitute a block, *with lead ball w^i_β. Any previously existing blocks which contained w^i_β become undefined.*

The point of this process is that (barring initialization of α), at any subsequent α-stage at which the restraint r at gate α is less than w^i_β, all the balls in the block will be able to pass gate α simultaneously. For instance, in the case where α is a $\mathcal{U}_{e,2k+1}$-gate, either no ball in the block is targeted for F_1 (if i is odd) or none of them is targeted for $G_{2k+1} \oplus E_1$ (if i is even and $j \neq 2k+2$). As noted above, the case $i = 2k+2$ is an exception, but then the ball w^i_β is targeted for G_{2k+2}, allowing G_{2k+2} to compute $\Phi^{F_1}_e$, so $\mathcal{U}_{e,2k+1}$ will still be satisfied. Finally, if $i = 0$ then no $\mathcal{T}$-requirement applies to G_i, so no trace is required. Therefore the entire block will be able to drop down to the next gate of the pinball machine simultaneously without violating requirement $\mathcal{U}_{e,2k+1}$. If more than one block of balls is waiting at a gate, we allow the block with highest-priority subscript to pass first; if several blocks have the same subscript, then the one with the largest lead ball goes first, since the larger lead ball will be a trace for the smaller lead ball.

If ρ is an $\mathcal{R}_k$-node, we extend the functional Ξ_ρ it builds. If $s' = s''$, then $\Xi_\rho^{G_k \oplus P}[s'+1]$ is empty, and we let $y = -1$; otherwise we let $y = \max(\mathrm{dom}(\Xi_\rho^{G_k \oplus P}))[s'+1]$. We check whether there is any $x \leq y$ such that $\Xi_\rho^{G_k \oplus P}(x)[s]$ does not converge to $Q(x)[s]$. If there is no such x, we define $\Xi_\rho^{G_k \oplus P}(y+1)[s+1] = Q(y+1)[s]$ with large use. If there is, then for each such x we act as follows:

- If x has entered $Q[s]$ since s', then it must have done so on behalf of an $\mathcal{M}$-node above ρ, and this node will have enumerated an element into P, allowing us to redefine $\Xi_\rho^{G_k \oplus P}(x)[s+1] = 1$. We do so, since now $Q(x)[s] = 1$, and we leave the use of the computation unchanged.
- If $Q(x)[s] = Q(x)[s']$, then there must have been a change in $G_k \oplus P$ since stage $s'+1$ on the use of the computation $\Xi_\rho^{G_k \oplus P}(x)[s'+1]$. Therefore, we simply redefine $\Xi_\rho^{G_k \oplus P}(x)[s+1] = Q(x)[s]$. If either $x \in G_k[s] - G_k[s']$ or $x = x_\alpha$ for some $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node $\alpha \supset \rho$ such that $w_\alpha^k \in G_k[s] - G_k[s']$ (as defined below), then we choose the use of this computation to be large. Otherwise, we retain the previous use.

$\mathcal{R}_k$ has only one outcome, namely ∞, and we end this substage, making $\rho\hat{\ }\langle\infty\rangle$ eligible to act at the next substage.

If ρ is an $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node, then there may be a witness element x_ρ already defined, which we will use to try to make $\Lambda^{W_j \oplus P} \neq Q$. We will also ask whether the stage $s+1$ is ρ-expansionary, defined as follows. Let

$$m(\rho, s) = \max\{y \leq s : (\forall x < y)\ \Upsilon_\rho^{G_i}(x)\downarrow = W_j(x)[s]\}$$

$$l(\rho, s) = \max\{z \leq s : (\forall x < z)\ \Lambda^{W_j \oplus P}(x)\downarrow = Q(x)[s] \text{ with use} < m(\rho, s)\}.$$

The stage $s+1$ is ρ-expansionary if ρ is eligible to act at $s+1$ and $l(\rho, s) > l(\rho, t)$ for every t with $s'' < t < s$ at which ρ was eligible to act. Thus, we are considering not the actual length of agreement between $\Lambda^{W_j \oplus P}[s]$ and $Q[s]$, but rather that portion of the length of agreement which we can guarantee by putting sufficient restraint on $W_j \oplus P$. Of course, we cannot restrain W_j directly, but we achieve this purpose by putting restraint on G_i and noting that the use of $\Lambda^{W_j \oplus P}[s]$ is less than the length of agreement between $\Upsilon_\rho^{G_i}[s]$ and $W_j[s]$. We define $r+1$ to be the greatest ρ-expansionary stage such that $s'' < r < s$. (If there has been no such stage, then $s' = s''$, and we set $r = s''$.)

1. If $x_\rho \in Q[s]$, then we end this substage, making $\rho\hat{\ }\langle f\rangle$ eligible to act at the next substage. (This preserves any finite win we may have achieved through Substep 6 at a previous stage since s''.)
2. If $s+1$ is not ρ-expansionary and x_ρ is not defined, then we end this substage, making $\rho\hat{\ }\langle w\rangle$ eligible to act at the next substage.

3. If $s+1$ is ρ-expansionary but no witness element x_ρ existed at stage $r+1$, then we check whether there exists $x \in \mathrm{dom}(\Theta_\rho^{W_j})[r+1]$ such that $x \in G_i[s] - G_i[r]$ and $W_j[s]$ has not changed on the use $\theta_\rho^{W_j}(x)[r]$ since stage r.

   (a) If there is no such x, then we extend $\mathrm{dom}(\Theta_\rho^{W_j})[s+1]$ up to $l(\rho, s) - 1$ by defining $\Theta_\rho^{W_j}(x)[s+1] = G_i(x)[s]$, with use $\theta_\rho^{W_j}(x)[s+1] = \lambda^{W_j \oplus P}(x)[s]$, for each x for which $\Theta_\rho^{W_j}(x)[s]$ is not already defined. (Possibly this defines $\Theta_\rho^{W_j}(x)[s+1] \neq \Theta_\rho^{W_j}(x)[r+1]$ for certain x, but only if $W_j[s]$ has changed on the use of the computation at r.) We make $\rho\hat{\ }\langle\infty\rangle$ eligible to act at the next substage, and end this substage.

   (b) If some $x \in \mathrm{dom}(\Theta_\rho^{W_j})[r+1]$ has entered $G_i[s]$ since stage r, without any corresponding W_j-change as above, we choose x_ρ be the least such x. For each $k \neq i$ such that some $\beta_k \subset \rho$ is an $\mathcal{R}_k$-node, we set $w_\rho^k = \xi_{\beta_k}^{G_k \oplus P}(x_\rho)[s]$ and assign to it a large trace e_ρ^l targeted for the set E_l, where $l = 0$ if k is odd and $l = 1$ if k is even. In order to preserve $(W_j \oplus P)\restriction \lambda^{W_j \oplus P}(x_\rho)[s]$, we initialize all nodes above ρ and end this substage, with no successor eligible to act at the next substage.

4. If x_ρ is defined and no balls with subscript ρ are currently on the pinball machine, but some ball w_ρ^k is defined and has not yet entered the machine, then for the least such k, we check whether w_ρ^k is certified at stage s (using the same definition as for $\mathcal{D}$-nodes, from page 23). If so, then we allow w_ρ^k and its trace to enter the machine at node ρ, in accordance with Instruction 4.3; if not, we do nothing. In either case we end this substage, with no successor eligible to act at the next substage.

5. If x_ρ is defined and some ball w_ρ^k has entered the pinball machine but is not yet in G_k, then we end this substage, with no successor eligible to act at the next substage.

6. If x_ρ is defined but not in $Q[s]$, and every ball w_ρ^k currently defined has entered $G_k[s]$, then we enumerate x_ρ into $Q[s+1]$. For every k such that some $\beta_k \subset \rho$ is an $\mathcal{R}_k$-node, we enumerate $\xi_{\beta_k}^{G_k \oplus P}(x_\rho)[s]$ into $P[s+1]$. We initialize every node $\supset \rho$ and terminate this substage, with no node eligible to act at the next substage.

   (The initializations when x_ρ was defined guarantee that either $\Lambda^{W_j \oplus P}(x_\rho)\downarrow = 0 \neq Q(x_\rho)[s]$ or W_j has changed in such a way that $W_j \neq \Upsilon^{G_i}$. Each possibility yields a finite win on requirement $\mathcal{M}_{i,j,\Lambda,\Upsilon}$. For each $k \neq i$, the enumerations of w_ρ^k into G_k guarantee that $\xi_{\beta_k}^{G_k \oplus P}(x_\rho)[s]$ has been chosen large since x_ρ was defined. Also, before x_ρ was chosen as x_ρ, it entered G_i, and at the next β_i-stage $t+1$, $\xi_{\beta_i}^{G_i \oplus P}(x_\rho)[t+1]$ was chosen large. Since $\lambda^{W_j \oplus P}(x_\rho)$ has not changed since before x_ρ entered G_i, our P-enumerations at this stage do not affect the convergence $\Lambda^{W_j \oplus P}(x_\rho)\downarrow = 0[s]$. Moreover, now each β_k (including $k = i$) will be allowed to redefine $\Xi_{\beta_k}^{G_i \oplus P}(x_\rho) = 1$ at the next β_k-stage, since now $Q(x_\rho)[s+1] = 1$.)

If ρ is an $\mathcal{N}_{e,\Phi}$-node, we define the length of agreement for ρ at this stage by:

$$l(\rho, s) = \max\{x : (\forall y < x)\Phi^R(y)\downarrow = W_e(y)[s]\}.$$

The stage $s+1$ is *ρ-expansionary* if ρ is eligible to act at s and $l(\rho, s) > l(\rho, t)$ for every t with $s'' < t < s$ at which ρ is eligible to act.

If $s+1$ is not ρ-expansionary, we end this substage, with $\rho\hat{}\langle w\rangle$ eligible to act at the next substage. Otherwise, for each $y < l(\rho, s)$ for which $\Gamma_\rho^{W_e \oplus B}(y)[s]$ is undefined, let $\Gamma_\rho^{W_e \oplus B}(y)[s+1] = C(y)[s]$. To define the use $\gamma_\rho^{W_e \oplus B}(y)[s+1]$, we ask if any of the following apply:

- $\gamma_\rho^{W_e \oplus B}(y)[s'+1]$ was not defined; or
- $\gamma_\rho^{W_e \oplus B}(z)[s'+1] \in B[s]$ for some $z \leq y$ (which happens if some $\mathcal{P}$-node $\supseteq \rho\hat{}\langle\infty\rangle$ has enumerated it into B in order to allow $\gamma_\rho^{W_e \oplus B}(z)[s'+1]$ to increase); or
- for some $z \leq y$, some node $\supseteq \rho\hat{}\langle\infty\rangle$ in T has requested that $\gamma_\rho^{W_e \oplus B}(z)[s'+1]$ be increased.

If so, we choose $\gamma_\rho^{W_e \oplus B}(y)[s+1]$ to be large. If none of the conditions applies, then apparently no node above ρ has tried to destroy the functional Γ_ρ, so we set $\gamma_\rho^{W_e \oplus B}(y)[s+1] = \gamma_\rho^{W_e \oplus B}(y)[s'+1]$, in order to keep $\gamma_\rho^{W_e \oplus B}(y)$ from approaching ∞. We then make $\rho\hat{}\langle\infty\rangle$ eligible to act at the next substage, and end this substage.

If ρ is a $\mathcal{P}_{i,\Psi}$-node, we first check if any balls with subscripts $\supset \rho$ are presently waiting at any gate below gate ρ. If so, then we end this substage, with no successor eligible to act at the next substage. Otherwise ρ continues to try to satisfy $\mathcal{P}_{i,\Psi}$, and any $\mathcal{N}$-requirement active along ρ may be injured by the action of ρ. Let

$$K = \{k : \mathcal{N}_k \text{ is active along } \rho \text{ via some } \alpha_k\}.$$

Define e_k and Φ_k such that $\mathcal{N}_k = \mathcal{N}_{e_k,\Phi_k}$, and for brevity write Γ_k for the function $\Gamma_{\alpha_k}^{W_{e_k} \oplus B}[s]$, the current version of the functional being built by α_k ($k \in K$), with associated use function $\gamma_k = \gamma_{\alpha_k}^{W_{e_k} \oplus B}[s]$. We will also define a restraint $r(\rho, j, k, s+1)$ associated to each $k \in K$, denoting the restriction which ρ places on elements targeted for G_j. (Restraining G_j, coupled with the expansionary outcome of α_k, will help ensure that W_{e_k} does not change, or else will ensure that if it does change, we can trace the source of the change to some set other than G_j.) The restraint finally enforced by ρ on such elements will be

$$r(\rho, j, s+1) = \max_{k \in K} r(\rho, j, k, s+1).$$

(Any restraint $r(\rho, j, k, s+1)$ which is not mentioned is assumed to retain its value from stage $s'+1$, or is reset to 0 if $\rho\hat{}\langle a_k\rangle$ was initialized at or since stage $s'+1$.)

Subcase 1: If no witness z_ρ is presently defined for $\mathcal{P}_{i,\Psi}$, then pick a large number z and designate it as the witness element z_ρ for $\mathcal{P}_{i,\Psi}$. Let $\rho\hat{}\langle w\rangle$ be eligible to act at the next substage, and end this substage.

Subcase 2: If $z_\rho \in C[s]$, then let $\rho\hat{}\langle f\rangle$ be eligible to act at the next substage, and end this substage.

Subcase 3: If $z_\rho \notin C[s]$ and ρ enumerated any elements into B at stage $s'+1$ (using Subcase 6(b)), then we set $\tilde{k}_{s+1} = \tilde{k}_{s'+1}$, $w_{s+1} = w_{s'+1}$, and $e = e_{\tilde{k}_{s+1}}$ and redefine the same functional $\Delta_{\rho,\tilde{k}_{s+1}}$ which we extended at that stage (using the notation from Subcase 6(b) below):

$$\Delta^{G_i}_{\rho,\tilde{k}_{s+1}} \restriction w_{s+1}[s+1] = W_e \restriction w_{s+1}[s].$$

If this involves adding any axioms to $\Delta_{\rho,\tilde{k}_{s+1}}$, we choose the use to be large. (Sublemma 5.17 will ensure that our redefinition of $\Delta_{\rho,\tilde{k}_{s+1}}$ is allowed.) We also set $r(\rho, i, \tilde{k}_{s+1}, s+1) = \varphi^R(\gamma_{\tilde{k}_{s+1}}(z_\rho))[s]$, which (along with the restraints set at stage $s'+1$) guarantees that $\alpha_{\tilde{k}_{s+1}}$ will preserve $W_e \restriction \gamma^{W_e \oplus B}_{\tilde{k}_{s+1}}(z_\rho)[s]$ until the next ρ-stage. We then end this substage, with $\rho\hat{}\langle a_{\tilde{k}_{s+1}}\rangle$ eligible to act at the next substage. (This is the stage where we complete the business begun in Subcase 6(b) at stage $s'+1$. Between $s'+1$ and s, any ball targeted for G_i can pass gate ρ without injuring the outcome $a_{\tilde{k}_{s+1}}$, since our B-enumeration at stage $s'+1$ already destroyed the convergence of $\Psi^{G_i \oplus B}(z_\rho)[s']$. Also, this is the only subcase in which $\rho\hat{}\langle a_{\tilde{k}_{s+1}}\rangle$ is made eligible. If $\rho\hat{}\langle a_k\rangle$ is on the true path, then after stage s'' we will cycle forever from Subcase 4 to 5 to 6 to 3 and back to 4, with $\tilde{k}_{s+1} = k$ infinitely often and $\tilde{k}_{s+1} < k$ only finitely often, and in Subcases 6(b) and 3 we will build a total function $\Delta^{G_i}_{\rho,k} = W_{e_k}$ to satisfy $\mathcal{N}_k$.)

Subcase 4: If $z_\rho \notin C[s]$ and ρ made no B-enumeration at stage $s'+1$ and z_ρ is not yet realized (i.e. either $\Psi^{G_i \oplus B}(z_\rho)[s]\uparrow$ or $\Psi^{G_i \oplus B}(z_\rho)[s]\downarrow \neq 0$), then we let $\rho\hat{}\langle w\rangle$ be eligible to act at the next substage, and end this substage.

Subcase 5: If $z_\rho \notin C[s]$ and $\Psi^{G_i \oplus B}(z_\rho)[s]\downarrow = 0$ and ρ made no B-enumeration at stage s', then consider each $\mathcal{N}_{e,\Phi}$-node $\beta \subset \rho$ (for each e and Φ) such that $\mathcal{N}_{e,\Phi}$ is *not* active along ρ via β. Let τ_β be the greatest node $\subset \rho$ such that $\mathcal{N}_{e,\Phi}$ is active along τ_β via β; there must be such a node τ_β, and it must be a $\mathcal{P}$-node. If for some such β we have $\gamma^{W_e \oplus B}_\beta(z_{\tau_\beta})\downarrow \leq \psi^{G_i \oplus B}(z_\rho)[s]$ then we let $\rho\hat{}\langle w\rangle$ be eligible to act at the next substage, and end this substage.

(If ρ is on the true path, then for each such β the requirement $\mathcal{N}_{e,\Phi}$ is either satisfied or destroyed at τ_β. Hence $\gamma^{W_e \oplus B}_\beta(z_{\tau_\beta})$ will eventually be redefined to be $> \psi^{G_i \oplus B}(z_\rho)[s]$. We wait for this to take place, because we do not want to enter Subcase 6 until we are certain that the convergence $\Psi^{G_i \oplus B}(z_\rho)\downarrow = 0$ will not be disrupted even if τ_β decides to enumerate $\gamma^{W_e \oplus B}_\beta(z_{\tau_\beta})$ into B.)

Subcase 6: Otherwise $z_\rho \notin C[s]$ and $\Psi^{G_i \oplus B}(z_\rho)[s]\downarrow = 0$ and ρ made no B-enumeration at stage $s'+1$ and there is no node β as described in Subcase 5. We choose $\tilde{k}_{s+1}$ to be the greatest $k \in K$ such that

$$\gamma_k(z_\rho) \leq \psi^{G_i \oplus B}(z_\rho)[s]. \tag{1}$$

If no k satisfies this condition, then let $\tilde{k}_{s+1} = -1$.

If $\tilde{k}_{s+1} = -1$, we enumerate $z_\rho \in C[s+1]$ (to satisfy $\mathcal{P}_{i,\Psi}$). We enumerate into $B[s+1]$ every number in the set

$$\{\gamma_\beta^{W_e \oplus B}(z_\rho)[s] : \beta \subset \rho \;\&\; \beta \text{ is an } \mathcal{N}\text{-node}\},$$

in order to allow the corresponding functionals Γ_β to change their value on the argument z_ρ. To preserve the computation $\Psi^{G_i \oplus B}(z_\rho)\!\downarrow = 0[s]$, we initialize every node $\succ \rho$. We set all restraints $r(\rho, j, k, s+1)$ to 0, since no further restraint is necessary after these initializations. Then we end this substage, with no successor eligible to act at the next substage. (Now that we have enumerated z_ρ into C, each such β will wait until the next stage at which it is eligible and then adjust Γ_β to compute C correctly. Our B-enumeration ensures that these changes will be possible.)

If $-1 < \tilde{k}_{s+1} < \tilde{k}_{s'+1}$ (or if $-1 < \tilde{k}_{s+1}$ and $\tilde{k}_{s'+1}$ was not defined), we request that the node $\alpha_{\tilde{k}_{s+1}}$ increase the use $\gamma_{\tilde{k}_{s+1}}(z_\rho)$ at the next opportunity, and revoke any corresponding request for $\alpha_{\tilde{k}_{s'+1}}$, since that request must have been fulfilled in order for $\tilde{k}$ to have decreased. We also set

$$r(\rho, i, \tilde{k}_{s+1}, s+1) = \max(\psi^{G_i \oplus B}(z_\rho)[s], \varphi^R(\gamma_{\tilde{k}_{s+1}}(z_\rho))[s]),$$

and for all $j \neq i$ we set $r(\rho, j, \tilde{k}_{s+1}, s+1) = 0$ (since at this stage, balls targeted for G_j with $j \neq i$ may pass node ρ without injuring our strategy for satisfying $\mathcal{P}_{i,\Psi}$). We then end this substage, with no successor eligible to act at the next substage.

Finally, if $\tilde{k}_{s+1} = \tilde{k}_{s'+1} > -1$, we write $\tilde{k} = \tilde{k}_{s+1}$, $e = e_{\tilde{k}}$, $\Phi = \Phi_{\tilde{k}}$, $\sigma = \rho\hat{\ }\langle a_{\tilde{k}}\rangle$, and $\gamma = \gamma_{\tilde{k}} (= \gamma_{\alpha_{\tilde{k}}}^{W_e \oplus B})$ for simplicity, and let $t+1$ be the greater of $s''+1$ and the last stage at which $\Delta_{\rho,\tilde{k}}$ was extended. We select the appropriate step among the following.

a. If some G_j with $j \neq i$ changed on $\varphi^R(\gamma(z_\rho))[s']$ between stage s' and stage s, then we initialize all nodes to the right of σ and end this substage, with no successor eligible to act at the next substage.

b. Otherwise, we let

$$w_{s+1} = \min(\gamma(z_\rho), 1 + \mathrm{dom}(\Delta_{\rho,\tilde{k}}^{G_i}))[t+1]$$

(here regarding $\mathrm{dom}(\Delta_{\rho,\tilde{k}}^{G_i})[t+1]$, a finite initial segment of ω, as an integer). We define

$$\Delta_{\rho,\tilde{k}}^{G_i} \upharpoonright w_{s+1}[s+1] = W_e \upharpoonright w_{s+1}[s].$$

If this involves adding any axioms to $\Delta_{\rho,\tilde{k}}$, we choose the use to be large. Sublemma 5.17 will ensure that these redefinitions are allowed.

We enumerate $\gamma(z_\rho)[s'+1]$ into $B[s+1]$, making $\Gamma^{W_e \oplus B}_{\alpha_{\tilde{k}}}(z_\rho)[s+1]$ undefined, so that $\alpha_{\tilde{k}}$ will increase the use $\gamma(z_\rho)$ at the next $\alpha_{\tilde{k}}$-stage. (Notice that $\gamma(z_\rho)[s'+1]$ did not already lie in $B[s]$. Only numbers in the ranges of the γ-functions are ever enumerated into B, and such numbers are always chosen large.) By (1), $\Psi^{G_i \oplus B}(z_\rho)[s+1]$ also becomes undefined. We set $r(\rho, i, \tilde{k}, s+1) = 0$ and $r(\rho, j, \tilde{k}, s+1) = \varphi^R(\gamma(z_\rho))[s]$ for all $j \neq i$, to ensure that until the next stage at which z_ρ is realized, $W_e \restriction w_{s+1}$ can only change on account of a G_i-change, which will allow us to redefine $\Delta^{G_i}_{\rho,\tilde{k}}$ wherever needed. We also revoke our request for $\alpha_{\tilde{k}}$ to increase the use $\gamma_{\alpha_{\tilde{k}}}(z_\rho)$. We initialize all nodes to the right of σ and end this substage, with no successor eligible to act at the next substage.

This completes the instruction for $\mathcal{P}$-nodes.

If ρ is a $\mathcal{U}_{e,i}$-node, let $m = 0$ if i is even and $m = 1$ if i is odd, and set:

$$l(\rho, s) = \max\{x : (\forall y < x)\Phi_e^{G_i \oplus E_m}(y)\downarrow = \Phi_e^{F_m}(y)\downarrow [s]\}.$$

The stage $s+1$ is *ρ-expansionary* if $s = 0$ or $l(\rho, s) > l(\rho, t)$ for all stages $t+1$ with $s'' < t < s$ at which ρ was eligible to act. We define $r(\rho, s) = 0$ if $s+1$ is ρ-expansionary, while otherwise $r(\rho, s)$ is the greatest number used in the construction up until the last ρ-expansionary stage. If no ball with a subscript $\supseteq \rho\hat{\ }\langle p_{r(\rho,s)}\rangle$ is waiting at any gate $\subsetneq \rho$ at stage s, then we make $\rho\hat{\ }\langle p_{r(\rho,s)}\rangle$ eligible to act at the next substage, and end this substage. If any such ball is waiting at any gate below ρ, then we initialize all nodes to the right of $\rho\hat{\ }\langle p_{r(\rho,s)}\rangle$, but end this substage without making any nodes eligible to act at the next substage.

This completes the instructions for the substages of the stage $s+1$. Once we have completed all s substages, or reached a substage at which no new node is made eligible to act at the next substage, we proceed to the final two steps of the stage: satisfying the $\mathcal{L}$-requirements and allowing balls on the pinball machine to drop to lower gates.

First we consider the $\mathcal{L}$-requirements. If $\mathcal{L}_k = \mathcal{L}_{\Phi,x}$, then k is the priority of that requirement. For each $\alpha \in T$, define $n(\alpha, s)$ to be the number of times that α has been initialized (up to stage s) by other nodes on T. (We do *not* count any initializations by $\mathcal{L}$-requirements themselves in this total.) For the least $k \leq s$ such that $\Phi^R(x)[s]\downarrow$ and $\Phi^R(x)[s-1]$ either diverges or converges with a different use, the requirement $\mathcal{L}_k$ initializes every $\alpha \in T$ satisfying:

$$\ulcorner \alpha \urcorner + n(\alpha, s) > k.$$

(Here $\ulcorner \alpha \urcorner \in \omega$ is a code for the node α, with T viewed as a subtree of $\omega^{<\omega}$.) This guarantees that none of the α initialized will later injure $\mathcal{L}_k$.

Finally, we use a pinball-style approach to determine which ball(s) currently on the pinball machine can pass the gate at which they are currently waiting. Choose the highest-priority α such that there is a gate ρ which was eligible at the current stage $s+1$ such that:

- there is a block of balls waiting at ρ, with lead ball w^j_α or e^j_α; and
- if ρ is a $\mathcal{U}$-gate, then the lead ball of the block is $> r(\rho, s)$; and
- if ρ is a $\mathcal{P}$-gate, then the lead ball of the block either is of the form e^j_α or is $> r(\rho, j, s+1)$; and
- no ball which passed gate ρ at any earlier stage is currently waiting at any gate below ρ.

If there is no such α, then end the stage. If α exists, then the corresponding ρ is unique (by the last condition), and we choose the greatest lead ball with subscript α currently waiting at gate ρ. We allow all balls in its block to pass gate ρ, initialize all nodes $\succ \alpha$, and follow Instruction 4.5 below for the balls in the block. Once the balls pass gate ρ, they are no longer in the same block. (For convenience, we usually think of the node α as having performed the initialization of the nodes $\succ \alpha$, even though α itself may not have been eligible at this stage.)

Instruction 4.5 (Dropping to a new gate) *1. For each ball e^k_α which passed gate ρ, we drop e^k_α to the highest $\mathcal{U}_{e,2l+k}$-gate $\tau \subsetneq \rho$ (for any l), if such a τ exists. Its block at gate τ consists only of itself.*

2. For each ball w^k_α which passed gate ρ, we drop w^k_α to the highest gate $\tau \subsetneq \rho$, if such a τ exists. For the time being, its block at gate τ consists only of itself, but traces may be added later.

3. If there is no such τ, then we enumerate the ball into its target set (w^k_α into $G_k[s+1]$ or e^k_α into $E_k[s+1]$). If this ball was a trace for another ball w^j_α which does not enter G_j at this same stage, then we add new traces for w^j_α in accordance with Instruction 4.4, to form a new block at the gate at which w^j_α is currently waiting. (If w^j_α is waiting at a $\mathcal{U}_{e,j-1}$-gate, this process will involve dropping it to a lower gate or into G_j.)

If the ball was not a trace, then it was of the form w^k_α. Either it was enumerated into G_k for the sake of some $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node α with $i \neq k$, so as to allow an $\mathcal{R}_k$-node $\beta \subset \alpha$ to redefine its functional $\Xi_\beta^{G_k \oplus P}$, helping α achieve a finite win; or it was a witness element for a $\mathcal{D}$-requirement assigned to α, in which case that $\mathcal{D}$-requirement is now satisfied.

Notice that under these instructions, no ball w^j_α can end up at a lower gate than its trace.

This completes the construction.

5 Verification of the Construction

To prove that the structure of our tree allows every node to be satisfied, we first need a sublemma.

Sublemma 5.1 *At every node ρ on T, if $\mathcal{N}_j$ is active or satisfied along ρ and $i < j$, then $\mathcal{N}_i$ is active or satisfied along ρ also.*

Proof. We use induction on the level of the node ρ. Suppose $\mathcal{N}_j$ is active or satisfied along ρ, and let $i < j$. Write $\eta = \rho^-$.

Case 1. Suppose $\mathcal{N}_j$ was active or satisfied along η. Then by induction so was $\mathcal{N}_i$. The only way for $\mathcal{N}_i$ not to be active or satisfied along ρ is if a $\mathcal{P}$-requirement was assigned to η and $\rho = \eta\hat{}\langle a_l\rangle$, for some l such that $\mathcal{N}_l$ is active along η via some α along which $\mathcal{N}_i$ is neither active nor satisfied. Since $\alpha \subsetneq \rho$, the inductive hypothesis ensures that $\mathcal{N}_j$ was neither active nor satisfied along α either. But then $\mathcal{N}_j$ cannot be active or satisfied along ρ, contradicting the assumption of the sublemma.

Case 2. Otherwise $\mathcal{N}_j$ was neither active nor satisfied along η, so in order to become active or satisfied at ρ it must have been assigned to η. Since $\mathcal{N}_i$ has higher priority than $\mathcal{N}_j$, this implies that $\mathcal{N}_i$ was already active or satisfied at η. But with an $\mathcal{N}$-requirement assigned to η, every requirement active or satisfied at η will still be active or satisfied at ρ, including $\mathcal{N}_i$. ■

Lemma 5.2 *For every path h through T and every requirement $\mathcal{N}_l$, there exists an $\mathcal{N}_l$-node $\alpha \subset h$ such that either:*

- *$\mathcal{N}_l$ is active via α along every β with $\alpha \subseteq \beta \subset h$; or*
- *there exists $\sigma \subset h$ such that $\mathcal{N}_l$ is active via α along every β with $\alpha \subseteq \beta \subseteq \sigma$, and satisfied via α along every β with $\sigma \subsetneq \beta \subset h$.*

Proof. Fix h, and assume by induction that the lemma holds for every $\mathcal{N}$-requirement of higher priority than $\mathcal{N}_l$. This yields a node $\alpha_i \subset h$ for each $i < l$, as well as nodes $\sigma_i \subset h$ for certain $i < l$, and we take ξ to be the largest of all these nodes (both α_i's and σ_i's). Then no $\mathcal{N}_i$ with $i < l$ is assigned to any node on h extending ξ.

Case 1. Suppose first that there exists an $\mathcal{N}_l$-node $\alpha \subset h$ above ξ. $\mathcal{N}_l$ must be active or satisfied along the immediate successor of α on h. We argue inductively that $\mathcal{N}_l$ must be active or satisfied via α along every $\beta \supset \alpha$ on h. Let $\rho = \beta^-$. Then the only way $\mathcal{N}_l$ could possibly fail to be active or satisfied at β via α is if a $\mathcal{P}$-requirement is assigned to ρ, and $\beta = \rho\hat{}\langle a_m\rangle$ for some m such that $\mathcal{N}_m$ is active along ρ via some η. According to the construction, $\mathcal{N}_m$ is then satisfied along β, so by the inductive hypothesis on l, we have $m > l$. But then, in order for $\mathcal{N}_m$ to have been assigned to η, $\mathcal{N}_l$ must have been active or satisfied along η. Hence $\mathcal{N}_l$ remains active or satisfied along β via α, by Definition 4.2.

Case 2. Otherwise $\mathcal{N}_l$ is not assigned to any node on h above ξ. Then $\mathcal{N}_l$ must have been assigned to some node below ξ (since otherwise it would eventually be assigned to some node above ξ, as no higher-priority requirement can be assigned to more than one node on h above ξ). So let α be the greatest $\mathcal{N}_l$-node $\subseteq \xi$. If $\mathcal{N}_l$ were neither active nor satisfied via α along any node on h above α, then a new node on h above α would be chosen as an $\mathcal{N}_l$-node, contrary to hypothesis. Thus in both of these two cases, $\mathcal{N}_l$ is either active or satisfied via the chosen α along every node β with $\alpha \subset \beta \subset h$.

Finally, we note that $\mathcal{N}_l$ cannot switch from satisfied via α at a node ρ to active at any of its immediate successors β. According to Definition 4.2, if $\mathcal{N}_l$ is satisfied via α at ρ, then either $\mathcal{N}_l$ is satisfied via α at β, or $\beta = \rho\hat{\ }\langle a_m \rangle$ for some m and $\mathcal{N}_l$ is neither active nor satisfied at β. Since this does not happen at any β with $\alpha \subset \beta \subset h$, we see that either $\mathcal{N}_l$ is active via α along every such β, or it is satisfied via α along every such β, or it is active via α along $\alpha\hat{\ }\langle \infty \rangle$, switches to satisfied via α along some higher $\sigma \subset h$, and then stays satisfied via α along all extensions of σ on h. In each of these cases, the lemma holds for $\mathcal{N}_l$. ∎

The *true path* g through T is defined inductively. It begins at the root of T, and for each $\rho \subset g$, we extend g to include the leftmost immediate successor τ of ρ such that τ is eligible to act at infinitely many stages. The existence of such a τ will be shown by induction in Lemma 5.9. To begin this induction, however, we need some sublemmas first.

$\mathcal{D}$- and $\mathcal{M}$-nodes are the only nodes that ever try to enumerate balls into the sets G_k. To see that these enumerations do occur, we need the following sublemmas.

Sublemma 5.3 *If the ball w_α^i is ever chosen as x_σ by an $\mathcal{M}$-node σ, then $\sigma\hat{\ }\langle \infty \rangle \subseteq \alpha$.*

Proof. The node σ has three immediate successors, corresponding to its outcomes f, ∞, and w. To be chosen as x_σ at a stage $s+1$, w_α^i must have been enumerated into G_i since the last $\sigma\hat{\ }\langle \infty \rangle$ stage $r+1$. If $\alpha \prec \sigma$, this enumeration would have initialized σ, in which case σ would not have chosen any x_σ at $s+1$. If $\sigma\hat{\ }\langle f \rangle \subseteq \alpha$, then α was never eligible until after x_σ was selected. If α lies to the right of $\sigma\hat{\ }\langle \infty \rangle$, then w_α^i must have been chosen after the last $\sigma\hat{\ }\langle \infty \rangle$-stage, hence could not lie in $\mathrm{dom}(\Theta_\sigma^{W_j})[s]$ and would not have been chosen as x_σ. Finally, $\alpha \neq \sigma$, since σ does not target balls for any set G_k until it chooses x_σ. Hence $\sigma\hat{\ }\langle \infty \rangle \subseteq \alpha$. ∎

The next sublemma will be used extensively throughout the rest of our proofs. It guarantees that balls entering the pinball machine are sufficiently large not to injure any higher-priority requirements.

Sublemma 5.4 *Let $s'' + 1$ be the greatest stage $< s+1$ at which α was initialized, and let $t+1$ be the least α-stage $> s''+1$. If a ball w_α^k enters the pinball machine at stage $s+1$, then w_α^k is greater than any number used in the construction up to stage t.*

Proof. If α is a $\mathcal{D}$-node, then its current witness was chosen at stage $t+1$, so w^k_α was chosen large at some stage $\geq t+1$. If α is an $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node, then $w^k_\alpha = \xi^{G_k \oplus P}_\beta(x_\alpha)[s']$ for some $\mathcal{R}_k$-node $\beta \subset \alpha$ and some α-stage $s'+1 < s+1$, and $x_\alpha = w^i_{\alpha'}$ for some $\alpha' \supseteq \alpha\hat{\ }\langle\infty\rangle$ (by Sublemma 5.3). If α' is a $\mathcal{D}$-node, then we are done, since $w^k_\alpha > x_\alpha$ and α' is initialized every time α is. If not, then we continue by induction. Eventually we must reach a witness or trace for a $\mathcal{D}$-node, since only finitely many nodes have been eligible up to stage s. ∎

Sublemma 5.5 *For a $\mathcal{D}$-node $\alpha \subset g$, if w^i_α is realized at a stage after which α is never again initialized, then eventually w^i_α will enter G_i. For an $\mathcal{M}$-node $\alpha \subset g$, if w^i_α is a ball targeted for G_i by α at a stage after which α is never again initialized, then eventually w^i_α will enter G_i.*

Proof. Every $\mathcal{M}_{i',j',\Lambda,\Upsilon}$-node σ with $\sigma\hat{\ }\langle\infty\rangle \subset \alpha$ has $\sigma\hat{\ }\langle\infty\rangle \subset g$. At each $\sigma\hat{\ }\langle\infty\rangle$-stage $s+1$, the domain of $\Theta^{W_{j'}}$ is extended to the new length of agreement $l(\rho, s)$. Hence for each ball w^j_α ($j \leq i$) there exists a stage $s+1$ such that w^j_α is certified at all stages $t+1 \geq s+1$. So w^j_α will eventually enter the pinball machine and drop to the highest gate below α, as dictated by Instruction 4.3. (For $\mathcal{M}$-nodes α, this involves an easy induction on $j \leq i$. For $\mathcal{D}$-nodes, it only applies with $j = i$.)

Leaving α and i fixed, we argue by induction, first on gates $\rho \subset \alpha$ and then on $j \leq i$, that every ball w^j_α (including traces for other balls) which reaches a gate $\rho \subset \alpha$ must eventually pass that gate. This will prove the sublemma.

Suppose w^j_α is currently waiting at a $\mathcal{U}$-gate ρ and let $\rho\hat{\ }\langle p_r\rangle$ be the immediate successor of ρ on g. Since α is never again initialized, no ball with subscript $\prec \alpha$ will ever move again. By induction on ρ, every current trace for w^j_α at any gate below ρ will eventually enter its target set. (If $j = 0$, no trace is ever assigned, and similarly for traces targeted for E_0 or E_1.) By Sublemma 5.4, w^j_α and all its traces must be $> r$, since $\rho\hat{\ }\langle p_r\rangle \subset \alpha$. There may be a trace w^{j-1}_α for w^j_α which is also waiting at gate ρ but in a different block from w^j_α. However, by induction w^{j-1}_α eventually passes gate ρ and enters G_{j-1}, with Instruction 4.4 assigning a new trace (or two) to w^j_α. The new trace(s) lie in the same block as w^j_α, so after that, the next time $\rho\hat{\ }\langle p_r\rangle$ is eligible, w^j_α and its new trace(s) will pass ρ and drop to lower gates, in accordance with Instruction 4.5. Thus, by induction on ρ, w^j_α will eventually enter G_j.

Now suppose w^j_α has been waiting at a $\mathcal{P}_{i',\Psi}$-gate ρ since the last σ-stage, where $\sigma \subseteq \alpha$ is the immediate successor of ρ below α. If $\sigma = \rho\hat{\ }\langle f\rangle$, then there is a stage s_0+1 after which ρ sets all its restraints to 0. We also note that each time a restraint $r(\rho, j, k', s)$ is changed, all nodes to the right of $\rho\hat{\ }\langle a_{k'}\rangle$ are initialized. Hence if $\sigma = \rho\hat{\ }\langle w\rangle$, then no restraint is redefined after the last initialization of σ, so by Sublemma 5.4, w^j_α is larger than all such restraints and is allowed to pass gate ρ. Finally, if $\sigma = \rho\hat{\ }\langle a_k\rangle$, then $w^j_\alpha > r(\rho, j, k', s+1)$ for all s and all $k' < k$ by Sublemma 5.4, and all restraints $r(\rho, j, k', s+1)$ with $k' > k$ are reset to 0 whenever $\tilde{k}_{s+1} = k$. We know that σ is eligible infinitely often. If $i' = j$, then $r(\rho, j, k, s+1)$ is set to 0 infinitely often in Subcase 6(b) with $\tilde{k}_{s+1} = k$; if not, it is set to 0 each time $\tilde{k}_{s+1} = k \neq \tilde{k}_{s'+1}$ in Subcase 6. Thus $\liminf_s r(\rho, j, s) < w^j_\alpha$. This completes the induction. ∎

Lemma 5.6 *Let $\rho \subset g$ be an $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node such that $\Upsilon^{G_i} = W_j$ and $\Lambda^{W_j \oplus P} = Q$, and let $\rho\hat{\ }\langle\infty\rangle \subset \gamma$. Fix a ball w^i_γ, chosen by γ at a stage after which ρ is never again initialized. Let s_0+1 be the least ρ-expansionary stage with $w^i_\gamma \in \mathrm{dom}(\Theta^{W_j}_\rho)[s_0+1]$, and let s_1+1 be the greatest ρ-expansionary stage before which w^i_γ has not yet entered the pinball machine. Let s_2+1 be the greatest ρ-expansionary stage $<$ the stage at which w^i_γ either enters G_i or is cancelled, and let s_3+1 be the least ρ-expansionary stage $> s_2+1$. Then:*

1. *For every ρ-expansionary stage $t+1$ with $s_0 \leq t < s_1$, let $t'+1$ be the least ρ-expansionary stage $> t+1$. Then we have*

$$P\restriction l\ [t] = P\restriction l\ [t'],$$

where $l = \theta^{W_j}(w^i_\gamma)[t+1]$.

2. *For every ρ-expansionary stage $t+1$ with $s_1 \leq t < s_2$, we have*

$$(W_j \oplus P)\restriction l_1\ [t] = (W_j \oplus P)\restriction l_1\ [t'],$$

with t' as above, and where $l_k = \lambda^{W_j \oplus P}(w^i_\gamma)[s_k]$ (for $k = 1, 2$). Hence $l_1 = l_2$.

3. *If w^i_γ is chosen as x_ρ at stage s_3+1, then for every ρ-expansionary stage $t+1 \geq s_3+1$, we have*

$$(W_j \oplus P)\restriction l_1\ [t] = (W_j \oplus P)\restriction l_1\ [s_1]$$

with l_1 as above. Hence $\Lambda^{W_j \oplus P}(w^i_\gamma) = 0$.

If w^i_γ is never cancelled nor ever enters G_i, then the items above hold with $s_2 = s_3 = \infty$. If w^i_γ never enters the pinball machine, they hold with $s_1 = \infty$.

In fact we will use this lemma to show that w^i_γ *cannot* be chosen as x_ρ, so that (assuming $\Upsilon^{G_i} = W_j$ and $\Lambda^{W_j \oplus P} = Q$) the third item never actually applies.

Proof. For the first item, let $t+1$ be a ρ-expansionary stage with $s_0 \leq t < s_1$. We will proceed by induction on such t. Only $\mathcal{M}$-nodes enumerate elements into P, so suppose an $\mathcal{M}_{k,j',\Lambda',\Upsilon'}$-node σ enumerates some x into P at a stage $s+1$ such that $t+1$ is the greatest ρ-expansionary stage $\leq s+1$. For this s, the node σ must be unique, since every node $\succ \sigma$ is initialized at $s+1$. (This also forces $\gamma \preceq \sigma$, since γ is not initialized before s_1+1.) We wish to show that $x > l$.

Now the least number enumerated into P by σ at $s+1$ is of the form $\xi^{G_k \oplus P}_\beta(x_\sigma)[s]$, for the $\mathcal{R}_k$-node $\beta \subset \sigma$, and σ also enumerates x_σ into Q at stage $s+1$. If σ lay to the right of $\rho\hat{\ }\langle\infty\rangle$, then x_σ and its use $\xi^{G_k \oplus P}_\beta(x_\sigma)[s]$ would both have been chosen after stage

$t+1$ (since the ball x_σ would have been cancelled at stage $t+1$, by Sublemma 5.3), hence could not be $< l$.

So assume $\rho\hat{\ }\langle\infty\rangle \subseteq \sigma$. Now $x_\sigma = w_\alpha^{k'}$ for some $\alpha \supset \sigma$, by Sublemma 5.3, so $\gamma \prec \alpha$. Let w_α^k be the original ball released by α (so either $w_\alpha^{k'} = w_\alpha^k$, or $w_\alpha^{k'}$ is a trace for w_α^k, or a trace for a trace for w_α^k, etc.) Sublemma 5.4 shows that $w_\gamma^i < w_\alpha^k$. By the construction for $\mathcal{D}$- and $\mathcal{M}$-nodes, w_α^k did not enter the pinball machine until after the first ρ-expansionary stage $t_0 + 1 \leq t + 1$ with $w_\alpha^k \in \mathrm{dom}(\Theta_\rho^{W_j})[t_0]$, by which stage $w_\gamma^i \in \mathrm{dom}(\Theta_\rho^{W_j})[t_0]$ as well, since at every $\rho\hat{\ }\langle\infty\rangle$-stage this domain is an initial segment of ω. Thus $s_0 < t_0$. Let $l_0 = \lambda^{W_j \oplus P}(w_\gamma^i)[t_0]$, and let u_0 be the use of the computation $\Upsilon^{G_i}(l_0)[t_0]$. Then $\xi_\beta^{G_k \oplus P}(x_\sigma)[s]$ must be $> l_0$, having been chosen large after w_α^k entered G_k, hence after t_0.

Now if $W_j \restriction l_0$ has changed between $t_0 + 1$ and t, then there must have been a corresponding change in $G_i \restriction u_0$ between those same stages, since $t_0 + 1$ and $t + 1$ are both ρ-expansionary. So some ball $w_{\alpha'}^i < u_0$ entered G_i between $t_0 + 1$ and t. Now $w_{\alpha'}^i$ must have been chosen by stage t_0, by Sublemma 5.4, and so $\alpha' \prec \alpha$, since $w_{\alpha'}^i$ was not cancelled at $t_0 + 1$. But the entry of $w_{\alpha'}^i$ into G_i took place after $t_0 + 1$, hence after w_α^k entered G_k, since otherwise it would have cancelled w_α^k.

Now we apply the same argument to $w_{\alpha'}^i$ as to w_α^k. If t_0' is the greatest ρ-expansionary stage before $w_{\alpha'}^i$ (or the ball for which it was a trace) entered the machine, with corresponding uses l_0' and u_0', then $w_\gamma^i \in \mathrm{dom}(\Theta_\rho^{W_j})[t_0']$, and $s_0 < t_0' < t_0$. and so $\xi_\beta^{G_k \oplus P}(x_\sigma)[s] > l_0'$. But any change to $W_j \restriction l_0'$ between t_0' and t would require a corresponding change in $G_i \restriction u_0'$ between those same stages, by a ball $w_{\alpha''}^i$ entering G_i, and so forth. Since $G_i[t]$ is finite, this process must terminate. Thus eventually we find a stage $t_0^{(n)}$, with corresponding $l_0^{(n)}$ and $u_0^{(n)}$, for which $\xi_\beta^{G_k \oplus P}(x_\sigma)[s] > l_0^{(n)}$ and no change occurred in $W_j \restriction l_0^{(n)}$ between $t_0^{(n)} + 1$ and t. But by the inductive hypothesis on t, $P \restriction l_0^{(n)}[t_0^{(n)}] = P \restriction l_0^{(n)}[t]$ as well, and so in fact $l_0^{(n)} = l$. Hence $\xi_\beta^{G_k \oplus P}(x_\sigma)[s] > l$, completing our induction on t.

For the second item, let $t + 1$ be a ρ-expansionary stage with $s_1 \leq t < s_2$. Now all nodes $\alpha \succ \gamma$ are initialized at stage $s_1 + 1$ when w_γ^i enters the pinball machine. Hence no such α ever again enumerates any element $< l_1$ into P, nor any ball $< u_1$ into G_i, where u_1 is the use of the computation $\Upsilon^{G_i}(l_1)[s_1]$. Also, no node $\prec \gamma$ could enumerate any ball into G_i or any element into P without cancelling w_γ^i, which would contradict $t' \leq s_2$. Finally, γ itself never has two distinct balls on the pinball machine which are both targeted for the same set G_i, and γ cannot make any P-enumerations until w_γ^i has entered G_i. For P, this shows that $P \restriction l_1\ [t] = P \restriction l_1\ [s_1]$. For G_i, it shows that $G_i \restriction u_1\ [t] = G_i \restriction u_1\ [s_1]$, and since $t + 1$ is ρ-expansionary,

$$W_j \restriction l_1\ [t] = \Upsilon^{G_i} \restriction l_1\ [t] = \Upsilon^{G_i} \restriction l_1\ [s_1] = W_j \restriction l_1\ [s_1].$$

The conclusion that $l_2 = l_1$ is immediate, by induction on t, so this completes the proof of the second item.

For the third item, assuming $s_2 < \infty$, we know $w_\gamma^i \in G_i[s_3]$. First we show that $P \restriction l_1\ [s_2] = P \restriction l_1\ [s_3]$. Nodes $\succ \gamma$ were initialized after $s_1 + 1$, hence cannot enumerate elements $\leq l_1$ into P. Among nodes $\prec \gamma$, those $\prec \rho$ could not enumerate elements into P without initializing ρ, and those $\succ \rho$ do not enumerate into P at s_2+1 (since this would initialize γ when w^i_γ has yet to enter G_i) and are not eligible after that until s_3+1, hence cannot enumerate any elements into P.

Now we induct on ρ-expansionary stages $t+1 \geq s_3+1$. Since w^i_γ is chosen as x_ρ at stage s_3+1, all nodes $\supseteq \rho\hat{}\langle\infty\rangle$ are initialized at s_3+1, and the only nodes above ρ that are ever again eligible are those $\supseteq \rho\hat{}\langle f\rangle$. These nodes have not been eligible since the last initialization of ρ, so they will never enumerate any elements $\leq l_3$ into P, nor any elements $\leq u_3$ into G_i. Since $W_j = \Upsilon^{G_i}$, this yields our result for $(W_j \oplus P)$ in the third item.

It is now clear that the final value $\Lambda^{W_j\oplus P}(w^i_\gamma)$ will be the value $\Lambda^{W_j\oplus P}(w^i_\gamma)[s_2]$. Since $w^i_\gamma \notin G_i[s_2]$, we must have $w^i_\gamma \notin Q[s_2]$. Since s_2+1 is ρ-expansionary, this forces $\Lambda^{W_j\oplus P}(w^i_\gamma)[s_2] = 0$. ■

Lemma 5.7 *The requirements $\mathcal{T}_j$ are all satisfied by our construction.*

Proof. We show that $G_{2i+1} \leq_T G_{2i} \oplus E_0$, as required by $\mathcal{T}_{2i}$. (The proof for $\mathcal{T}_{2i+1}$ is analogous.) To compute whether $n \in G_{2i+1}$, we run the following steps:

1. Check whether n is targeted for G_{2i+1} at or before stage n, either as a witness for some $\mathcal{D}$- or $\mathcal{M}$-requirement or as a trace. If not, halt and conclude that $n \notin G_{2i+1}$.

2. If n is targeted for G_{2i+1} by stage n, then when it was so targeted, it must have had a trace appointed. Use the oracle to check whether this trace ever entered $G_{2i} \oplus E_0$. If it never entered them, conclude that $n \notin G_{2i+1}$.

3. If the trace did enter $G_{2i} \oplus E_0$, find the stage $s+1$ at which it did so, and check if $n \in G_{2i+1}[s+1]$. If so, conclude that $n \in G_{2i+1}$. If not, then another trace must have been appointed at stage s. Repeat Step 2 with this new trace.

We claim that this process must eventually terminate with the correct answer. The conclusion in Step 1 is justified by Sublemma 5.4, and the conclusion in Step 3 is abundantly clear. For Step 2, we note that the construction does appoint a trace when n is targeted for G_{2i+1}, and each time such a trace enters its target set, either n itself simultaneously enters G_{2i+1} or another trace is appointed. Furthermore, this new trace becomes part of the same block as n, so they will pass the current gate simultaneously. Hence n must advance by at least one gate down the tree before that trace can enter its target set, which implies that only finitely many traces for n will ever be appointed. (In particular, if $n = w^{2i+1}_\rho$, then after its first trace, it can only have as many traces appointed as there are gates below ρ on T.) Thus the process does eventually terminate.

Finally, notice that at each gate α on the tree, the blocks waiting at gate α are prioritized so that if n and a trace for n lie in different blocks waiting at gate α, then each is the lead ball of its block, and the trace will get to pass the gate first. Hence at every stage until a trace enters its target set, the trace will be waiting at a gate $\subseteq$ the gate at which the ball itself is waiting. Therefore n cannot enter its target set G_{2i+1} unless all its traces have entered their target sets by the same stage. This proves the correctness of our conclusion in Step 2 of the above procedure. ∎

Lemma 5.8 *Each requirement $\mathcal{L}_{\Phi,x}$ is satisfied by our construction, and initializes other nodes at only finitely many stages.*

Proof. Let $s''+1$ be a stage after which no $\mathcal{L}_j$ with $j<k$ ever initializes any node, and write $\mathcal{L}_k=\mathcal{L}_{\Phi,x}$. If $\Phi^R(x)[s]\downarrow$ with the same use at every $s\geq s''$, or if $\Phi^R(x)[s]\uparrow$ for all such s, then $\mathcal{L}_k$ is satisfied and never again initializes any nodes. Otherwise, $\mathcal{L}_k$ initializes cofinitely many nodes at some stage $s+1\geq s''+1$. Thereafter, none of those nodes will put any number $<\varphi^R(x)[s]$ into R. Among the finitely many remaining nodes α, $\mathcal{D}$- and $\mathcal{M}$-nodes may injure $\mathcal{N}_k$, but each of them can put only finitely many numbers into R without being initialized itself. (A witness ball for α drops by at least one gate every time a new trace targeted for R is assigned.) Such an α will not be initialized by any other $\mathcal{L}$-requirement after $s''+1$, and if other nodes initialize α infinitely often, then eventually $n(\alpha,s)$ becomes so large that $\mathcal{L}_k$ will initialize α along with everything else. Hence we will reach a stage at which $\mathcal{L}_k$ initializes every node except those which will never again injure it, and thereafter $\mathcal{L}_k$ is satisfied and never needs to initialize any more nodes. ∎

Lemma 5.9 *The true path g is infinite, and every node on it is initialized only finitely often.*

Proof. Suppose the node ρ lies on g. Now ρ may have infinitely many immediate successors, so we must show that one of them will be eligible infinitely often. Since the immediate successors of ρ are well-ordered, there will be a leftmost one τ eligible infinitely often, and that τ will then lie on g.

By induction we assume that ρ is initialized only finitely often by nodes $\prec\rho$ (and never by nodes $\succeq\rho$). Hence $n(\rho)=\lim_s n(\rho,s)$ exists, and only requirements $\mathcal{L}_k$ with $k<\ulcorner\rho\urcorner+n(\rho)$ will ever initialize ρ. By Lemma 5.8, therefore, ρ will only be initialized finitely often. Let $s''+1$ be the last stage at which ρ is initialized. Now we argue that ρ itself initializes its successors only finitely often, and that one of its immediate successors will be eligible infinitely often (so g does not terminate at ρ).

If ρ is an $\mathcal{N}$- or $\mathcal{R}$-node, then ρ has only finitely many immediate successors. Every time ρ acts, one of its successors will be eligible, and such a ρ never initializes any of its successors, so the lemma is clear.

A $\mathcal{P}_{i,\Psi}$-node ρ also has only finitely many immediate successors, but may initialize its successors. We claim that this only happens once after stage $s''+1$, however. It occurs at only a stage $s+1$ such that z_ρ has been realized and $\tilde{k}_{s+1}=-1$, and the initialization preserves the computation $\Psi^{G_i \oplus B}(z_\rho)[s] = 0$. Thereafter $\rho\hat{\ }\langle f\rangle$ will be eligible whenever ρ is, and no further initialization occurs.

Even if no initializations occur after $s'' + 1$, however, there may be infinitely many stages after $s'' + 1$ at which ρ is eligible but no immediate successor of ρ is eligible. Such stages occur in Subcase 6 of the construction for ρ. To complete the induction for $\mathcal{P}$-nodes, therefore, we need the following sublemma.

Sublemma 5.10 *In this situation, let k be the least number such that $\mathcal{N}_k = \mathcal{N}_{e,\Phi}$ is active along ρ via some α and $\tilde{k}_{s+1} = k$ at infinitely many ρ-stages $s + 1$. Then $\sigma = \rho\hat{\ }\langle a_k\rangle$ is eligible at infinitely many stages.*

Proof. $\rho\hat{\ }\langle f\rangle$ can never be eligible after $s'' + 1$, or $\tilde{k}$ would never again be defined. By minimality of k, there is a stage $s_0 + 1$ after which no node to the left of σ is ever eligible. Hence at stages $s + 1 > s_0 + 1$, α must never have been able to obey the request from Subcase 6 and increase $\gamma_\alpha^{W_{e_k} \oplus B}(z_\rho)[s]$ on its own. Therefore, the only increases come at stages $s + 1$ when Subcase 6(b) applies with $\tilde{k}_{s+1} = k$ and $\gamma_\alpha^{W_{e_k} \oplus B}(z_\rho)[s]$ is enumerated into B. After each such B-enumeration, σ will be eligible at the next ρ-stage. Hence if σ were eligible only finitely often, then $\gamma_\alpha^{W_{e_k} \oplus B}(z_\rho)$ would only be redefined finitely often, so would converge to some c. However, since $\alpha\hat{\ }\langle\infty\rangle \subset g$, Lemma 5.8 ensures that Φ^R is total, so there will exist ρ-stages $s + 1$ with $\tilde{k}_{s+1} = k$ at which $R \upharpoonright \varphi^R(c)[s]\downarrow$. At such stages we have no G_j-change on $\varphi^R(c)[s]$, so we enter Subcase 6(b), and at the next ρ-stage we will be in Subcase 3 and σ will be eligible again, yielding a contradiction. ∎

If ρ is a $\mathcal{D}_{i,\Omega}$-node, then ρ initializes all its successors when the witness element w^i_ρ is defined, again if w^i_ρ is realized, again at every ρ-stage until w^i_ρ enters the pinball machine, and again each time any ball with subscript ρ moves down the pinball machine. Once we reach a stage after which ρ is never initialized again, the next witness w^i_ρ will never be cancelled, and by Sublemma 5.5, if it is realized, then it will eventually enter its target set, after which ρ will never again initialize any of its successors.

Among the immediate successors of ρ, $\rho\hat{\ }\langle f\rangle$ will be eligible infinitely often if w^i_ρ enters G_i after the last initialization of ρ. If this w^i_ρ never enters G_i, then by Sublemma 5.5 it must never have been realized, so $\rho\hat{\ }\langle w\rangle$ is eligible infinitely often.

An $\mathcal{M}$-node ρ has only finitely many immediate successors, and the only stages $> s'' + 1$ at which ρ either fails to make one of them eligible or initializes its successors are those stages $s + 1$ (if any) at which Substeps 3(b), 4, 5, or 6 of the construction for $\mathcal{M}$-nodes apply. To reach any of these substeps after $s'' + 1$, ρ must enter Substep 3(b) first. After that, x_ρ is permanently defined. By Sublemma 5.5, ρ can only stay in Substep 4 for finitely many steps for each of the finitely many balls w^k_ρ, and then can only stay in Substep 5 for finitely many steps for each such ball. Finally ρ spends exactly one ρ-stage in Substep 6, at which x_ρ enters Q. Thereafter $\rho\hat{\ }\langle f\rangle$ will be eligible (via Substep 1) at every ρ-stage, and ρ will make no further initializations after its one stage in Substep 6. Thus the lemma holds at $\mathcal{M}$-nodes.

Finally, if ρ is a $\mathcal{U}_{e,i}$-node, we let $r = \liminf_s r(\rho, s)$ and $\sigma = \rho\hat{\ }\langle p_r\rangle$. The existence of this r follows from the definition of $r(\rho, s)$ in the construction: if there are infinitely many ρ-expansionary stages, then $r = 0$; otherwise r equals the greatest number appearing in the construction at or before the greatest ρ-expansionary stage. Clearly, once we reach a stage $s'' + 1$ after which $r(\rho, s) \geq r$, ρ will never again initialize any node above σ. We claim that $\sigma \subset g$.

Now the proof of Sublemma 5.5 actually showed slightly more than was stated:

Sublemma 5.11 *Let $\rho \subset g$ be a $\mathcal{U}_{e,2i}$-gate, and choose r as above. Any ball $w^j_\alpha > r$ waiting at gate ρ at a stage $s + 1$ after which α is never again initialized must eventually enter G_j.* ■

From this it follows that there will be infinitely many σ-stages. If not, then eventually all of the (finitely many) balls emanating from above σ which had passed gate ρ would either enter their target sets or be cancelled by initialization of their source nodes. (All such balls are $> r$ by Sublemma 5.4.) At the next stage at which $r(\rho, s) = r$, σ would then be eligible again. By induction, then, we have established Lemma 5.9. ■

Lemma 5.12 *Every $\mathcal{D}$-, $\mathcal{M}$-, $\mathcal{P}$-, $\mathcal{R}$-, and $\mathcal{U}$-requirement is satisfied by our construction.*

Proof. Every one of these requirements is assigned to some unique node on the true path g. ($\mathcal{N}$-requirements, which may be assigned to several nodes, are handled in Lemma 5.16.) We argue by induction along g, proving that the requirement assigned to each $\rho \subset g$ is satisfied by the sets we construct. Assume this holds for every $\sigma \subsetneq \rho$, and let $s'' + 1$ be the last stage at which ρ is initialized.

- Suppose ρ is a $\mathcal{D}_{i,\Omega}$-node. Once we reach the first ρ-stage $> s'' + 1$, the construction selects a witness element w^i_ρ which will remain fixed through all subsequent stages. If this witness element is ever realized, then by Sublemma 5.5 we see that $w^i_\rho \in G_i$, so that $\mathcal{D}_{i,\Omega}$ is satisfied. If it is never realized, then $\Omega^{H_i}(w^i_\rho)$ either diverges or converges to a value $\neq 0$. However, in this case w^i_ρ never enters the pinball machine on behalf of ρ, and it cannot simultaneously be a witness or trace for any other node, since such witnesses are always chosen large (including balls from $\mathcal{M}$-nodes, whose values were originally chosen as the uses of Ξ-functionals). Hence $w^i_\rho \notin G_i$, satisfying $\mathcal{D}_{i,\Omega}$.

- Suppose ρ is an $\mathcal{R}_k$-node. Then at each ρ-stage we either extend the functional Ξ_ρ to a larger domain, or add new axioms so as to redefine it on some value x in its current domain. However, the use of $\Xi^{G_k \oplus P}_\rho(x)$ is only changed when x enters G_k or when $x = x_\alpha$ for some $\alpha \supset \rho$ and w^k_α enters G_k. Since each of these can happen only once after $\Xi^{G_k \oplus P}_\rho(x-1)$ has converged, the function $\Xi^{G_k \oplus P}_\rho$ must be total.

  If $s' + 1 < s + 1$ are consecutive ρ-stages and a number x enters Q at a stage t with $s' + 1 \leq t < s + 1$, we have $x = x_\alpha$ for some $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node $\alpha \succ \rho$ (since if $\alpha \prec \rho$,

ρ would be initialized). If α lies to the right of ρ, then x_α must have been chosen by a node to the right of ρ at a stage $> s'+1$, so x_α cannot be in the domain of $\Xi_\rho^{G_k \oplus P}[s'+1]$. Otherwise $\rho \subset \alpha$, and by the construction for $\mathcal{M}$-nodes, α will have enumerated the use $\xi_\rho^{G_k \oplus P}(x)[s'+1]$ into P at the same stage that x entered Q. Since $s+1$ is the first ρ-stage since then, ρ is allowed to redefine $\Xi_\rho^{G_k \oplus P}(x)[s+1] = 1$. Thus $\Xi_\rho^{G_k \oplus P} = Q$, satisfying $\mathcal{R}_k$.

- Suppose ρ is an $\mathcal{M}_{i,j,\Lambda,\Upsilon}$-node, and suppose $W_j = \Upsilon^{G_i}$ and $\Lambda^{W_j \oplus P} = Q$. Then there must be infinitely many ρ-expansionary stages. Now if a witness element x_ρ becomes defined at any stage $s+1 > s''+1$, then as noted in the construction we have $x_\rho = w_\gamma^i$ for some $\gamma \supset \rho\hat{\ }\langle\infty\rangle$ (by Sublemma 5.3), and x_ρ must have entered G_i by stage s but since the previous ρ-expansionary stage $r+1$. At stage $s+1$ we enter Substep 3(b) of the construction for ρ, and at the next ρ-stage we enter Substep 4. By Sublemma 5.5, the balls chosen by ρ in Substep 3(b) all eventually enter their target sets, so we eventually enumerate x_ρ into $Q[t+1]$ via Substep 6 at some stage $t+1$. However, applying Lemma 5.6 to the ball w_γ^i, we see that $\Lambda^{W_j \oplus P}(x_\rho) = 0$, contradicting the assumption that $\Lambda^{W_j \oplus P} = Q$.

  Hence x_ρ is never defined. But this means that $\Theta_\rho^{W_j} = G_i$ on all elements of ω, since in Step 3(a) of the construction for ρ we define it thus, with the same use as $\Lambda^{W_j \oplus P}$, for all $x < l(\rho, s)$ (and $\lim_s l(\rho,s) = \infty$). Moreover, we made sure in that step that any redefinition of Θ was allowed by some W_j-change, so Θ is indeed a computable functional. (Had there been no W_j-change, we would have defined an element to be x_ρ instead.) Thus $\mathcal{M}_{i,j,\Lambda,\Upsilon}$ is satisfied.

- Suppose ρ is a $\mathcal{U}_{e,2i}$-node, and that $\Phi_e^{G_{2i} \oplus E_0} = \Phi_e^{F_0}$ with domain ω. (The argument for $\mathcal{U}_{e,2i+1}$ is analogous.) Then the node $\sigma = \rho\hat{\ }\langle p_0\rangle$ will lie on g.

  We make the standard argument for a pinball construction. In the construction, we only allow a block to pass gate ρ if its lead ball is targeted for the infimum (as discussed below) or for E_1, or is $> r(\rho, s)$. (A ball is always smaller than its trace. Hence if the lead ball is $> r(\rho,s)$, then the entire block consists of balls $> r(\rho, s)$.) A single block can injure both $\Phi_e^{G_{2i} \oplus E_0}$ and $\Phi_e^{F_0}$ only if it contains a ball targeted for the infimum G_{2i+1}. Otherwise, we protect the uninjured side by initializing all nodes of lower priority than the source node of the balls in the block and refusing to allow any other ball to pass the gate until all the balls of the first block have either entered their target sets or disappeared due to initialization of their source nodes, and until the injured computation has recovered and achieved a longer length of agreement with the uninjured computation. (Notice that ρ has the correct guesses about $\liminf_s r(\alpha, s)$ for every $\mathcal{U}$-gate $\alpha \subsetneq \rho$ and about $\liminf_s r(\alpha, i, s)$ for every $\mathcal{P}$-gate $\alpha \subsetneq \rho$ and every i, so any ball emanating from a node above ρ is large enough that such an α will eventually allow that ball to pass. Thus no ball which passed gate ρ will have to wait permanently at any gate below ρ, so we know that eventually each block waiting at gate ρ will be allowed to pass.)

In Instruction 4.4, however, we allowed a ball w_α^{2i+1} targeted for G_{2i+1} to pass gate ρ at stage $s+1$ even if it was $\leq r(\rho, s)$. (No other instruction allows balls $\leq r(\rho, s)$ to pass gate ρ, except balls targeted for E_1, which will not injure either side of the computation. Also, Instruction 4.4 applies when w_α^{2i+1} needs a new trace, so w_α^{2i+1} passes gate ρ by itself; its block at stage $s+1$ contained no other balls.) Therefore, when we attempt to compute $\Phi_e^{F_0}(x)$, we use a G_{2i+1}-oracle to look for the least σ-stage $s_0+1 > s''+1$ by which the length of agreement between $\Phi_e^{F_0}[s_0]$ and $\Phi_e^{G_{2i}\oplus E_0}[s_0]$ exceeds x and such that both computations on input x are G_{2i+1}-*correct*, i.e.

$$G_{2i+1}\restriction(u_0+1)[s_0] = G_{2i+1}\restriction(u_0+1),$$

where u_0 is the greater of the uses of the two computations $\Phi_e^{G_{2i}\oplus E_0}(x)[s_0]$ and $\Phi_e^{F_0}(x)[s_0]$. Set $h(x) = \Phi_e^{G_{2i}\oplus E_0}(x)[s_0]$.

(Such a stage s_0 must exist. $\Phi_e^{F_0}(x)$ and $\Phi_e^{G_{2i}\oplus E_0}(x)$ both converge by some stage s with some use u, and there must be infinitely many σ-stages $> s$ by which G_{2i+1} stabilizes on u. Pick any $\mathcal{D}_{j,\Omega}$-node $\alpha \supset \rho$ such that $\alpha \subset g$, $j \in \omega$, the functional Ω evaluates to 0 on every input and every oracle, and α is never eligible until after $G_{2i+1} \restriction u$ has stabilized. Then α must subsequently enumerate an element into G_j. We claim that the least σ-stage $s_0+1 > s+1$ by which α has completed its enumerations into all sets G_j will satisfy the above conditions. Clearly the length of agreement exceeds x, so we must show that s_0 is G_{2i+1}-correct. Now α is never initialized again, since otherwise it would have to enumerate another element into G_j to get back to the outcome $\alpha\hat{}\langle f\rangle$, so no ball from any node $\prec \alpha$ ever moves after stage s_0+1. Since s_0+1 is a σ-stage, every node to the right of $\sigma = \rho\hat{}\langle p_0\rangle$ is initialized at stage s_0+1. Nodes $\succ \alpha$ above σ were initialized when the witness w_α^j entered G_j, and cannot have been eligible since then, because σ has not been eligible since then. Also, no ball from any node $\preceq \alpha$ can have been below ρ when w_α^j passed ρ, or can have moved since then. Hence no ball at all is at any gate $\subsetneq \rho$ at stage s_0+1, except those which wait there permanently.)

We note that no balls below ρ at stage s_0+1 ever move again. Balls from nodes $\prec \rho$ cannot move without initializing ρ, which is impossible since $s_0+1 > s''+1$. Balls from nodes to the right of σ are all cancelled at the σ-stage s_0+1, and in order for s_0+1 also to be a σ-stage, no ball from any node $\supseteq \sigma$ can be waiting at any node below ρ at stage s_0+1, since otherwise σ would not be eligible at s_0+1.

We argue by induction that at every σ-stage $s+1 > s_0+1$, at least one side of the computation is G_{2i+1}-correct – that is, either $\Phi_e^{F_0}(x)[s] = h(x)$ with use u_s such that $G_{2i+1}\restriction(u_s+1)[s] = G_{2i+1}\restriction(u_s+1)$, or $\Phi_e^{G_{2i}\oplus E_0}(x)[s] = h(x)$ with use satisfying the same condition. Assume this holds for all σ-stages $t+1$ with $s_0 \leq t \leq s'$, where $s'+1$ is the last σ-stage before $s+1$.

Suppose that $\Phi_e^{F_0}(x)[s'] = h(x)$ is G_{2i+1}-correct with use $u_{s'}$. (The analogous argument will hold if the other side was G_{2i+1}-correct, as shown below.) The induction is trivial unless some ball $\leq u_{s'}$ enters F_0 before stage $s+1$, so suppose $w = w_\alpha^{2k+1}$ is the first such ball to do so, entering G_{2k+1} at a stage $t+1$ with $s' < t \leq s$. Since no ball from any node $\prec \rho$ ever moves after $s''+1$, we must have $\alpha \succeq \sigma$. With $w \leq u_{s'}$, Sublemma 5.4 ensures that w must have been chosen before stage $s'+1$, and hence $\sigma \subseteq \alpha$, since all nodes to the right of σ were initialized at stage $s'+1$. Thus w cannot have been below ρ at the σ-stage $s'+1$, so w must have passed ρ at a stage $t'+1 \geq s'+1$. By G_{2i+1}-correctness, w was not targeted for the infimum G_{2i+1}, so we must have $r(\rho, t') < w \leq u_{s'} \leq s'$, and thus $r(\rho, t') = 0$. This forces $t'+1$ to be a σ-stage $< s+1$, so in fact $t'+1 = s'+1$. Notice that due to the instructions for gate ρ, no other ball from above ρ can have been below ρ at stage $s'+1$, or can have passed ρ between stages $s'+1$ and t. Traces may have been chosen for w after it passed ρ, but they would all be chosen $> u_{s'}$, and will all enter their target sets before w enters G_{2k+1}. After stage $t+1$, the restraint $r(\rho, \cdot)$ will be set to prevent any other ball from above σ from passing gate ρ, until the next time the length of agreement recovers and exceeds $l(\rho, s')$ – which must be the next σ-stage, namely $s+1$. Thus w was the only ball chosen before $s'+1$ to pass gate ρ between stages $s'+1$ and s.

Now we claim that if $\Phi_e^{F_0}(x)[s]$ is no longer G_{2i+1}-correct, then the computation $\Phi_e^{G_{2i} \oplus E_0}(x)[s] = h(x)$ is G_{2i+1}-correct. The preceding paragraph shows that no ball entered $G_{2i} \oplus E_0$ between stages $s'+1$ and s, except balls which were chosen large after $s'+1$, so that

$$\Phi_e^{G_{2i} \oplus E_0}(x)[s] = \Phi_e^{G_{2i} \oplus E_0}(x)[s'] = \Phi_e^{F_0}(x)[s'] = h(x).$$

The first two of these computations have the same use u, and we claim that both are G_{2i+1}-correct. Suppose that some ball w_β^{2i+1} was chosen before stage $s'+1$ (since any ball chosen after $s'+1$ would be $> u$) and eventually enters G_{2i+1}. Then w_β^{2i+1} was waiting either at gate ρ or at a gate $\supseteq \sigma$ at stage $s'+1$ (since no ball was waiting below ρ at $s'+1$, balls from the right of σ were initialized then, and balls from nodes $\prec \rho$ never move again). Also, we have $\beta \preceq \alpha$, since w_β^{2i+1} was not cancelled when w_α^{2k+1} moved. Hence w_β^{2i+1} was chosen before w_α^{2k+1} was chosen, and so $w_\beta^{2i+1} < w_\alpha^{2k+1}$. (If $\beta = \alpha$, then $k < i$, since w_α^{2k+1} passed gate ρ first. Then w_α^{2k+1} would be a trace for w_β^{2i+1}, hence larger.) But $w_\alpha^{2k+1} < u_{s'}$, so this would contradict the G_{2i+1}-correctness of the computation $\Phi_e^{F_0}(x)[s']$, which was the inductive hypothesis.

The analogous argument, assuming $\Phi_e^{G_{2i} \oplus E_0}(x)[s'] = h(x)$ to be G_{2i+1}-correct with use $u_{s'}$, is similar. However, in the second paragraph, when we claim that no ball chosen before $s'+1$ entered F_0 between $s'+1$ and s, we must worry about balls w_β^{2i+1} targeted for the infimum, since such balls could pass ρ at a stage t between $s'+1$ and s despite a large restraint $r(\rho, t)$. The only way for this to happen is under Instruction 4.4, if the trace for w_β^{2i+1} entered its target set at stage $t+1$. This is possible only if that trace was the ball w_α^{2i} or e_α^0 which passed ρ at stage $s'+1$, which forces both $\beta = \alpha$ and $w_\beta^{2i+1} < u_{s'}$. This contradicts the G_{2i+1}-correctness of the computation $\Phi_e^{G_{2i}\oplus E_0}(x)[s']$, so in fact no such ball w_β^{2i+1} can have passed ρ between $s'+1$ and s. The rest of the argument goes through essentially unchanged, showing that $\Phi_e^{F_0}(x)[s] = h(x)$ must be G_{2i+1}-correct.

- Suppose ρ is a $\mathcal{P}_{i,\Psi}$-node. Then at the first ρ-stage after $s''+1$, a witness element z_ρ will be chosen and will remain fixed at all subsequent stages. We will need the following two sublemmas for our argument. The first one guarantees that if ρ is in Subcase 5 at infinitely many stages but only reaches Subcase 6 at finitely many stages, then the use $\psi^{G_i\oplus B}(z_\rho) \to \infty$.

Sublemma 5.13 *Suppose some $\alpha \subset g$ is an $\mathcal{N}_k$-node, with $\mathcal{N}_k = \mathcal{N}_{e,\Phi}$, but there is a node σ with $\alpha \subset \sigma \subset g$ such that $\mathcal{N}_{e,\Phi}$ is not active along σ via α. Then for all sufficiently large z, $\gamma_\alpha^{W_e\oplus B}(z)[s] \to \infty$ as $s \to \infty$. (More specifically, let τ be the immediate predecessor of the least such σ. Then $\alpha \subset \tau$, τ is a $\mathcal{P}$-node, and for each $z \geq z_\tau$ and each $n \in \omega$ there exists a stage s_0 such that at all stages $s \geq s_0$ we have either $\gamma_\alpha^{W_e\oplus B}(z)[s] > n$ or $\Gamma_\alpha^{W_e\oplus B}(z)[s]\uparrow$.)*

Proof. $\mathcal{N}_k$ is active via α along every immediate successor of α, so the τ described must lie above α. The node τ must be a $\mathcal{P}_{i,\Psi}$-node for some i and Ψ, since only at successors of $\mathcal{P}$-nodes can $\mathcal{N}_k$ change from active via α to inactive via α. If $\sigma \supseteq \tau\hat{\ }\langle a_k\rangle$, then Subcase 6(b) must apply infinitely often with $\tilde{k}_{s+1} = k$ in the construction for τ. Hence $\gamma_\alpha^{W_e\oplus B}(z_\tau)[s+1]$ is chosen large at infinitely many stages $s+1$, so the sublemma is satisfied for $z = z_\tau$. Otherwise $\sigma \supseteq \tau\hat{\ }\langle a_l\rangle$ for some $l < k$ with $\mathcal{N}_l$ assigned to some $\beta \subset \alpha$. In order for this σ to be eligible infinitely often, k must fail Condition (1) from page 29 infinitely often. But $\gamma_\beta^{W_{e_l}\oplus B}(z_\tau)[s+1]$ is chosen large at infinitely many stages $s+1$ since $\tau\hat{\ }\langle a_l\rangle \subset g$, and we must have $\psi^{G_i\oplus B}(z_\tau)[s] \to \infty$ as $s \to \infty$ to allow $\tilde{k} = l$ infinitely often. By Condition (1), therefore, $\gamma_\alpha^{W_e\oplus B}(z_\tau)[s] \to \infty$ as well.

By convention the use function $\gamma_\alpha^{W_e\oplus B}$ is increasing, so the result holds for all $z \geq z_\tau$. ∎

Sublemma 5.14 *Let $s+1 > s''+1$ be a ρ-stage at which the construction for ρ is in Subcase 6. Then $(G_i \oplus B)\lceil\psi^{G_i\oplus B}(z_\rho)[s]$ will be preserved (and $\Psi^{G_i\oplus B}(z_\rho)[s]\downarrow= 0$) until we enter either Subcase 6(b) or Subcase 2 of the construction for ρ.*

Proof. Let $t+1 > s+1$ be a ρ-stage such that ρ has not entered Subcase 2 or 6(b) since stage $s+1$. By induction on t, ρ has not entered any of Subcases 1, 3, 4, or 5 since stage $s+1$ either. (For Subcase 5, this follows because γ-uses never decrease from one stage to the next.)

For preservation of G_i, notice that no ball targeted for G_i was below gate ρ at stage $s+1$ (except possibly balls from nodes to the left of ρ, and such balls never move again). The construction set $r(\rho, i, \tilde{k}_s, s+1) \geq \psi^{G_i \oplus B}(z_\rho)[s]$, ensuring that no ball $\leq \psi^{G_i \oplus B}(z_\rho)[s]$ from above ρ has passed ρ since stage $s+1$, and all balls from nodes to the right of ρ are cancelled by initialization at $s+1$. (Balls from nodes $\prec \rho$ never move again, since ρ is never initialized again.)

Now a node must be eligible in order to enumerate an element into B. Hence no node to the left of ρ nor any node above ρ will violate our B-preservation, since nodes above ρ are never eligible when ρ is in Subcase 6. Each node σ to the right of ρ is initialized at $s+1$, so any element which σ enumerates into B at stage $t+1$ will be of the form $\gamma_\alpha^{W_e \oplus B}(z_\sigma)[t]$, hence $> z_\sigma[t] > \psi^{G_i \oplus B}(z_\rho)[s]$. A node $\tau \subset \rho$ never again enumerates anything into C (since doing so would initialize ρ), so its only B-enumeration can come when τ is in Subcase 6(b). Assume $\tau\hat{\ }\langle a_k\rangle \subset \rho$, and suppose that at stage $t+1$, τ enumerates some element $\gamma_\alpha^{W_e \oplus B}(z_\tau)[t]$ into B, with $\mathcal{N}_l = \mathcal{N}_{e,\Phi}$ assigned to $\alpha \subset \tau$. Then $k \leq l$, since otherwise ρ would be initialized at this stage, so $\mathcal{N}_l$ is not active via α along ρ, and τ is precisely the node τ_α described in Subcase 5. But then $\gamma_\alpha^{W_e \oplus B}(z_\tau) > \psi^{G_i \oplus B}(z_\rho)[t]$, since we are not in Subcase 5. Thus the B-enumeration by τ at stage $t+1$ does not violate the sublemma. ∎

If $\rho\hat{\ }\langle f\rangle$ ever becomes eligible at some stage $s+1 > s''+1$, then our initializations when z_ρ entered C will preserve the convergence $\Psi^{G_i \oplus B}(z_\rho)[s]\downarrow = 0 \neq C(z_\rho)$ forever after, satisfying $\mathcal{P}_{i,\Psi}$. (The same argument as in Sublemma 5.14 shows that no $\tau \prec \rho$ will injure this computation by any subsequent B-enumeration.)

If $\rho\hat{\ }\langle w\rangle \subset g$, then either Subcase 4 holds infinitely often (so $\Psi^{G_i \oplus B}(z_\rho)\downarrow \neq 0$ or diverges) or Subcase 5 holds infinitely often (so $\Psi^{G_i \oplus B}(z_\rho)\uparrow$, by Sublemma 5.13 and the conditions of Subcase 5). Sublemma 5.13 excludes the possibility of our remaining in Subcase 5 without eventually entering Subcases 2 or 3, both of which give outcomes to the left of $\rho\hat{\ }\langle w\rangle \subset g$.) Moreover, z_ρ never enters C, so $\mathcal{P}_{i,\Psi}$ holds.

Otherwise, z_ρ is realized infinitely often, and Sublemma 5.13 guarantees that each time it is realized, we will eventually enter Subcase 6 of the construction for ρ. If $\tilde{k} = -1$ at any subsequent ρ-stage, ρ will enumerate z_ρ into C, and $\rho\hat{\ }\langle f\rangle$ becomes eligible, as described above. Otherwise there is some k such that $\mathcal{N}_k = \mathcal{N}_{e,\Phi}$ is active along ρ via some α and $\sigma = \rho\hat{\ }\langle a_k\rangle \subset g$. Let $s_0+1 < s_1+1 < \cdots$ be all the σ-stages occurring after the last initialization of ρ. We claim that $\mathcal{P}_{i,\Psi}$ holds because in this case $\Psi^{G_i \oplus B}(z_\rho)$ must diverge. At each stage s_n+1, we are in Subcase 3 of the construction for the node ρ, so we have been in Subcase 6(b) at some stage $t+1 > s_{n-1}+1$. At that stage $t+1$, we enumerated $\gamma_k^{W_e \oplus B}(z_\rho)[t]$ into B, and at the next α-stage $s+1$, $\gamma_k^{W_e \oplus B}(z_\rho)[s+1]$ was chosen large. Since $\tilde{k}_{s_n+1} = k$, we know $\gamma_k^{W_e \oplus B}(z_\rho) \leq \psi^{G_i \oplus B}(z_\rho)[s_n]$, so $\psi^{G_i \oplus B}(z_\rho)[s_n] \to \infty$ as $n \to \infty$, satisfying $\mathcal{P}_{i,\Psi}$.

This completes the proof of Lemma 5.12. ∎

The following lemma ensures that the functionals $\Delta_{\rho,k}$ built at $\mathcal{P}$-nodes ρ with $\rho\hat{\ }\langle a_k\rangle \subset g$ are indeed computable.

Lemma 5.15 (Δ-Correction Lemma) *Let ρ be a $\mathcal{P}_{i,\Psi}$-node, and let $\sigma = \rho\hat{\ }\langle a_k\rangle$ lie on the true path g through T, with $\mathcal{N}_k = \mathcal{N}_{e,\Phi}$. Then there is an s_0 such that for every $w \in \omega$ and every σ-stage $s+1 \geq s_0+1$ such that $w \in \operatorname{dom}(\Delta^{G_i}_{\rho,k})[s+1]$, no ball w^i_β with $\delta^{G_i}_{\rho,k}(w)[s+1] < w^i_\beta < \varphi^R(w)[s]$ enters G_i from stage $s+1$ until the next σ-stage.*

Proof. Since a_k is an outcome of the node ρ, the requirement $\mathcal{N}_k$ must be active along ρ via some node $\tau \subsetneq \rho$. Now $\operatorname{dom}(\Delta^{G_i}_{\rho,k}) \subseteq \operatorname{dom}(\Phi^R)[t]$ at every stage t, since $\Delta_{\rho,k}$ is extended only at ρ-stages, all of which are τ-expansionary. Let s_0+1 be the last stage at which σ is initialized, so that no ball with subscript $\prec \sigma$ ever moves after stage s_0+1. Let s_1+1 be the first σ-stage $> s_0+1$ at which $\Delta^{G_i}_{\rho,k}(w)[s_1+1]$ is defined. Then the use $\delta^{G_i}_{\rho,k}(w)[s_1+1]$ is chosen large, hence is greater than $\varphi^R(w)[s_1]$.

We argue by induction on σ-stages $s+1$ that no ball w^i_α with $\delta^{G_i}_{\rho,k}(w) < w^i_\alpha < \varphi^R(w)[s]$ and $\sigma \preceq \alpha$ even exists at any σ-stage $\geq s_1+1$. (By the above remarks, this holds at stage s_1+1.) Since no ball with subscript $\prec \sigma$ moves after stage s_0+1 and no ball defined after a σ-stage $s+1$ can be $< \varphi^R(w)[s]$, this will prove the lemma.

For the inductive step, let $s+1 > s'+1$ be consecutive σ-stages $\geq s_1+1$, and fix $\alpha \succeq \sigma$. Now any new witness or trace chosen at an intervening stage must be greater than $\varphi^R(w)[s']$, so the induction will be trivial unless $\varphi^R(w)[s] > \varphi^R(w)[s']$. This implies that some element $y < \varphi^R(w)[s']$ entered R after stage s'. (Recall that $R = \oplus_k G_k$.) Then y must have been appointed as a witness or trace before stage $s'+1$, by some node $\beta \supseteq \sigma$.

If y entered G_i, then by our induction on $s'+1$, we have $y \leq \delta^{G_i}_{\rho,k}(w)[s'+1]$ as well, so at stage $s+1$ we redefine $\Delta^{G_i}_{\rho,k}(w)$ and set $\delta^{G_i}_{\rho,k}(w)[s+1] > \varphi^R(w)[s]$, completing the induction for $s+1$.

Otherwise $y = w^j_\beta$ entered G_j for some $j \neq i$. If $\alpha \succ \beta$ on T, then α was initialized when y entered G_j and is not eligible again before stage $s+1$, so any new ball with subscript α at stage $s+1$ will be greater than $\varphi^R(w)[s]$, by Sublemma 5.4.

If $\alpha \prec \beta$, then no ball y' with subscript α has moved or been chosen since w^j_β was chosen (since otherwise β would have been initialized). Sublemma 5.4 then ensures that such a y' is $< w^j_\beta$, hence $< \varphi^R(w)[s']$, and the inductive hypothesis guarantees that either $y' < \delta^{G_i}_{\rho,k}(w)[s'+1]$ or y' is targeted for a set other than G_i.

Finally we consider the case $\alpha = \beta$. Since w^j_β entered G_j after stage s', it must have passed gate ρ at a stage $> s'$, since otherwise σ could not have been eligible at $s'+1$. When w^j_β passed gate ρ, all its traces either had already entered their target sets or were targeted for E_0 or E_1. Moreover, if w^j_β was a trace for another ball w^{j+1}_β, then w^{j+1}_β must have been waiting at a gate $\supseteq \sigma$ at stage $s'+1$. (Two balls targeted for R cannot pass a $\mathcal{P}$-gate simultaneously.) The ball w^{j+1}_β may have dropped as far as gate ρ when w^j_β entered G_j, but the construction does not allow it to pass ρ until the next σ-stage $s+1$. When w^j_β entered G_j, a new trace (or traces) was appointed for w^{j+1}_β, but these new traces will each be targeted for either E_0, E_1, or G_j, not for G_i, and will begin at the same gate at which w_β^{j+1} is currently waiting. Hence none of these traces will move until stage $s+1$, so no new trace will be targeted for G_i until at least stage $s+1$. This proves our claim, and the lemma follows. ■

Lemma 5.16 *Every requirement $\mathcal{N}_k = \mathcal{N}_{e,\Phi}$ is satisfied by our construction.*

Proof. Let g be the true path, and let α be the node described by Lemma 5.2 for $\mathcal{N}_k$. If $\alpha\hat{}\langle w\rangle \subset g$, then $W_e \neq \Phi^R$, so $\mathcal{N}_k$ holds. Otherwise we have two cases.

Case 1: There exists a $\mathcal{P}_{i,\Psi}$-node $\rho \supset \alpha$, for some i and Ψ such that $\sigma = \rho\hat{}\langle a_k\rangle \subset g$. By the construction and our choice of α, $\mathcal{N}_k$ must be satisfied via α along every node $\supseteq \sigma$ on g. Once we have reached a stage $s''+1$ after which σ is never again initialized, Subcase 6(b) guarantees that the domain of the function $\Delta_{\rho,k}^{G_i}$ built by ρ will be extended by at least one element between every pair of σ-stages, subject only to the restriction that every element in the domain at stage $s+1$ must be $< \gamma_k^{W_e\oplus B}(z_\rho)[s+1]$. As noted in the proof of satisfaction of $\mathcal{P}_{i,\Psi}$ above, $\gamma_k^{W_e\oplus B}(z_\rho)[s] \to \infty$ as $s \to \infty$, so for each $n \in \omega$, $\Delta_{\rho,k}^{G_i}(n)[s+1]$ is defined at infinitely many stages $s+1$. The $\mathcal{L}$-requirements will then ensure that $\Delta_{\rho,k}^{G_i}$ is total, so that $W_e \leq_T G_i$, given the following.

Sublemma 5.17 *In the situation above, $\Delta_{\rho,k}$ is a computable functional, with $\Delta_{\rho,k}^{G_i} = W_e$.*

Proof. In Subcases 3 and 6(b) we always redefine $\Delta_{\rho,k}^{G_i}$ to equal W_e on its domain. We must prove that $\Delta_{\rho,k}$ is computable, i.e. that these redefinitions are allowed.

If the construction is in Subcase 3 at the ρ-stage $s+1$, let $s'+1$ be the most recent ρ-stage (at which we must have been in Subcase 6(b)). If any $y < w_{s'+1}$ has entered W_e since stage $s'+1$, then some change in $\varphi^R(y)$ must have taken place since $s'+1$ to allow it, where Φ is the functional assigned to $\mathcal{N}_k$. With the restraints $r(\rho, j, k, s'+1) \geq \varphi^R(y)[s']$ for all $j \neq i$, this means that $G_i\restriction\varphi^R(y)[s']$ has changed since s'. Indeed, by Lemma 5.15, $G_i\restriction\delta_{\rho,k}^{G_i}(y)[s']$ must have changed, so our redefinition of $\Delta_{\rho,k}^{G_i}(y)[s+1]$ is allowed.

In Subcase 6(b) at stage $s+1$, if some $y < w_{s+1} \leq \gamma_k^{W_e\oplus B}(z_\rho)[t+1]$ has entered W_e since the last stage $t+1$ at which we were in Subcase 3 with $\tilde{k}_{t+1} = k$, then y must have been allowed to enter W_e by some change in $R\restriction\varphi^R(y)[t]$. Now we set $r(\rho, i, k, t+1) = \varphi^R(\gamma_k(z_\rho))[t]$, and this restraint has stayed at least that large at all subsequent stages up through $s+1$. Hence some $G_j\restriction\varphi^R(z_\rho)[t]$ with $j \neq i$ must have changed after z_ρ was realized. However, the change would have happened before the previous ρ-stage $s'+1$ (since otherwise we would be in Subcase 6(a) at stage $s+1$) and after the stage $t'+1 > t+1$ at which we re-entered Subcase 6 with $\tilde{k}_{t'+1} = k$ and reset $r(\rho, j, k, t'+1) = 0$. However, at stage $t'+1$ we also requested that $\alpha = \alpha_k$ increase $\gamma_\alpha^{W_e\oplus B}(z_\rho)[t']$. Hence by stage $s+1$, α would have recognized the W_e-change and obeyed our request, setting $\gamma_\alpha^{W_e\oplus B}(z_\rho)[s+1] > \psi^{G_i\oplus B}(z_\rho)[s]$. This would contradict Condition (1), since our restraints have preserved $\psi^{G_i\oplus B}(z_\rho)[t']$ since we entered Subcase 6. Thus no such G_j-changes can have taken place, so W_e has not changed, and Subcase 6(b) only extends the domain of the functional $\Delta_{\rho,k}$ without redefining it on any arguments. ■

Case 2: By Lemma 5.2, if Case 1 does not hold, then $\mathcal{N}_k$ is active via α along every node on g above α. Then at every α-expansionary stage we extend the domain of $\Gamma_\alpha^{W_e \oplus B}$ by another element. Once the use $\gamma_\alpha^{W_e \oplus B}(x-1)$ stabilizes, $\gamma_\alpha^{W_e \oplus B}(x)$ increases only when:

- x enters C; or
- $x = z_\rho$ for some eligible $\rho \supset \alpha$ at which $\mathcal{N}_k$ is satisfied; or
- $x = z_\rho$ for some eligible $\rho \supset \alpha$ at a stage at which $\tilde{k} = k$ in the construction at ρ.

In the second case, ρ cannot lie on g, so either ρ is eligible only finitely often or z_ρ is eventually cancelled by initialization and redefined to be $> x$. In the third case, if $\rho \subset g$, then there are only finitely many such stages. Hence the use will increase only finitely often, and $\Gamma_\alpha^{W_e \oplus B}$ is total.

It only remains to show that this function computes C correctly – which is clear for any argument $x \notin C$. Now no node $\prec \alpha$ ever enumerates any element into C without initializing α, and after each initialization we start building a new Γ_α, so the version of Γ_α constructed after the last initialization of α will never be injured by those nodes. Among nodes $\rho \succ \alpha$, only $\mathcal{P}$-nodes ever enumerate any elements into C. When such a ρ does so, the element is the witness z_ρ, and it enters C at a stage $s+1$ with $\tilde{k}_{s+1} = -1$ in the construction for that ρ.

If ρ lies to the right of α, then z_ρ is cancelled each time α is eligible. If such a z_ρ is enumerated into C, therefore, then there were no α-stages between the definition of z_ρ at some stage $s+1$ and its entry into C. Since z_ρ was chosen large, it cannot have been in the domain of $\Gamma_\alpha^{W_e \oplus B}[s]$, nor can it have entered that domain since stage $s+1$. When $\Gamma_\alpha^{W_e \oplus B}(z_\rho)$ is finally defined, therefore, it will be correct.

So suppose α lies below the $\mathcal{P}_{i,\Psi}$-node ρ, and ρ enumerates z_ρ into C at stage $s+1$ (using Subcase 6 of the construction for ρ). Since $\tilde{k}_{s+1} = -1$ and Subcase 5 did not apply, either $\gamma_\alpha^{W_e \oplus B}(z_\rho)[s]\uparrow$ or $\gamma_\alpha^{W_e \oplus B}(z_\rho) > \psi^{G_i \oplus B}(z_\rho)[s]$. In the latter case ρ enumerates $\gamma_\alpha^{W_e \oplus B}(z_\rho)[s]$ into $B[s{+}1]$. In either case, therefore, α will be able to define $\Gamma_\alpha^{W_e \oplus B}(z_\rho)[t{+}1]$ correctly at the next α-stage $t+1$. Hence $\Gamma_\alpha^{W_e \oplus B} = C$, satisfying $\mathcal{N}_k$. ∎

6 References

Bernays, P. and Schoenfikel, M. [1928], Zum Entscheidungsproblem der mathematischen Logik, *Math. Ann.* **99**, 342-372.

Börger, E., Grädel, E. and Gurevich, Y. [1997], *The Classical Decision Problem*, Springer, Berlin.

Calhoun, W. C. [1993], Incomparable prime ideals of recursively enumerable degrees., 9th International Congress of Logic, Methodology and Philosophy of Science (Uppsala, 1991). *Ann. Pure Appl. Logic* **63**, 39–56.

Calhoun, W. C. and Lerman, M. [2001], Embedding finite lattices into the ideals of computably enumerable Turing degrees. *J. Symbolic Logic* **66**, 1791–1802.

Dreben, B. and Goldfarb, W. [1979], The decision problem, *Solvable Classes of Quantificational Formulas*, Addison-Wesley, Reading MA.

Epstein, R. L. [1979], *Degrees of Unsolvability: Structure and Theory*, **LNM** 759, Springer-Verlag, Berlin.

Friedberg, R. M. [1957], Two recursively enumerable sets of incomparable degrees of unsolvability, *Proc. Nat. Ac. Sci.* **43**, 236-238.

Harrington, L. and Shelah, S. [1982], The undecidability of the recursively enumerable degrees (research announcement), *Bull. Am. Math. Soc.*, N. S. **6**, 79-80.

Jockusch, C. G. Jr. and Slaman, T. A. [1993], On the Σ_2-theory of the upper semi-lattice of Turing degrees, *J. Symbolic Logic* **58**, no. 1, 193–204.

Kleene, S. C. and Post, E. L. [1954] The upper semi-lattice of degrees of recursive unsolvability, *Ann. of Math.* (2) **59**, 379–407.

Lachlan, A. H. [1966], The impossibility of finding relative complements for recursively enumerable degrees, *J. Symb. Logic* **31**, 434-454.

Lachlan, A. H. [1968], Distributive initial segments of the degrees of unsolvability, **14**, 457-472.

Lachlan, A. H. [1975], A recursively enumerable degree which will not split over all lesser ones, *Ann. Math. Logic* **9**, 307-365.

Lempp, S., Nies, A. and Slaman, T. A. [1998], The Π_3-theory of the computably enumerable Turing degrees is undecidable, *Trans. Am. Math. Soc.* **350**, 2719-2736.

Lerman, M. [1971], Initial segments of the degrees of unsolvability, *Ann. of Math.* (2) **93**, 365-389.

Lerman, M. [1973], Admissible ordinals and priority arguments, in *Cambridge Summer School in Mathematical Logic*, LNM **337**, A. R. D. Mathias and H. Rogers jr. eds., Springer-Verlag, Berlin, 311-344.

Lerman, M. [1983], *Degrees of Unsolvability*, Springer-Verlag, Berlin.

Lerman, M. [2000] A necessary and sufficient condition for embedding principally decomposable finite lattices into the computably enumerable degrees, *Annals of Pure and Applied Logic* 101, 275–297.

Lerman, M. and Shore, R. A [1988], Decidability and invariant classes for degree structures, *Trans. Am. Math. Soc.* **310**, 669-692.

Lerman, M., Shore, R. A. and Soare, R. I. [1984], The Elementary Theory of the Recursively Enumerable Degrees is not $\aleph_0$-categorical, *Advances in Mathematics* **53**, 301-320.

Minsky, M. L. [1961], Recursive unsolvability of Post's problem of tag and other topics in the theory of Turing machines, *Ann. Math.* **74**, 437-454.

Muchnik, A. A. [1956], On the unsolvability of the problem of reducibility in the theory of algorithms, *Dokl. Akad. Nauk SSSR* N.S. **108**, 29-32.

Nies, A. [2003], Parameter definability in the recursively enumerable degrees, to appear in *Journal of Mathematical Logic.*

Nerode, A. and Shore, R. A. [1997], *Logic for Applications*, Springer, New York, , 2nd ed.

Nies, A., Shore, R. A. and Slaman, T. A. [1998], Interpretability and definability in the recursively enumerable degrees, *Proc. London Math. Soc.* (3) **77**, 241-291.

Ramsey, F. P. [1930], On a problem of formal logic, *Proc. Lond. Math. Soc.* (2) **30**, 338-384.

Sacks, G. E. [1963], Recursive enumerability and the jump operator, *Trans. Am. Math. Soc.* **108**, 223-239.

Sacks, G. E. [1964], The recursively enumerable degrees are dense, *Ann. of Math. (2)* **80**, 300-312.

Shepherdson, J. and Sturgis, H. [1963], Computability of recursive functions, *J. Assoc. Comp. Mach* **10**, 217-255.

Shore, R. A. [1978], On the $\forall\exists$-sentences of α-recursion theory, in *Generalized Recursion Theory II*, J. E. Fenstad, R. O. Gandy and G. E. Sacks eds., *Studies in Logic and the Foundations of Mathematics* **94**, North-Holland, Amsterdam, 331-354.

Shore, R. A. [1981],The theory of the degrees below $0'$, *Journal of the London Mathematical Society* **24** (1981), 1-14.

Shore, R. A. [1999], The recursively enumerable degrees, in *Handbook of Computability Theory*, *Studies in Logic and the Foundations of Mathematics* **140**, North-Holland, Amsterdam, 169–197.

Simpson, S. G. [1977], First order theory of the degrees of recursive unsolvability, *Ann. Math. (2)*, **105**, 121-139.

Slaman, T. A. and Soare, R. I. [2001], Extension of embeddings in the computably enumerable degrees, *Annals of Mathematics*, **153**, 1-43.

Soare, R. I. [1987], *Recursively Enumerable Sets and Degrees*, Springer-Verlag, Berlin.

Spector, C. [1956], On degrees of recursive unsolvability, *Ann. Math. (2)* **64**, 581-592.

Yates, C. E. M. [1966], On the degrees of index sets, *Trans. Am. Math. Soc.* **121**, 309-328.