A Discrete Splitting Theorem for the 2-REA Degrees

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1 Introduction

The following definitions are taken from [Cooper, 2000].

Definition 1.1
- Given a, b, and d, say d is *splittable over a avoiding b* iff if $a, b < d$ and $b \not\leq a$, then there exist $d_0, d_1 < d$ for which $a < d_0, d_1$, $b \not\leq d_0$ and $b \not\leq d_1$, and $d = d_0 \vee d_1$.
- Further, d is *discretely* splittable over a avoiding b iff each such d_i is greater than or equal to a minimal cover m_i of a such that $m_i \not\leq d_{1-i}$.
- d is *relatively unsplittable* iff there are a and b such that d is not discretely splittable over a avoiding b.

Recall, a set is *d-computably enumerable (d.-c.e.)* or equivalently *d-recursively enumerable (d-r.e.)* iff it can be written as the set theoretic difference of two recursively enumerable sets.

A set W is 1-*REA relative to, or in,* A if and only if it is recursively enumerable in A and above it in Turing degree. W is $(n+1)$-*REA in* A if and only if it is $1-REA$ relative to some U which is $n-REA$ in A. A degree w is *n-REA relative to, or in,* a if it contains a set W which is $n-REA$ relative to some set $A \in a$. If $a = 0$, we just say that W and w are *n-REA*.

Cooper [2000] states the following as *The Main Theorem.*

> There exists a relatively unsplittable d-c.e. degree. That is, there is a d-c.e. set $D = W_i - W_j$ and sets $A, B \leq_T D$ such that $\deg(D)$ is not discretely splittable over $\deg(A)$ avoiding $\deg(B)$.

In Theorem 2.1, we refute Cooper's assertion. Since they depended on this assertion, Cooper's claims to have defined the Turing jump and to have defined the relation of "recursively enumerable in" are also refuted.

As Lachlan has shown that every d-r.e. set is of 2-*REA* degree (see [Jockusch and Shore, 1984] for a proof of a more general fact), it suffices to prove that every $2-REA$ degree d is discretely splittable over every a avoiding any b to show that there is no $d-r.e.$ set as claimed in the Main Theorem of [Cooper, 2000].

Our proof is minor perturbation of the one in [Shore and Slaman, n.d.], where we refuted Cooper's [1990] claim to have defined the Turing jump in $\mathcal{D}$. In fact, we prepared this note by cutting and pasting from [Shore and Slaman, n.d.].

2 A discrete splitting theorem

Theorem 2.1 *For every 2-REA degree* d *and every* a *and* b*,if* $d > a$, $d > b$, *and* $a \not\geq b$, *then* d *is discretely splittable over* a *avoiding* b.

Note that Theorem 2.1 directly contracts Cooper's Main Theorem.

The proof of Theorem 2.1 is divided into two cases. Suppose that d, a and b are given as above.

In the first case d is recursively enumerable relative to a. As Cooper [2000] points out, it follows from results of Sacks [1961] and Cooper and Epstein [1987] that d is discretely splittable over a avoiding b.

In the case that d is not recursively enumerable in a, Theorem 2.1 follows from the following fact.

Theorem 2.2 *Let* D *and* A *be given so that* D *is 2-REA,* $D >_T A$ *and* D *is not recursively enumerable relative to* A*. Then there are sets* X_0, X_1, M_0, M_1 *and* Y_0, Y_1, N_0, N_1 *recursive in* D *with the following properties.*

1. *$D \equiv_T X_0 \oplus X_1$ and $D \equiv_T Y_0 \oplus Y_1$.*
2. *M_0, M_1, N_0, and N_1 are minimal covers of A.*
3. *For any i and j from $\{0, 1\}$, and other set B, if $X_i \oplus M_i \geq_T B$ and $Y_j \oplus N_j \geq_T B$ then $A \geq_T B$.*
4. *For any i from $\{0, 1\}$, $X_i \oplus M_i \not\geq_T M_{1-i}$ and $Y_i \oplus N_i \not\geq_T N_{1-i}$.*

In the case when d is not recursively enumerable relative to a, we can obtain Theorem 2.1 as follows. Suppose we are given a d-r.e. set D and sets A and B of degree d, a and b, respectively. Theorem 2.2 applies, and we take X_0, X_1, M_0, M_1 and Y_0, Y_1, N_0, N_1 having the properties that it asserts. If neither $X_0 \oplus M_0 \geq_T B$ nor $X_1 \oplus M_1 \geq_T B$, then they constitute a discrete splitting of the degree of D over that of A avoiding the degree of B. Otherwise, B is recursive in at least one of $X_0 \oplus M_0$ or $X_1 \oplus M_1$. By the third clause of Theorem 2.2, since B is not recursive in A, B is not recursive in either $Y_0 \oplus N_0$ or $Y_1 \oplus N_1$, and their degrees constitute a discrete splitting of the degree of D over that of A avoiding the degree of B.

Remark 2.3 As in [Shore and Slaman, n.d.], Theorem 2.1 extends to the class of n-*REA* degrees.

The remainder of this note is devoted to the proof of Theorem 2.2. We will identify sets with their characteristic functions. For example, $D(n) = 0$ is synonymous with $n \notin D$, and $D(n) = 1$ is synonymous with $n \in D$.

Since D has 2-*REA* degree, there is a recursively enumerable set U such that $D \geq_T U \geq_T \emptyset$ and D has recursively enumerable degree relative to U. The condition that the degree of D is not recursively enumerable relative to A implies that D is 2-*REA* in a nontrivial sense: $D >_T U >_T \emptyset$. Now let U be fixed as above and assume that D is a representative of the given d-r.e. degree such that D is recursively enumerable relative to U. Of course, since the degree of D is not recursively enumerable relative to A, D is not recursive in $A \oplus U$ and U is not recursive in A.

The minimal covers of A. We will produce the sets as required in two phases. First, fix M_0, M_1, N_0, and N_1 so that each is recursive in $A \oplus U$, each is a minimal cover of A, and any two of which have incomparable Turing degrees. There are such sets since $A \oplus U$ is recursively enumerable relative to A and strictly above A. For example, see [Epstein, 1979]

We ensure that our sets satisfy the remaining claims in Theorem 2.2 by a priority construction.

Join requirements. We directly ensure that $X_0 \oplus X_1$ and $Y_0 \oplus Y_1$ compute D by enforcing the following conditions.

$$(\forall n)[n \in D \text{ if and only if } X_0(n) = X_1(n)] \tag{1}$$

$$(\forall n)[n \in D \text{ if and only if } Y_0(n) = Y_1(n)]. \tag{2}$$

Note that we can specify X_i or Y_j arbitrarily and still define X_{1-i} or Y_{1-j}, respectively, to satisfy Equations 1 and 2.

Infima requirements. We consider requirements P of the following form, in which i and j belong to $\{0, 1\}$, Φ and Ψ are Turing functionals, and B is a free variable.

$$\text{If } \Phi(X_i \oplus M_i) = \Psi(Y_j \oplus N_j) = B, \text{ then } A \geq_T B. \tag{3}$$

Let $(P_i : i \in \mathbb{N})$ be a recursive enumeration of all requirements of the above form. To fix our notation, let P_k be the following requirement.

$$\text{If } \Phi_k(X_{i_k} \oplus M_{i_k}) = \Psi_k(Y_{j_k} \oplus N_{j_k}) = B_k, \text{ then } A \geq_T B_k. \tag{4}$$

We organize our construction so that for all P_k, either one of $\Phi_k(X_{i_k} \oplus M_{i_k})$ or $\Psi_k(Y_{j_k} \oplus N_{j_k})$ is not total, they are not equal, or their common value is recursive in A. During the construction, we search for an argument at which we can make $\Phi_k(X_{i_k} \oplus M_{i_k})$ and $\Psi_k(Y_{j_k} \oplus N_{j_k})$ unequal. If no such opportunity appears and both functions are total, then we conclude that their common value must be recursive in A.

Definition 2.4 1. A *finite condition* on one of the X_i's or Y_j's is a 0-1 valued function whose domain is a finite initial segment of $\mathbb{N}$. That is a finite condition on a set is a specification of a finite initial segment of the values of that set. Two conditions are *compatible* if they agree on the domain they have in common.

2. A *P_k-split* consists of a natural number n, a pair of finite conditions p_0 and p_1, and a pair of computations in Φ_k and Ψ_k of lengths less than the domains of p_0 and p_1 such that $\Phi_k(n, p_0 \oplus M_{i_k})$ and $\Psi_k(n, p_1 \oplus N_{j_k})$ are defined by these computations and have different values.

3. We say that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ *extend* the conditions in the above P_k-split if X_{i_k} extends p_0 and Y_{j_k} extends p_1.

4. We say that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are *compatible* with the conditions in the above P_k-split if $X_{i_k} \upharpoonright x$ and p_0 are compatible and $Y_{j_k} \upharpoonright x$ and p_1 are compatible.

Note, if $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ extend the conditions in a P_k-split, then $\Phi_k(X_{i_k} \oplus M_{i_k})$ and $\Psi_k(Y_{j_k} \oplus N_{j_k})$ take different values at the number n mentioned in that split.

Cone avoiding requirements. We consider requirements Q^X of the following form, in which i and is an element of $\{0, 1\}$ and Φ is a Turing functional.

$$\Phi(X_i \oplus M_i) \neq M_{1-i}. \tag{5}$$

We consider the corresponding family Q^Y of requirements, $\Phi(Y_i \oplus N_i) \neq N_{1-i}$, on the Y_i's and N_i's analogously but separately. Let $(Q^X_i : i \in \mathbb{N})$ be a recursive enumeration of all requirements of the above form. To fix our notation, let Q^X_k be the requirement

$$\Phi_k(X_{i_k} \oplus M_{i_k}) \neq M_{1-i_k}. \tag{6}$$

During the construction, we search for an argument at which we can make $\Phi_k(X_{i_k} \oplus M_{i_k})$ unequal to M_{1-i_k}. If no such opportunity appears, then we conclude that $\Phi_k(X_{i_k} \oplus M_{i_k})$ is not total.

Definition 2.5 1. A Q^X_k*-split over* E consists of a natural number n, a finite condition p and a computation in Φ_k of length less than the domain of p such that $\Phi_k(n, p \oplus M_{i_k})$ is defined by this computation and has value different from $M_{1-i_k}(n)$.

2. We say that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ *extend* the conditions in the above Q^X_k-split if $X_{i_k} \upharpoonright x$ extends p.

3. We say that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are *compatible* with the condition in the above Q^X_k-split if $X_{i_k} \upharpoonright x$ and p are compatible.

Note, if $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ extend the conditions in a P_k-split, then $\Phi_k(X_{i_k} \oplus M_{i_k})$ and M_{1-i_k} take different values at the number n mentioned in that split.

We define the corresponding notions for the Q^Y requirements analogously.

Computing the X_i's and Y_j's from D. We let $(R_k : k \in \omega)$ be a recursive listing of the above requirements in order type ω. We fix a universal enumeration of their associated splits which is recursive relative to $A \oplus U$.

Definition 2.6 1. Let $D[s]$ denote the set of numbers less than or equal to s which are enumerated in D relative to $A \oplus U$ by computations of length less than or equal to s.

2. For each $x \in \mathbb{N}$, let s_x be the least stage s such that $D[s]$ and D are equal on all numbers less than or equal to x, that is $D[s] \restriction x+1 = D \restriction x+1$.

We compute the X_i's and Y_j's from D by recursion as follows.

1. To compute these sets at argument x, first compute their restrictions to all numbers less than x. If x is 0, then these restrictions are null functions.

2. Compute s_x.

3. Say that k *requires attention at* x if k is less than or equal to x and the following conditions hold.

   (a) $X_0 \restriction x$, $X_1 \restriction x$, $Y_0 \restriction x$, and $Y_1 \restriction x$ do not extend the conditions in any R_k-split enumerated at or before stage s_x.

   (b) There is an R_k-split enumerated at or before stage s_x such that $X_0 \restriction x$, $X_1 \restriction x$, $Y_0 \restriction x$, and $Y_1 \restriction x$ are compatible with the conditions in that split.

   If there is no k which requires attention at x, then let $X_0(x)$ and $Y_0(x)$ both equal 1 and let $X_1(x)$ and $Y_1(x)$ both equal $D(x)$. Thus, $X_0(x) = X_1(x)$ if and only if $Y_0(x) = Y_1(x)$ if and only if $D(x) = 1$.

   If there is a k which requires attention at x, then let σ be the R_k-split which appeared first in our universal enumeration of splits among those R_k-splits of highest priority, i.e. with the smallest k, which are compatible with $X_0 \restriction x$, $X_1 \restriction x$, $Y_0 \restriction x$, and $Y_1 \restriction x$. Define $X_{i_k}(x)$ and/or $Y_{j_k}(x)$ to so as to be compatible with the condition or conditions appearing in σ. This leaves at least one of each pair $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ undefined at x. Define the sets whose values at x have not yet been determined to ensure the equivalences $X_0(x) = X_1(x)$ if and only if $Y_0(x) = Y_1(x)$ if and only if $D(x) = 1$.

Here is a short summary of the algorithm. Use the enumeration of D relative to $A \oplus U$ to produce a finite set of splits for various R_k's, and then define the X_i's and Y_j's at x compatibly with the split for the highest priority requirement which requires attention.

By inspection, the X_i's and the Y_j's are recursive in D.

Remark 2.7 The function mapping $D \upharpoonright x$ to s_{x-1}, $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ is recursive in $A \oplus U$.

Proof: Suppose we are given $D \upharpoonright x$ and by induction have computed $X_0 \upharpoonright x-2$, $X_1 \upharpoonright x-2$, $Y_0 \upharpoonright x-2$, and $Y_1 \upharpoonright x-2$. We can run the enumeration of D relative to $A \oplus U$ until the stage s_{x-1} at which each element of $D \upharpoonright x$ has been enumerated. The universal enumeration of R_k splits is recursive in $A \oplus U$, and so we can compute the set of splits enumerated at or before stage s_{x-1} using $A \oplus U$. The values of the X_i's and Y_j's at $x-1$ are effectively determined from this collection of splits and the value of D at $x-1$, data which is now seen to be available to $A \oplus U$ from $D \upharpoonright x$. ∎

Satisfaction of the requirements.

Lemma 2.8 *For each k, there are only finitely many x such that R_k requires attention at x.*

Proof: We verify Lemma 2.8 by a Friedberg-style priority argument. By induction on k, choose x_0 so that for all x greater than x_0 no requirement of index less than k requires attention at x. Suppose that R_k requires attention at some x_1 greater than x_0. Then the values of X_0, X_1, Y_0, and Y_1 at x will be defined to be compatible with the conditions in the first R_k-split for which this is possible. Since no requirement of lower index can require attention, the same R_k-split will be used during subsequent stages until reaching an x_2 such that the X_i's and Y_j's restrictions to x_2 extend that R_k-split. But then R_k cannot require attention for any argument greater than x_2. ∎

Lemma 2.9 *Suppose that $\Phi_k(X_{i_k} \oplus M_{i_k}) = \Psi_k(Y_{j_k} \oplus N_{j_k})$ and their common value B_k is not recursive in A. Then for all x there is an s greater than x with the following properties.*

1. *$s \geq s_x$.*

2. *There is a P_k-split which is enumerated at or before stage s such that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are compatible with the conditions in that split.*

Proof: For a contradiction, fix x so that there is no s which satisfies the conclusions of Lemma 2.9.

Then every t greater than s_x satisfies the first conclusion of Lemma 2.9, so there cannot be any P_k-split such that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are compatible with the conditions in that split.

Consider any conditions p_0 and p_0^* extending $X_{i_k} \upharpoonright x$. If there were an n such that $\Phi_k(n, p_0 \oplus M_{i_k})$ and $\Phi_k(n, p_0^* \oplus M_{i_k})$ are defined and give incompatible values, then one of them could be paired with the initial segment of Y_{j_k} used to compute $\Psi_k(n, Y_{j_k} \oplus N_{j_k})$ to produce a P_k-split.

Now, conclude that B_k is recursive in M_{i_k}: Given n, find p_0 extending $X_{i_k} \upharpoonright x$ such that $\Phi_k(n, p_0 \oplus M_{i_k})$ is defined. The value of $\Phi_k(n, p_0 \oplus M_{i_k})$ must equal $B_k(n)$, since it must equal $\Phi_k(n, X_{i_k} \oplus M_{i_k})$.

Similarly, B_k is recursive in N_{j_k}. Since M_{i_k} and N_{j_k} are incomparable minimal covers of A, $A \geq_T B_k$ is the desired contradiction. ■

The proof of the following lemma is exactly analogous to the previous one: assuming that there is no such s as stated, one concludes that $M_{i_k} \geq_T M_{1-i_k}$ or that $N_{i_k} \geq_T N_{1-i_k}$.

Lemma 2.10 1. *Suppose that $\Phi_k(X_{i_k} \oplus M_{i_k}) = M_{1-i_k}$. Then for all x there is an s greater than x with the following properties.*

   (a) *$s \geq s_x$.*

   (b) *There is a Q_k^X-split which is enumerated at or before stage s such that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are compatible with the conditions in that split.*

2. *Suppose that $\Phi_k(Y_{i_k} \oplus N_{i_k}) = N_{1-i_k}$. Then for all x there is an s greater than x with the following properties.*

   (a) *$s \geq s_x$.*

   (b) *There is a Q_k^Y-split which is enumerated at or before stage s such that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are compatible with the conditions in that split.*

Lemma 2.11 *1. For all k, if $\Phi_k(X_{i_k} \oplus M_{i_k}) = \Psi_k(Y_{j_k} \oplus N_{j_k})$, then their common value is recursive in A.*

2. For all k, $\Phi_k(X_{i_k} \oplus M_{i_k}) \neq M_{1-i_k}$.

3. For all k, $\Phi_k(Y_{i_k} \oplus N_{i_k}) \neq N_{1-i_k}$.

Proof: We give the proof of the first of these claims. The other two are proven similarly.

For a contradiction, suppose that $\Phi_k(X_{i_k} \oplus M_{i_k}) = \Psi_k(Y_{j_k} \oplus N_{j_k})$ and that their common value is not recursive in A. We will show that D is recursive in $A \oplus U$.

Suppose that P_k is the requirement R_{k^*}. By Lemma 2.8, fix t_0 so that for all k_0^* less than or equal to k^* and all x greater than t_0, $R_{k_0^*}$ does not require attention at x.

Take $D \upharpoonright t_0$ as a given finite amount of data. We compute the value of D at arguments $x > t_0$ using the already computed $D \upharpoonright x$ as follows.

1. Compute s_{x-1}, $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$. (See Remark 2.7 and text preceding it for the effectiveness of this step relative to $A \oplus U$.)

2. Compute the least t such that the following conditions are satisfied.

   (a) t is greater than the maximum of x and s_{x-1}.

   (b) There is a P_k-split which is enumerated at or before stage t such that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are compatible with the conditions in that split.

   We let t_x denote this t.

3. Return the value of $D[t_x]$ at x as the value of D at x.

By Lemma 2.9, there is a stage t and a P_k-split which is enumerated at or before stage t such that $X_0 \upharpoonright x$, $X_1 \upharpoonright x$, $Y_0 \upharpoonright x$, and $Y_1 \upharpoonright x$ are compatible with the conditions in that split. Thus, the above procedure will find some t_x and will return some value for D at x.

We chose t_0 so that no strategy with index less than or equal to k^* requires attention after stage t_0. In particular, P_k cannot require attention at x. But then, the P_k-split which is enumerated before or during stage t_x can not have been enumerated before or during stage s_x. Consequently, s_x is less than

t_x and $D(x) = D[t_x](x)$. Thus, the calculation relative to $A \oplus U$ correctly returns the value of D at x.

The conclusion that D is recursive in $A \oplus U$ is the desired contradiction, which proves the first claim of Lemma 2.11. ■

Now, Theorem 2.2 follows directly.

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