Jumps of Σ_2^0-high e-degrees and properly Σ_2^0 e-degrees

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Abstract

We show that the Σ_2^0 high e-degrees coincide with the high e-degrees. We also show that not every properly Σ_2^0 e-degree is high.

1 Introduction

Enumeration reducibility is the notion of relative enumerability of sets: a set A is *enumeration reducible* (or simply *e-reducible*) to a set B, in symbols, $A \leq_e B$, if there is an effective procedure for enumerating A given *any* enumeration of B. Formally, we define $A \leq_e B$ if there is some computably enumerable set Φ (called in this context an *enumeration operator* or simply an *e-operator*) such that

$$A = \{x : (\exists \text{ finite } D)[\langle x, D\rangle \in \Phi \,\&\, D \subseteq B]\}$$

(throughout the paper we identify finite sets with their canonical indices). We denote by $\equiv_e$ the equivalence relation generated by the preordering relation $\leq_e$ and $\deg_e(A)$ denotes the equivalence class (or the *e-degree*) of A. The partially ordered structure of the e-degrees is denoted by $\mathfrak{D}_e$; its partial ordering is denoted by $\leq$. $\mathfrak{D}_e$ is, in fact, an upper semilattice with least element $\mathbf{0}_e$.

One of the most interesting features of the e-degrees is that they extend the structure $\mathfrak{D}_T$ of the Turing degrees ([Med55] and [Rog67]). Indeed if we define $\iota : \mathfrak{D}_T \to \mathfrak{D}_e$ by $\iota(\deg_T(A)) = \deg_e(\chi_A)$ (where $\deg_T(A)$ is the Turing degree of the set A and χ_A denotes the characteristic function of A) then ι

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is the desired embedding. (In fact, it preserves joins and least element.) The verification that ι is well defined relies on the following lemma:

Lemma 1.1 *For every total function f and g, we have*

$$f \leq_T g \Leftrightarrow f \leq_e g.$$

Proof. See e.g. [Rog67, p. 153]. □

One can define (see below) a jump operation $'$ on the e-degrees, and therefore introduce the notions of a *low* e-degree (i.e. an element of the class $\mathbf{L}_1 = \{\mathbf{a} \leq \mathbf{0}'_e : \mathbf{a}' = \mathbf{0}'_e\}$); and that of a *high* e-degree (i.e. an element of $\mathbf{H}_1 = \{\mathbf{a} \leq \mathbf{0}'_e : \mathbf{a}' = \mathbf{0}''_e\}$). Moreover, since ι preserves jump, we have that low Turing degrees are mapped to low e-degrees and high Turing degrees are mapped to high e-degrees.

A nice, useful characterization of the class $\mathbf{L}_1$ of the low e-degrees is given in [MC85]: $\mathbf{a} \in \mathbf{L}_1$ if and only if $\mathbf{a}$ contains a set A such that, for every $B \leq_e A$, $B \in \Delta^0_2$. Thus $\mathbf{a}$ and all the e-degrees below $\mathbf{a}$ consist entirely of Δ^0_2 sets. In this paper, we characterize the class $\mathbf{H}_1$ of high e-degrees by a result analogous to the one characterizing the high Turing degrees as those containing a set with an approximation whose associated computation function dominates every total recursive function (Theorem 2.1). The relevant definitions of a Σ^0_2 approximation and a computation function are given below. The e-degrees of sets with such approximations are known as the Σ^0_2-high e-degrees (Definition 1.7). This characterization answers Question 7.3 of [Coo90].

Since the e-degrees below $\mathbf{0}'_e$ are exactly the e-degrees consisting of Σ^0_2 sets, in view of the above cited characterization of the low e-degrees, a natural question to ask is where an e-degree $\mathbf{a} \leq \mathbf{0}'_e$ which contains no Δ^0_2 set (such an e-degree is called a *properly* Σ^0_2 e-degree) lies in the low/high hierarchy. A natural conjecture might be that the properly Σ^0_2 e-degrees are all in $\mathbf{H}_1$. Cooper and Copestake ([CC88]) show that there exist properly Σ^0_2 e-degrees that are Σ^0_2-high, and thus lie in $\mathbf{H}_1$ by Theorem 2.1. However, in Theorem 3.1 we show that the properly Σ^0_2 e-degrees are not contained in $\mathbf{H}_1$.

Our notations and terminology are mostly based on [Soa87]. The reader is referred to [Coo90] for an introduction and extensive bibliography on enumeration reducibility. We will be mostly working with Σ^0_2 sets. We recall that a Σ^0_2 *approximation* to a Σ^0_2 set A is computable sequence of computable sets $\{A^s : s \in A^s\}$ such that $A = \{x : (\exists t)(\forall s \geq t)[x \in A^s]\}$. See [LS92] for an introduction to Σ^0_2 approximations, and for a proof that every Σ^0_2 set has a *good* Σ^0_2 approximation $\{A^s : s \in A^s\}$, i.e. a computable sequence of computable (in fact, finite) sets such that $\{s : A^s \subseteq A\}$ is infinite.

Let X be any set of natural numbers; if x is a number, then $X^{[x]} = \{z \in X : (\exists y)[z = \langle x, y\rangle]\}$, and $X \restriction x = \{y \in X : y < x\}$. If σ is a string and

$x < |\sigma|$ (where $|\sigma|$ denotes the length of σ), then $\sigma \upharpoonright x$ denotes the initial segment of σ having length x; likewise, if f is a function, then $f \upharpoonright x$ denotes the initial segment of f having length x.

Let $\{\varphi_i\}_{i \in \omega}$ be the standard enumeration of all partial computable functions with corresponding enumerations $\{W_i\}_{i \in \omega}$ and $\{\Phi_i\}_{i \in \omega}$ of the computably enumerable sets and the enumeration operators, respectively. Let us fix, as in [Soa87, p. 16], computable approximations $\{\varphi_{i,s}\}_{i,s \in \omega}$ to the partial computable functions. Without loss of generality, we may assume that if $\varphi_{i,s}(x) \downarrow$ then $\varphi_{i,s}(x) < s$. Correspondingly, we get computable finite approximations $\{W_{i,s}\}_{i,s \in \omega}$ and $\{\Phi_{i,s}\}_{i,s \in \omega}$ to the computably enumerable sets and the enumeration operators, respectively.

Let

$$K^A = \{x : x \in \Phi_x^A\}.$$

Lemma 1.2 *[McE85] Let A, B be sets; then*

$$A \leq_e B \Leftrightarrow A \leq_1 K^B \Leftrightarrow K^A \leq_1 K^B.$$

Define the *jump* of a set A ([McE85]) to be the set $J_e(A) = \chi_{K^A}$. (Note that we identify functions with their graphs.) Clearly $J_e(A) \equiv_e K^A \oplus \overline{K^A}$. If $\mathbf{a}$ is an e-degree, then we can define $\mathbf{a}'$ as $\deg_e(J_e(A))$ for any $A \in \mathbf{a}$ since, by the previous lemma,

$$A \equiv_e B \Rightarrow K^A \oplus \overline{K^A} \equiv_e K^B \oplus \overline{K^B}$$

this gives a well defined unary operation on the e-degrees. Moreover, $\mathbf{a} < \mathbf{a}'$ for every e-degree $\mathbf{a}$.

The following lemma records two important properties of the jump operation.

Lemma 1.3 *[McE85] For every set A,*

1. *$\iota((\deg_T(A))') = (\deg_e(A))'$;*
2. *if A is* total *(i.e. the graph of some total function), then $J_e(A) \equiv_e \overline{K^A}$.*

Definition 1.4 A set A is called *e-high* if $A \in \Sigma^0_2$ and $J_e^{(2)}(\emptyset) \leq_e J_e(A)$. An e-degree $\mathbf{a}$ is called *high*, if $\mathbf{a}$ contains an e-high set (hence $\mathbf{a}' = \mathbf{0}_e''$).

By Lemma 1.3, the embedding ι preserves highness, i.e. it maps high T-degrees to high e-degrees.

The following is a useful characterization of the e-high sets.

Lemma 1.5 *For every set A,*

$$A \textit{ is e-high } \Leftrightarrow Tot \leq_e J_e(A) \Leftrightarrow Tot \leq_T K^A.$$

Proof. First notice that, by Lemma 1.3, A is e-high if and only if $\overline{K^{\overline{K}}} \leq_e J_e(A)$. Moreover, A is e-high if and only if, by Lemma 1.3, $\chi_{K^{\overline{K}}} \leq_e \chi_{K^A}$, if and only if, by Lemma 1.1, $\overline{K^{\overline{K}}} \leq_T K^A$. On the other hand, $\overline{K^{\overline{K}}} \equiv_1 Tot$. Indeed, $\overline{K^{\overline{K}}} \leq_1 Tot$ follows from the fact that Tot is Π^0_2-complete. To show that $Tot \leq_1 \overline{K^{\overline{K}}}$ simply observe that $\overline{Tot} \in \Sigma^0_2$, hence $\overline{Tot} \leq_e \overline{K}$, and thus, by Lemma 1.2, $\overline{Tot} \leq_1 K^{\overline{K}}$.

It follows that A is e-high if and only if $Tot \leq_e J_e(A)$ if and only if $Tot \leq_T K^A$. $\square$

Finally notice the following:

Lemma 1.6 *For every total function f, $J_e(f) \equiv_T f'$.*

Proof. Since K^f is computably enumerable in f, we have $K^f \leq_1 f'$, and so $\overline{K^f} \leq_T f'$. On the other hand, f' is computably enumerable in f, hence, by the totality of f, $f' \leq_e f$, and thus $f' \leq_1 K^f$, by Lemma 1.2, which shows that $f' \leq_1 J_e(f)$.

Hence we conclude that $J_e(f) \equiv_T f'$. $\square$

McEvoy ([McE85]) defines the notion of a Σ^0_2-high e-degree:

Definition 1.7 A Σ^0_2-*high* approximation $\{A^s\}_{s\in\omega}$ to a set A is a Σ^0_2 approximation such that the function

$$c(x) = \mu s(s > x \ \& \ A^s \restriction x \subseteq A)$$

(called the *computation function for* A relative to the given approximation) is total and dominates every computable function. A set A is called Σ^0_2-*high*, if it has a Σ^0_2-high approximation. Finally an e-degree is said to be Σ^0_2-*high*, if it contains a Σ^0_2-high set.

Using Lemma 1.3 (1), it is shown in [McE85] that if $\mathbf{a} \leq \mathbf{0}'_e$ is total (hence a degree of the form $\iota(\mathbf{b})$, for some Turing degree $\mathbf{b} \leq_T \mathbf{0}'_T$), and $\mathbf{a}$ is high, then $\mathbf{a}$ is Σ^0_2-high. On the other hand, the class of Σ^0_2-high e-degrees properly extends the class of all high total e-degrees (in fact, there exist quasi-minimal Σ^0_2-high e-degrees, [McE85]).

In the next section we show that the high e-degrees and the Σ^0_2-high e-degrees coincide. This answers Question 7.3 of [Coo90].

2 The Σ^0_2-high e-degrees coincide with the high e-degrees

Theorem 2.1 *An e-degree* $\mathbf{a}$ *is Σ^0_2-high if and only if* $\mathbf{a}$ *is high.*

Proof. ($\Rightarrow$) Let A be Σ^0_2-high; let $\{A^s\}_{s\in\omega}$ be a Σ^0_2-high approximation to A, and let c be the computation function for A relative to this approximation.

By Lemma 1.5 it is enough to show that $Tot \leq_e J_e(A)$. We claim that, for every i,

$$\begin{aligned} i \in Tot &\Leftrightarrow (\forall y)[\varphi_i(y)\downarrow] \\ &\Leftrightarrow (\exists x)(\exists s)[(\forall y < x)\varphi_{i,s}(y)\downarrow \ \& \\ &\qquad (\forall y \geq x)(\forall t > y)[A^t \restriction y \subseteq A \Rightarrow \varphi_{i,t}(y)\downarrow]]. \end{aligned}$$

Indeed, if φ_i is total, then the function

$$\hat{\varphi}_i(x) = \mu s(\,\varphi_{i,s}(x)\downarrow)$$

is a total computable function, and thus dominated by c. Let x be such that $c(y) > \hat{\varphi}_i(y)$ for every $y \geq x$; then

$$(\forall y \geq x)(\forall t > y)[A^t \restriction y \subseteq A \Rightarrow \varphi_{i,t}(y)\downarrow].$$

This establishes the left-to-right implication in the claimed equivalence. The right-to-left implication is trivial.

Now let

$$B = \{\langle i, x\rangle : (\exists y \geq x)(\exists t > y)[A^t \restriction y \subseteq A \ \&\ \varphi_{i,t}(y)\uparrow]\}.$$

Clearly $B \leq_e A$. Hence, by Lemma 1.2, $B \leq_1 K^A$, via, say, the computable function h.

Then

$$\begin{aligned} i \in Tot &\Leftrightarrow (\exists x)(\exists s)[(\forall y < x)\varphi_{i,s}(y)\downarrow \ \&\ \langle i, x\rangle \notin B] \\ &\Leftrightarrow (\exists x)(\exists s)[(\forall y < x)\varphi_{i,s}(y)\downarrow \ \&\ h(\langle i, x\rangle) \in \overline{K^A}]. \end{aligned}$$

It follows that $Tot \leq_e \overline{K^A}$, hence $Tot \leq_e J_e(A)$, as desired.

($\Leftarrow$) Assume that A is e-high. Then, by Lemma 1.5, we have that $Tot \leq_T K^A$. Let $Z = K^A$ and let $\{Z^s\}_{s\in\omega}$ be a good Σ^0_2-approximation to K^A. Let ψ be some Turing functional such that $Tot = \psi^Z$.

We now define an enumeration operator Θ by stages, and show that Θ^Z is Σ^0_2-high. Since the Σ^0_2-high e-degrees are closed upwards in Σ^0_2-e-degrees (see [BCS97]), it follows that $\deg_e(A)$ is Σ^0_2-high as well.

Construction:

Stage 0: Let $\Theta_0 = \emptyset$.

Stage $s+1$: For every $i \leq s$, we distinguish the following two cases:

(a) if $\psi_s^{Z^s}(i) = 0$, then for every $x \leq s$, we enumerate the axiom

$$\langle\langle i, x\rangle, Z^s\rangle \in \Theta_{s+1};$$

(b) otherwise, do nothing.

Let Θ^{s+1} consist of Θ^s plus all the axioms enumerated at stage $s+1$ and let $\Theta = \bigcup_{s\in\omega} \Theta_s$. We now prove that Θ is the desired enumeration operator by verifying a series of claims.

Verifications:
Claim 1 For every number i,

$i \in Tot \Rightarrow (\Theta^Z)^{[i]}$ finite;

$i \notin Tot \Rightarrow (\Theta^Z)^{[i]} = \omega^{[i]}$.

Proof. If $i \in Tot$ then at all sufficiently large good stages of the approximation $\{Z^s\}_{s\in\omega}$ we do nothing on behalf of i (i.e. case (b) of the definition of Θ applies to i). To see this, assume that $i \in Tot$, and let $\sigma \subset \chi_Z$ be such that $\psi^\sigma(i) = 1$. Let t_0 be such that

$$(\forall x < |\sigma|)[\sigma = 1 \Rightarrow (\forall s \geq t_0)[x \in Z^s]].$$

Then

$$(\forall s \geq t_0)[s \text{ is good } \Rightarrow \psi_s^{Z^s}(i) = 1].$$

It follows that at stages $s \geq t_0$, if (a) holds then s is not good, hence $\Theta_s^{Z^s} \not\subseteq \Theta^Z$. Therefore $(\Theta^Z)^{[i]}$ is finite.

If $i \notin Tot$, then at all sufficiently large good stages, case (a) of the definition of Θ applies to i. Indeed, if $i \notin Tot$, then $\psi^Z(i) = 0$. One thus argues as in the preceding case, but starting with a string $\sigma \subset \chi_Z$ such that $\psi^\sigma(i) = 0$.

Since $\Theta_s^{Z_s} \subseteq \Theta^Z$, for all good stages s, it follows in this case that $(\Theta^Z)^{[i]} = \omega^{[i]}$. $\square$

Now let $Y = \Theta^Z$. We want to show that Y has a Σ_2^0-high approximation $\{\hat{Y}^s\}_{s\in\omega}$. Let $\{Y^s\}_{s\in\omega}$ be any good Σ_2^0-approximation to Y. Given a partial function φ and a number u, define $\varphi \upharpoonright u \downarrow$, if $\varphi(v)\downarrow$ for all $v < u$.

Define

$$\langle i, x\rangle \in \hat{Y}^s \Leftrightarrow [\langle i, x\rangle \in Y^s \vee \varphi_{i,s} \upharpoonright \langle i, x+1\rangle \uparrow].$$

Claim 2 $\{\hat{Y}^s\}_{s\in\omega}$ is a Σ_2^0-approximation to Y.
Proof. Let $\hat{Y} = \{y : (\exists t)(\forall s \geq t)[y \in \hat{Y}^s]\}$. If $i \notin Tot$ then $Y^{[i]} = \omega^{[i]} = \hat{Y}^{[i]}$. On the other hand, assume that $i \in Tot$. If $\langle i, x\rangle \in Y$, then clearly $\langle i, x\rangle \in \hat{Y}$. If $\langle i, x\rangle \notin Y$, then at all sufficiently large stages $\varphi_{i,s} \upharpoonright \langle i, x+1\rangle \downarrow$ and so when $\langle i, x\rangle \notin Y^s$, we have that $\langle i, x\rangle \notin \hat{Y}^s$. $\square$

Next, let c be the computation function for Y relative to the Σ_2^0 approximation $\{\hat{Y}^s\}_{s\in\omega}$. The following claim completes the proof of the Theorem.

Claim 3 The function c is total and dominates all total computable functions.
Proof. Let us first show that c is total. To this end, let $z \in \omega$ be given. Let $t > z$ be a stage such that

$$(\forall \langle i, x\rangle < z)[\varphi_i \restriction \langle i, x+1\rangle \downarrow \Leftrightarrow \varphi_{i,t} \restriction \langle i, x+1\rangle \downarrow].$$

Then if $s \geq t$ is a good stage of the enumeration $\{Y^s\}_{s\in\omega}$, we have that $\hat{Y}^s \restriction z \subseteq Y$. Therefore $c(z)$ is defined.

Now consider any total φ_i. Let x be such that $\langle i, y\rangle \notin Y$, for every $y \geq x$. Let $z \geq \langle i, x\rangle$, and let y be the least number such that $\langle i, y\rangle \leq z < \langle i, y+1\rangle$. Let s be the least stage such that $\varphi_{i,s}(z)\downarrow$, hence $\varphi_{i,t} \restriction \langle i, y+1\rangle \uparrow$ for every $t < s$. Then $\langle i, y\rangle \in \hat{Y}^t$, for every $t < s$. Therefore $\varphi_i(z) < s \leq c(z)$. $\square$

3 Jumps of properly Σ_2^0 e-degrees

A Σ_2^0 e-degree $\mathbf{a}$ is called *properly* Σ_2^0 ([CC88]) if $\mathbf{a}$ contains no Δ_2^0 set. Copestake and Cooper, [CC88, Theorem 1], show that there exist e-degrees that are properly Σ_2^0 and Σ_2^0-high. Since every high computably enumerable Turing degree corresponds, under the embedding ι, to a high e-degree, it follows that not every Σ_2^0-high e-degree is properly Σ_2^0-high. (A trivial counterexample is $\mathbf{0}'_e = \deg_e(\overline{K})$). It is shown in [MC85] that $\deg_e(A)$ is low if and only if $B \in \Delta_2^0$, for every $B \leq_e A$. This characterization of the low e-degrees seems to suggest the possibility that the properly Σ_2^0 e-degrees are all high. We show in this section that this is not the case.

Theorem 3.1 *Let C be such that C is computably enumerable in $\emptyset'$, $\emptyset' \leq_T C <_T \emptyset''$ and $C' \equiv_T \emptyset'''$. Then there exists a set A of properly Σ_2^0 e-degree, such that $J_e(A) \leq_e \chi_C$.*

Corollary 3.2 *There exist properly Σ_2^0 e-degrees that are not high.*

Proof. Let C and A be as in the previous theorem. If A were e-high, then $J_e^{(2)}(\emptyset) \leq_e \chi_C$, from which, by totality, $J_e^{(2)}(\emptyset) \leq_T \chi_C$; but $J_e^{(2)}(\emptyset) \equiv_T \emptyset''$, by Lemma 1.6. Hence $\emptyset'' \leq_T C$, contradiction. $\square$

3.1 Proof of Theorem 3.1

Let C satisfy the hypotheses of the theorem; let $C = W^K$, for some computably enumerable set W. For every t, let $\kappa_t = \chi_{K^t} \restriction k(t)$, where k is some

$1-1$ computable function such that $K = range(k)$ and $K^t = \{k(s)|s \leq t\}$. Define a Σ^0_2 approximation $\{C^t\}_{t\in\omega}$ to C by letting

$$C^t = W_t^{\kappa_t},$$

As $C' \equiv_T K''$, there is an $f \leq_T C$ that dominates all Δ^0_2 total functions (see e.g. [Ler83, p. 85]). Let $f = \Psi^C$, for some Turing functional Ψ, be such a function.

We need the following lemma:

Lemma 3.3 *There exists a computable sequence $\{B_i^s\}_{i,s\in\omega}$ of finite sets such that, if*

$$B_i = \{x : (\exists t)(\forall s \geq t)[x \in B_i^s]\}$$

then

1. *for every $B \in \Delta^0_2$, there is an i such that $B =^* B_i$ and, for almost all x, $\lim_s B_i^s(x)$ exists;*
2. *the relation $x \in B_i$ (as one of x and i) is computable in C.*

Proof. Given u and X, with $X = K$, or $X = K^v$ for some $v \geq u$, we say that u is X-*true* if $\kappa_u \subseteq \chi_X$. We will use the fact that for every $B \in \Delta^0_2$ there exists some i such that $\chi_B = \varphi_i^K$. Roughly speaking, we will have $x \in B_i^s$ if there exists some K^s-true stage $t < w$, with $\varphi_{i,t}^{\kappa_t}(x) = 1$, where w is the least K^s-true stage such that $\Psi_w^\sigma(x)\downarrow$, for some $\sigma \subset \chi_C$. Then we use the fact that Ψ^C dominates all Δ^0_2 functions to verify that, for all but finitely many x, there exists a K-true stage t such that $\varphi_{i,t}^{\kappa_t}(x) = 1$ and $t < \Psi^C(x) < w$. The main difficulty here is that one can not find, in a computable way, the right w at s. For every i, x, s, we will therefore define the values of a finite set $B_i^s \subseteq \omega$, a finite set $L(x,s) \subseteq \omega \times 2^{<\omega}$ and a linear ordering $<_{x,s}$ on $L(x,s)$. We "assign preconditions" to elements of $\omega \times 2^{<\omega}$ subject to the following rules: $L(x,s)$ may contain only pairs $\langle r, \rho\rangle$ with preconditions which have been satisfied at some stage $u \leq s$. At stage s, we will choose the $<_{x,s}$-first element $\langle r,\rho\rangle$ of $L(x,s)$. We will argue that infinitely many times we choose the correct $\langle w,\sigma\rangle$ and we eventually choose only pairs $\langle r,\rho\rangle$ with $r \geq w$.

Let i, x be given. The formal definitions are given by induction on s.

Stage 0: Define $B_i^0 = \emptyset$ and $L(x,0) = <_{x,0} = \emptyset$. No $\langle r,\rho\rangle$ has a precondition at 0.

Stage $s+1$: If $x \geq s+1$ then $x \notin B_i^{s+1}$; otherwise, we distinguish two cases:

- if $L(x,s) = \emptyset$, then
$$x \in B_i^{s+1} \Leftrightarrow x \in B_i^s;$$

- otherwise, let $\langle w, \sigma \rangle$ be the $<_{x,s}$-least element of $L(x, s)$. Then,

  (a) if there is no $t < w$ such that t is K^{s+1}-true, then $x \notin B_i^{s+1}$;

  (b) otherwise, for the least such t, $x \in B_i^{s+1}$ if and only if $\varphi_{i,t}^{K_t}(x) = 1$.

In the latter case, i.e. when $L(x, s) \neq \emptyset$, we extract $\langle w, \sigma \rangle$ from $L(x, s+1)$ and cancel the related precondition. Hence $\langle w, \sigma \rangle$ has no precondition at any stage $v \geq s + 1$ prior to the smallest stage $v' > s + 1$ (if any) at which we again assign a precondition to $\langle w, \sigma \rangle$.

We *assign a precondition* to each pair $\langle r, \rho \rangle$ such that

1. r is K^{s+1}-true;
2. $\Psi_r^{\rho}(x)\downarrow$;
3. $\rho \subseteq \chi_{C^r}$;
4. ρ of minimal length, satisfying 2. and 3. (i.e. if $\Psi_r^{\rho'}(x)\downarrow$ and $\rho' \subseteq \chi_{C^r}$ then $\rho \subseteq \rho'$; notice that $|\rho'| < r$, for each such ρ', by the definition of the use function as in [Soa87, p. 49]);
5. $\langle r, \rho \rangle$ does not have a precondition at $s + 1$.

At any $v > s + 1$, we say that this precondition becomes *satisfied at* v if

$$(\forall i < |\rho|)[\rho(i) = 0 \Rightarrow (\exists t)[s + 1 \leq t \leq v \,\&\, i \notin C^t]].$$

Let

$$\begin{aligned} L(x, s+1) = {} & (L(x, s) - \{\langle w, \sigma \rangle\}) \cup \\ & \{\langle r, \rho \rangle : \langle r, \rho \rangle \text{ has a precondition that becomes satisfied at } s + 1\}. \end{aligned}$$

Finally, we order $L(x, s+1)$ as follows: if $\langle r, \rho \rangle, \langle r', \rho' \rangle \in L(x, s+1)$, then let $\langle r, \rho \rangle <_{x,s+1} \langle r', \rho' \rangle$ if either

1. $\langle r, \rho \rangle, \langle r', \rho' \rangle \in L(x, s)$ and $\langle r, \rho \rangle <_{x,s} \langle r', \rho' \rangle$, or
2. $\langle r, \rho \rangle \in L(x, s)$ and $\langle r', \rho' \rangle \notin L(x, s)$, or
3. $\langle r, \rho \rangle, \langle r', \rho' \rangle \notin L(x, s)$ and $r < r'$.

We now check that the sequence B_i^s has the desired properties.

Claim Let x be given, let σ be the least string such that $\sigma \subset \chi_C$ and $\Psi^{\sigma}(x)\downarrow$. Let w be the least K-true stage such that $\Psi_w^{\sigma}(x)\downarrow$. Then

1. at infinitely many stages s we extract $\langle w, \sigma\rangle$ from $L(x,s)$;
2. there exists a stage t_0 such that we do not extract any pair $\langle r, \rho\rangle$ with $r < w$ from $L(x,s)$ at any stage $s \geq t_0$.

Proof. Since $\sigma \subset \chi_C$, it is clear that there are infinitely many stages at which the requirements (1-4) for assigning a precondition to $\langle w, \sigma\rangle$ are fulfilled. Moreover, once assigned at a stage s_0, there exists a stage $s_1 > s_0$ such that the precondition becomes satisfied at s_1 and so is then in $L(x,s)$ until extracted. As there are only finitely many elements of $L(x, s_1)$ before $\langle w, \sigma\rangle$ in the ordering and no new ones can later be inserted before it $\langle w, \sigma\rangle$ is eventually extracted. Hence, there exist infinitely many stages s such that $\langle w, \sigma\rangle \in L(x,s)$ and we extract $\langle w, \sigma\rangle$ from $L(x,s)$ at infinitely many stages.

Let $t < w$, and assume for a contradiction that at infinitely many stages s, we extract $\langle t, \rho_s\rangle$, for some string ρ_s. Thus $|\rho_s| < t$ by the definition of the use function, since $\Psi_t^{\rho_s}(x)\downarrow$. Then there exist a ρ, with $|\rho| < t$, and infinitely many stages u_s at which we assign a precondition to $\langle t, \rho\rangle$ which becomes satisfied at some stage $v_s \leq s$ and $\langle t, \rho\rangle \in L(x, v)$ for every v such that $v_s \leq v \leq s$. Then t is K-true. Let t_0 be a stage such that

$$(\forall s \geq t_0)(\forall i < t)[i \in C \Rightarrow i \in C^s].$$

By the minimality of σ and w, and since $C^t \subseteq C$ and t is K-true, it follows that there exists some $i < |\rho| < t$ such that $i \in C$ and $\rho(i) = 0$. But no pair $\langle r, \rho\rangle$ with $\rho(i) = 0$, for some $i < |\rho|$ such that $\chi_C(i) = 1$, can have a precondition assigned to $\langle r, \rho\rangle$ at some stage $u \geq t_0$ which becomes later satisfied. $\Box$

We now conclude the proof of the lemma. Let $B \in \Delta_2^0$, and let i be such that $\chi_B = \varphi_i^K$. Let

$$t(x) = \min\{t : t \text{ is } K\text{-true and } \varphi_{i,t}^{K_t}(x)\downarrow\}.$$

Then t is total and so a Δ_2^0 function. It follows that there exists some number x_0 such that $f(x) > t(x)$, for all $x \geq x_0$.

Given $x \geq x_0$, let w and σ be as in the previous claim (for x). Then $t(x) < f(x) < w$ (since $f(x) = \Psi_w^\sigma(x)$). Moreover, if t_0 is as in the proof of the previous claim, then, for every pair $\langle r, \rho\rangle$ such that $\langle r, \rho\rangle$ is extracted from $L(x,s)$ at any stage $s \geq t_0$, we have $t(x) \leq r$. Hence for all $x \geq x_0$, $\chi_{B_i}(x) = \varphi_i^K(x)$.

Since $f(x), \psi(x)$, and w can be computed by C, we easily conclude that the relation $x \in B_i$ is computable in C. $\Box$

Remark 3.4 Note that if $t(x) \geq w$, then $\lim_s B_i^s(x)$ need not exist, but, in any case, $x \notin B_i$, since at every large enough stage at which we extract $\langle w, \sigma\rangle$ from $L(x,s)$ we have $x \notin B_i^s$.

We now go back to the proof of the theorem. We will build a Σ_2^0 set A such that, for every Δ_2^0 set B, $A \not\equiv_e B$, and $K^A \leq_T C$. This implies that $\deg_e(A)$ is properly Σ_2^0 and, by Lemma 1.1, $J_e(A) \leq_e \chi_C$.

3.2 The strategies

The properly Σ_2^0-strategy. Let $\{\Phi_e, \Psi_e\}_{e \in \omega}$ be some effective listing of all pairs of e-operators. To make A of properly Σ_2^0 e-degree, it is enough to satisfy the following requirements, for every $e, i \in \omega$:

$$\mathcal{P}_{e,i}: \qquad A = \Phi_e^{B_i} \;\&\; B_i = \Psi_e^A \Rightarrow (\exists^\infty x)[\lim_s B_i^s(x) \uparrow]$$

where $\{B_i\}_{i \in \omega}$ and $\{B_i^s\}_{i,s \in \omega}$ are as given in Lemma 3.3.

Indeed, if we satisfy these requirements for every e, i, then $\deg_e(A)$ is properly Σ_2^0. Suppose, for the sake of a contradiction, that $A \equiv_e B$ and $B \in \Delta_2^0$. Then, by the previous lemma, $A = \Phi_e^{B_i}$ and $B_i = \Psi_e^A$ for some e, i, with $B =^* B_i$ and so $\lim_s B_i^s(x)$ does not exist for infinitely many x for the desired contradiction, since $\lim_s B_i^s(x)$ exists for almost all x.

The strategy to meet $\mathcal{P}_{e,i}$ is a slight modification of the canonical properly Σ_2^0 strategy as given in [CC88], and described as follows:

(a) appoint a witness x and let $x \in A$;

(w_1) wait for finite sets D, E such that $x \in \Phi_e^D$ and $D \subseteq \Psi_e^E$;

(b) fix $E - \{x\} \subseteq A$;

(w_2) wait for $D \subseteq B_i$;

(w_3) let $x \notin A$, wait for $D \not\subseteq B_i$;

(ℓ) let $x \in A$; go back to (w_2).

A triple x, D, E as above is called a *follower* of $\mathcal{P}_{e,i}$.

As described in [CC88, Theorem 1], for a given follower x, D, E this strategy may have the following outcomes: (w_1) yields $x \in A - \Phi_e^{B_i}$ or $y \in B_i - \Psi_e^A$ for some y; (w_2) corresponds to the case $D \subseteq \Psi^A$, $D \not\subseteq B_i$; (w_3) corresponds to the case $x \in \Phi^{B_i} - A$; finally, the infinitary outcome ℓ entails that $\lim_s B_i^s(y)$ does not exist for some $y \in D$.

The subrequirements $\mathcal{P}_{e,i,j}$. It follows by the analysis of the outcomes of the previous strategy that if $B_i^s(x)$ does have limit on every $x \in D$, then $A \neq \Phi_e^{B_i}$ or $B_i \neq \Psi_e^A$. The only complication here (see Remark 3.4) is that there might exist finitely many numbers x such that $\lim_s B_i^s(x)$ does not exist, thus, for some $y \in D$, $\lim_s B_i^s(y)$ need not exist. We cope with this difficulty by attacking $\mathcal{P}_{e,i}$ through infinitely many subrequirements $\mathcal{P}_{e,i,j}$, with $j \in \omega$. The strategy for $\mathcal{P}_{e,i,j}$ consists in looking for a follower x, D, E such that $D \upharpoonright j = B_i \upharpoonright j$: thus, for almost all j, if we appoint a follower x, D, E as before, we are bound to conclude that $B_i^s(y)$ exists on every $y \in D$. Thus $\mathcal{P}_{e,i}$ is satisfied through some subrequirement $\mathcal{P}_{e,i,j}$ (in fact cofinitely many such subrequirements). Before acting, the subrequirement $\mathcal{P}_{e,i,j}$ must therefore be provided with some knowledge of what numbers $x < j$ are in fact in B_i. This information is coded in the first component, $h(\sigma, s)$, of the outcome of the node corresponding to $\mathcal{P}_{e,i,j}$ in the tree of outcomes.

The strategy for $K^A \leq_T C$. For every i, we will look for a finite set D such that $i \in \Phi_i^D$. If such a D exists then we let $D \subseteq A$. Notice that we can determine computably in $\emptyset'$ and, thus, in C, whether or not such a finite set exists.

3.3 The tree of outcomes

For notation and terminology for strings and trees, the reader is referred to [Soa87]. The tree of outcomes is the smallest set T of strings σ such that

1. if $|\sigma|$ is even then $\sigma^\frown(h, r) \in T$, for every $h \in \omega$ and $r \in \{0, 1\}$;
2. if $|\sigma|$ is odd then $\sigma^\frown r \in T$, for every $r \in \{0, 1\}$.

The strings of even length are assigned to the (sub)requirements $\mathcal{P}_{e,i,j}$, according to some fixed priority listing. The first component, $h(\sigma, s)$, of the outcome of σ at stage s will be an assessment as to which numbers $x < j$ are in fact in B_i: at stage $s+1$, $h(\sigma, s)$ will be chosen to be the first element of a list $\mathcal{L}(\sigma, s)$ of numbers. Each element h of the list is the canonical index of a finite subset of $\{x : x < j\}$. Its position in the list measures how well the set $\{x : x < j\} - B_i$ is approximated by the finite set D_h. Having decided on the first component, h, of the outcome at σ, the strategy for $\mathcal{P}_{e,i,j}$ is ready to act at $\sigma^+ = \sigma^\frown h$. The outcome 1 at σ^+ corresponds to (w_1) or (w_2); the outcome 0 corresponds to (w_3) or (ℓ).

The strings of odd length are devoted to guaranteeing that $K^A \leq_T C$: if $|\sigma| = 2i + 1$ then we have outcome 0 if there exists (modulo higher priority constraints) some finite set D such that $i \in \Phi_i^D$; otherwise we have outcome 1.

Let $\hat{T} = T \cup \{\sigma\hat{\ }h : |\sigma| \text{ even } \& h \in \omega\}$. For $\sigma \in \hat{T}$, the parameter $\alpha(\sigma, s)$ is intended to record some finite set which we want to keep in A for the sake of our actions at σ; the parameter $\epsilon(\sigma, s)$ is meant to record some finite set of elements which we want to keep out of A.

The ordering $\preceq$ of T is determined in the usual way by the ordering of the outcomes given that we define $(h, r) < (h', r')$ if

$$h > h' \text{ or } [h = h' \,\&\, r < r'].$$

We extend $\preceq$ to $\hat{T}$ in the obvious way.

Finally, let $\{\xi_\sigma\}_{\sigma \in \hat{T}}$ be a computable partition of ω into infinite computable sets.

3.4 The construction

The construction proceeds by stages. At stage s we define a finite set A^s, a string δ_s, and the values of several parameters. Unless otherwise specified, at each stage each parameter retains the same value as at the preceding stage.

Stage 0: Define $\delta_0 = \emptyset$. For every $\sigma \in \hat{T}$, let

$$\alpha(\sigma, 0) = \epsilon(\sigma, 0) = \mathcal{L}(\sigma, 0) = \emptyset.$$

Let $x(\sigma, 0)$ and $p(\sigma, h, 0)$ be undefined for every $x, h \in \omega$. Finally, let $A^0 = \emptyset$.

Stage $s + 1$: Suppose that we have defined $\delta_{s+1} \restriction n$, where $n < s + 1$: let $\sigma = \delta_{s+1} \restriction n$. Our aim is to define a string σ^{++} which we will be $\delta_{s+1} \restriction n + 1$.

$|\sigma|$ **even.** Let $\mathcal{P}_{e,i,j}$ be the requirement assigned to σ. For simplicity, drop subscripts, and let $\Phi_e = \Phi$, $\Psi_e = \Psi$ and $B_i = B$.

Our first task is to define the first component, $h(\sigma, s+1)$, of the outcome. We define $h(\sigma, s+1)$ to be the least element of $\mathcal{L}(\sigma, s)$ if $\mathcal{L}(\sigma, s) \neq \emptyset$, otherwise $h(\sigma, s + 1) = 0$. Then we cancel the precondition for $h(\sigma, s + 1)$ by letting $p(\sigma, h(\sigma, s + 1), s + 1)\uparrow$.

To every h such that $\max D_h < j$ and h does not have a precondition, we assign the precondition $p(\sigma, h, s + 1)$ which becomes satisfied at some later stage $v > s + 1$ if, for every $x < j$ and $x \in D_h$, there exists u such that $s + 1 \leq u \leq v$ and $B^u(x) = 0$.

Define

$$\mathcal{L}(\sigma, s + 1) = (\mathcal{L}(\sigma, s) - \{h(\sigma, s + 1)\}) \cup \{h : h \text{ has a precondition that is satisfied at } s + 1\}$$

and order $\mathcal{L}(\sigma, s + 1)$ in the usual way: for every $h, h' \in \mathcal{L}(\sigma, s + 1)$, define $h <_{\sigma,s+1} h'$ if either

1. $h, h' \in \mathcal{L}(\sigma, s)$ and $h <_{\sigma,s} h'$, or
2. $h \in \mathcal{L}(\sigma, s)$ and $h' \notin \mathcal{L}(\sigma, s)$, or
3. $h, h' \notin \mathcal{L}(\sigma, s)$ and $h < h'$.

Let $\sigma^+ = \sigma\widehat{\ }h(\sigma, s+1)$.

Now we are ready to activate the strategy for $\mathcal{P}$.

Let $x = x(\sigma^+, s+1)$ be the least number in ξ_{σ^+} such that $x \notin \alpha(\rho, s+1)$, for every $\rho \prec \sigma^+$.

Case 1).

$$(\exists D)(\exists E)[x \in \Phi_s^D \,\&\, D \cap D_{h(\sigma,s+1)} = \emptyset \,\&\, D \subseteq \Psi_s^E \\ \&\, E \cap \bigcup\{\epsilon(\rho, s+1) : \rho \preceq \sigma\} = \emptyset].$$

Choose the least such pair D, E.

In this case, let $\alpha(\sigma^+, s+1) = E - \{x\}$:

1. if $D \subseteq B^s$, then let $\sigma^{++} = \sigma^+\widehat{\ }0$ and $\epsilon(\sigma^{++}, s+1) = \{x\}$;
2. otherwise, let $\sigma^{++} = \sigma^+\widehat{\ }1$ and $\alpha(\sigma^{++}, s+1) = \{x\}$.

Case 2). Otherwise, let $\sigma^{++} = \sigma^+\widehat{\ }1$ and $\alpha(\sigma^{++}, s+1) = \{x\}$.

$|\sigma|$ odd. Let $|\sigma| = 2i+1$. We distinguish two cases.

Case 1). $(\exists D)[i \in \Phi_i^D \,\&\, D \cap \bigcup\{\epsilon(\rho, s+1) : \rho \preceq \sigma\} = \emptyset]$.

In this case, let $\sigma^{++} = \sigma\widehat{\ }0$, and let $\alpha(\sigma, s+1) = D$ for the least such D.

Case 2). Otherwise, let $\sigma^{++} = \sigma\widehat{\ }1$.

Definition of A^{s+1}. At the end of stage $s+1$, let

$$A^{s+1} = (A^s \cup \bigcup\{\alpha(\rho, s+1) : \rho \preceq \delta_{s+1}\}) - \bigcup\{\epsilon(\rho, s+1) : \rho \preceq \delta_{s+1}\}.$$

3.5 Verification

The verification is based upon the following lemmas.

Lemma 3.5 *For every n, $\sigma_n = \liminf_s \delta_s \restriction n$ exists.*

Proof. Assume by induction that the claim is true of n. The only nontrivial case is when $|\sigma_n|$ is even, where, say, the requirement $\mathcal{P}_{e,i,j}$ is assigned to σ_n.

Let h be the canonical index of $\overline{B_i} \restriction j$. It is clear that, whenever we assign a precondition to h, then this precondition becomes satisfied at some later stage. Hence, at infinitely many stages s, $h \in \mathcal{L}(\sigma_n, s)$, and at infinitely many stages t, $h = h(\sigma_n, t)$. On the other hand, it is also clear that for almost all stages s, if $h' \in \mathcal{L}(\sigma_n, s)$, then $D_{h'} \subseteq D_h$, hence $h' \leq h$ by the usual coding of canonical sets. Therefore it follows that $\sigma_{n+1} = \sigma_n{}^\frown(h, r)$, for some $r \in \{0, 1\}$. $\square$

Let $f = \bigcup_{n \in \omega} \sigma_n$.

Lemma 3.6 *For every $\tau \in \hat{T}$, if $\tau \subset f$, then $\alpha(\tau) = \lim_s \alpha(\tau, s)$, $\epsilon(\tau) = \lim_s \epsilon(\tau, s)$ and $x(\tau) = \lim_s x(\tau, s)$ exist. Moreover, if $\tau = \sigma_n$ for some n, then the requirement assigned to σ_n is satisfied.*

Proof. By induction on n, we show that if $\tau = \sigma_n$ or $\tau = \sigma_n^+$, where $\sigma_{n+1} = \sigma_n^+{}^\frown r$ for some $r \in \{0, 1\}$ (of course $\sigma_n^+ = \sigma_n$ if n is odd), then $\lim_s \alpha(\tau, s)$, $\lim_s \epsilon(\tau, s)$ and $\lim_s x(\tau, s)$ exist, and the requirement assigned to σ_n is satisfied. The case $n = 0$ is trivial as are the existence of the required limits for all nodes to the left of the true path.

Assume that the claim is true of n. For every $\tau \in \hat{T}$ such that $\tau \preceq \sigma_n$, let $\alpha(\tau) = \lim_s \alpha(\tau, s)$, $\epsilon(\tau) = \lim_s \epsilon(\tau, s)$ and $x(\tau) = \lim_s x(\tau, s)$; and let t be a stage such that, for every $s \geq t$ and $\tau \preceq \sigma_n$, $\alpha(\tau) = \alpha(\tau, s)$, $\epsilon(\tau) = \epsilon(\tau, s)$ and $x(\tau) = x(\tau, s)$.

Suppose first that $|\sigma_n|$ is even, and let $\mathcal{P}_{e,i,j}$ be the requirement assigned to σ_n. Let $\sigma_{n+1} = \sigma_n{}^\frown(h, i)$, and let $\sigma_n^+ = \sigma_n{}^\frown h$. Then

$$x(\sigma_n^+) = \min x \in (\xi_{\sigma_n^+} - \bigcup_{\tau \preceq \sigma_n} \alpha(\tau)).$$

- If there are no finite sets D, E such that $D \cap D_h = \emptyset$, $x(\sigma_n^+) \in \Phi_e^D$, $E \cap \bigcup_{\tau \preceq \sigma_n} \epsilon(\tau) = \emptyset$ and $D \subseteq \Psi_e^E$, then $\sigma_{n+1} = \sigma_n^+{}^\frown 1$,

$$\lim_s \epsilon(\sigma_n^+, s) = \lim_s \epsilon(\sigma_{n+1}, s) = \emptyset,$$

  $\lim_s \alpha(\sigma_n^+, s) = \emptyset$, $\lim_s \alpha(\sigma_{n+1}, s) = \{x(\sigma_n^+)\}$ and $x(\sigma_n^+) \in A$. Moreover, either $x(\sigma_n^+) \in A - \Phi_e^{B_i}$, or $x(\sigma_n^+) \in \Phi_e^D$, for some $D \subseteq B_i$, but $D \not\subseteq \Psi_e^A$.

- If D, E exist, then we eventually choose the least such pair D, E, hence $\alpha(\sigma_n^+) = E - \{x(\sigma_n^+)\}$, $\epsilon(\sigma_n^+) = \emptyset$, and either (a) or (b) holds:

  (a) $\sigma_{n+1} = \sigma_n^+ {}^\frown 0$ and $\alpha(\sigma_{n+1}) = \emptyset$, $\epsilon(\sigma_{n+1}) = \{x(\sigma_n^+)\}$;

  (b) $\sigma_{n+1} = \sigma_n^+ {}^\frown 1$ and $\alpha(\sigma_{n+1}) = \{x(\sigma_n^+)\}$, $\epsilon(\sigma_{n+1}) = \emptyset$.

  In case (a) either

  (a_1) there exist infinitely many stages s such that $\sigma_n^+ {}^\frown 1 \subseteq \delta_s$, in which case, there exists some $y \in D$ such that $\lim_s B_i^s(y)$ does not exist; or

  (a_2) $D \subseteq B_i$ but $x(\sigma_n^+) \notin A$, giving $x(\sigma_n^+) \in \Phi_e^{B_i} - A$.

  In (b) we have $D \nsubseteq B_i$, but $E \subseteq A$, hence $D \subseteq \Psi_e^A$.

Remark 3.7 Notice that if j is such that $\lim_s B_i^s(y)$ exists for every $y \geq j$, then (a_1) does not occur, by Lemma 3.3.

If $|\sigma_n| = 2i + 1$ is odd and $i \in \Phi_i^D$, for some finite set D such that $D \cap \bigcup_{\tau \preceq \sigma_n} \epsilon(\tau) = \emptyset$, then $\sigma_{n+1} = \sigma_n {}^\frown 0$ and $\alpha(\sigma_{n+1}) = D$, for some such D; otherwise $\sigma_{n+1} = \sigma_n {}^\frown 1$ and $\alpha(\sigma_{n+1}) = \emptyset$. In either case $\epsilon(\sigma_{n+1}) = \emptyset$.

The proof of the lemma is now complete. $\Box$

Lemma 3.8 $K^A \leq_T C$.

Proof. We will show that, for every n, one can compute σ_n recursively in C.

Now, $\sigma_0 = \emptyset$. Assume by induction that we can compute σ_n and a stage s_n such that $\tau \nsubseteq \delta_s$, for every $s \geq s_n$ and $\tau \prec_L \sigma_n$ and each parameter at any $\tau \preceq \sigma_n$ has reached its limit by stage s_n. Assume first that $|\sigma_n|$ is even, and let $\mathcal{P}_{e,i,j}$ be the requirement assigned to σ_n. Since C can compute $B_i \restriction j$, it follows that C can compute the first component, $h = \lim_s h(\sigma_n, s)$, of the outcome of σ_n. Moreover, since $\emptyset' \leq_T C$, C can compute the least stage $s_n^+ \geq s_n$ such that $\tau \nsubseteq \delta_s$, for every $\tau \prec_L \sigma_n {}^\frown h$ (since for every $x \in B_i \restriction j$ and every t, one can compute in $\emptyset'$ whether there exists some $s \geq t$ such that $x \notin B_i^s$). Again, using $\emptyset'$ as an oracle, one can compute whether Case 1 or Case 2 of the construction holds, thus computing σ_{n+1} and the corresponding s_{n+1}.

A similar argument applies in the case $|\sigma_n| = 2i + 1$, for some i, since the oracle $\emptyset'$ can compute whether or not there exists some stage $s \geq s_n$ and some finite D such that $i \in \Phi_i^D$, and $D \cap \bigcup_{\tau \preceq \sigma_n} \epsilon(\tau, s_n) = \emptyset$.

It follows that $i \in K^A$ if and only if $\sigma_{2i+2} = \sigma_{2i+1} {}^\frown 0$, thus $K^A \leq_T C$. $\Box$

Remark 3.9 We expect that, by combining the above construction of A with a variant of the coding procedure and the associated guessing at outcomes used in the tree proof of the Sacks' jump inversion theorem, one can actually guarantee that $J_e(A) \equiv_e \chi_C$.

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