The Recursively Enumerable Degrees

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1. Introduction

Decision problems were the motivating force in the search for a formal definition of algorithm that constituted the beginnings of recursion (computability) theory. In the abstract, given a set A the decision problem for A consist of finding an algorithm which, given input n, decides whether or not n is in A. The classic decision problem for logic is whether a particular sentence is a theorem of a given theory T. Other examples arise in almost all branches of mathematics. In most settings one is almost immediately confronted by the notion of a *recursively (or computably) enumerable set* (the sets which can be listed (i. e. enumerated) by a computable (i. e. recursive) function): the theorems of a axiomatized theory, the solvable Diophantine equations, the true equations between words in a finitely presented group, etc. Typically such decision problems amount to deciding if a particular r. e. set is computable (recursive). Indeed, the first examples of unsolvable decision problems provided examples of nonrecursive r. e. sets: the theorems of predicate logic, the word problem for groups, the halting problem.

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(For technical convenience, we code all expressions in formal languages, groups, etc. as natural numbers and so restrict our attention to sets of natural numbers.)

One can say that all these sets are simply noncomputable. Another view sees them as more complicated or harder to compute than the recursive sets. This is the view that leads to the notion of relative computability (reducibility) introduced by Turing [1936], [1939] and Post [1936], [1944]. The equivalence classes under this notion of relative computability were first called the degrees of recursive unsolvability. As Church's Thesis became widely accepted the word "recursive" was dropped and they became simply the degrees of unsolvability. As Turing's model of computation became the standard one, they became the Turing degrees. In view of the centrality of Turing's notion as the basic general definition of computability, the unqualified notion of degree eventually became that of Turing degrees. Other notions of relative computability whether stronger or weaker, from one-one to truth-table to arithmetic to constructibility, are refereed to by specifying the reducibility.

The starting point for the investigation of this fundamental notion of relative computability was the r. e. degrees (those equivalence classes containing r. e. sets). The classic results of logic (such as Gödel's incompleteness theorem, Church's proof of the undecidability of predicate logic and Turing's unsolvability of the Halting problem) each proved that there was a nonrecursive r. e. degree. All the natural examples, however, of nonrecursive r. e. sets supplied by standard theories which could be proven undecidable (e.g. Peano arithmetic) or from other natural definitions of noncomputable r. e. sets, turned out to have the same complexity. They were all complete, i. e. of the same degree as the halting problem $K = \{e|\phi_e(e)\downarrow\}$. The obvious question, first proposed by Post [1944], was whether there are any other classes of decision problems under the equivalence of relative computability, i. e. are there any r. e. degrees other than that of the recursive sets, $\mathbf{0}$, and $\mathbf{0}'$, the degree of K?

Post attacked this problem by trying to define set theoretic properties of r. e. sets such as simplicity or hypersimplicity which would guarantee incompleteness as well as nonrecursiveness. Post concentrated on thinness properties of the complement of the r. e. set. This particular approach was doomed to failure (Yates [1965]), but Post's work initiated the study of the structure of the r. e. sets under set inclusion and the connections between their set-theoretic structure and computational complexity (see Soare [1997]). The solution to Post's problem, however, came from another approach.

Friedberg [1957] and Muchnik [1956] independently solved the problem by constructing intermediate r. e. degrees. The construction technique they introduced is called the priority method. It has been extensively studied, expanded and developed over the years. The priority method has proven useful in many areas of recursion theory and in applications to other areas of logic as well. Indeed, this technique has been called the hallmark of recursion theory. The principle arena of both its development and application has been in the study of r. e. sets and in particular of the r. e. degrees. The great strides that have been made in the past forty years in the understanding of the structure of the r. e. degrees have gone hand in hand with the development of new types of priority arguments. In this chapter we try to present the important results contributing to our overall picture of the structure $\mathcal{R}$ of the r. e. degrees with just a word or two about the associated proof techniques followed by appropriate references.

2. Structure and Decidability

$\mathcal{R}$, the structure of the r. e. degrees with the ordering $\leq$ induced by Turing reducibility is clearly a partial ordering with least element 0 and greatest element 1 represented by the degrees $\mathbf{0}$ and $\mathbf{0}'$, respectively. It is also easily seen to be an uppersemilattice *(usl)* with join (least upper bound), $\vee$, induced by disjoint union on representatives: $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$ where $\mathbf{c}$ is the degree of $A \bigoplus B = \{2x | x \in A\} \cup \{2x+1 | x \in B\}$ for any sets A, B of degree $\mathbf{a}, \mathbf{b}$ respectively. One way to describe such a structure (or to determine its complexity) is to decide which partial orderings or uppersemilattices (usls) can be embedded in it (preserving the appropriate relations).

Friedberg's and Muchnik's solutions to Post problem can be seen as the first such result: the p. o. with $0, 1$ and two other incomparable elements can be embedded into $\mathcal{R}$. The technique they introduced is now known as the finite injury priority method. (In addition to the original papers, clear expositions of (different views of) the method can be found in Rogers [1967 §10.2], Shoenfield [1971, §13-14] and Soare [1987, VI.1]. It has been used to prove many theorems about the r. e. degrees. In particular, it suffices to prove the following:

Theorem 2.1. (Embedding theorem; Muchnik [1958], Sacks [1963]) *Every countable partial ordering (and even usl) can be embedded into $\mathcal{R}$.*

In terms of the theory of $\mathcal{R}$, this result says that the existential ($\exists$) theory of $\mathcal{R}$ is decidable:

Theorem 2.2. *There is an effective decision procedure for deciding the truth of sentences of the form $\exists x_1 ... \exists x_n \phi$ where ϕ is a quantifier free formula in the language of partial orderings (usls).*

Proof. By the embedding theorem, any such sentence in the language of partial orderings ($\leq$) is true in $\mathcal{R}$ iff there is a partial ordering of size at most n (the number of distinct variables appearing in the sentence) in which it is true. It is clear that we can list all such orderings and then just check each one. If we consider the language of usls, then we must note that, in any usl $\mathcal{L}$, the substructure generated by a finite set $x_1, ... x_n$ is finite and of size at most 2^n. Thus to decide if a given existential sentence in the language of usls ($\leq, \vee$) is true in $\mathcal{R}$ it suffices to see if it is true in one of the finitely many usls of size 2^n. $\Box$

Algebraically, after embedding problems, the next set of questions about the structure of a partial ordering are the extension of embedding problems such as density: Given two elements $a < b$ of the p. o., can we always find another c, such that $a < c < b$? This particular question was answered by Sacks.

Theorem 2.3. (Sacks Density Theorem [1964]) *For every pair of nonrecursive r. e. degrees $\mathbf{a} < \mathbf{b}$ there is one $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.*

The proof of this theorem goes beyond the finite injury method by having both positive (put numbers into the set being constructed) and negative (keep numbers out of the set being constructed) infinitary requirements. It was the primary early example of what is now called the infinite injury method. (Interesting expositions of different approaches to this construction can be found in Shoenfield [1971, §16] and Soare [1987, VIII.4].) The construction requires simple coding (A into C), upward cone avoiding ($B \not\leq C$), downward cone avoiding ($C \not\leq A$) and some sort of control by the top set B to get $C \leq B$. The third of these four requirements also involves a coding procedure. The second was accomplished by an ingenious method (called Sacks preservation which, in its basic form, entails finite but unbounded action for both positive and negative requirements) introduced in an earlier important theorem:

Theorem 2.4. (Sacks Splitting Theorem [1963b]) *For every nonrecursive r. e. degree $\mathbf{a}$ there are r. e. degrees $\mathbf{b}, \mathbf{c} < \mathbf{a}$ such that $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$.*

In 1963, these results lead Shoenfield [1965] to make the sweeping conjecture that the r.e. degrees, $\mathcal{R}$, are a "dense" (or more formally in the language of model theory, a countably saturated) usl with least and greatest elements:

Conjecture 2.5. (Shoenfield [1965]) *For every pair $\mathcal{P} \hookrightarrow \mathcal{Q}$ of finite usls with $0, 1$ and every embedding $f : \mathcal{P} \to \mathcal{R}$, there is an extension g of f to an embedding of $\mathcal{Q}$ into $\mathcal{R}$.*

If true, this conjecture would have implied that the r. e. degrees had many of the familiar properties of structures like dense linear ordering or atomless Boolean algebras which satisfy the corresponding property for the appropriate family of structures (linear orderings and Boolean algebras). Such structures are countably categorical (i. e. there is a unique such countable structure up to isomorphism) and so (if axiomatizable) have decidable theories. They are countably homogeneous (every structure preserving map from one finite subset to another can be extended to an automorphism) and so there are continuum many automorphisms of the structure. A positive solution to Shoenfield's conjecture would thus have constituted an essentially complete characterization of the structure of the r. e. degrees.

In what may have seemed like an unfortunate development, however, the conjecture was refuted almost immediately by the construction of minimal pairs by Lachlan [1966] and Yates [1966]:

Theorem 2.6. (Lachlan [1966], Yates [1966])*: There are nonrecursive r. e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$, i. e. any degree recursive in both $\mathbf{a}$ and $\mathbf{b}$ is recursive.*

(Shoenfield's conjecture clearly implies that for any $\mathbf{a}, \mathbf{b} > \mathbf{0}$ there is a $\mathbf{c} > \mathbf{0}$ which is below both $\mathbf{a}$ and $\mathbf{b}$.)

This result was the beginning of a trend away from "simplicity" and towards "complexity". Its proof also introduced new techniques involving negative requirements that impose restraint that may be unbounded but that infinitely often drops back to a fixed finite value. After the original proofs two important expository approaches to these constructions were discovered that would become the basis for further developments in both results and technology. The first (Lachlan [1973], Soare [1987, IX.1]) exploits an inductive definition of nested expansionary stages to make the restraint drop back simultaneously. The second involves the very important idea of priority trees and constructions introduced in Lachlan [1975] for a more difficult theorem (Theorem 2.15). These latter techniques are exposited in Soare [1987, XIV].

First notice that a meet (greatest lower bound or infimum) operator $\wedge$ has been introduced. The existence of meets at first raises the hope that $\mathcal{R}$ might be a lattice. It is not.

Theorem 2.7. (Lachlan [1966a], Yates [1966]) *The r. e. degrees are not a lattice.*

The methods used to prove that there are minimal pairs also show that there is a strictly ascending sequence $\mathbf{c}_0 < \ldots < \mathbf{c}_n < \ldots$ with an *exact pair*, i. e. incomparable degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{c}_i \leq \mathbf{a}, \mathbf{b}$ for each i and every $\mathbf{c} \leq \mathbf{a}, \mathbf{b}$ is below some $\mathbf{c}_i$ (Yates [1966]). No such $\mathbf{a}$ and $\mathbf{b}$ can have an infimum. (If $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$, then for some i, $\mathbf{c} \leq \mathbf{c}_i$. As $\mathbf{c}_i < \mathbf{c} < \mathbf{a}, \mathbf{b}$, we have contradicted the assumption that $\mathbf{c}$ is the infimum of $\mathbf{a}$ and $\mathbf{b}$.)

Lachlan's proof of this result was quite different. It was based on a relativization of the following:

Theorem 2.8. (Nondiamond Theorem, Lachlan [1966a]) *There are no r. e. degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a}|\mathbf{b}$ *(i.e.* $\mathbf{a} \not\leq \mathbf{b}$ *and* $\mathbf{b} \not\leq \mathbf{a}$*),* $\mathbf{a} \vee \mathbf{b} = \mathbf{1}$, $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

This theorem is now known as the nondiamond theorem as it says that there is no embedding of the diamond shaped four element lattice ($0, 1$ plus two incomparable intermediate elements) into $\mathcal{R}$ preserving $0, 1$ and the lattice structure.

These results opened up a new chapter in the embedding problem for the r. e. degrees which has yet to be finished: Which lattices can be embedded into $\mathcal{R}$ (preserving the lattice structure, of course, but perhaps also 0 and/or 1? These initial results said that the diamond can be embedded preserving 0 but not preserving both 0 and 1. The techniques introduced to construct a minimal pair suffice to embed all countable distributive lattices:

Theorem 2.9. (Lachlan, Lerman, Thomason; see Soare [1987] IX.2) *Every countable distributive lattice can be embedded into* $\mathcal{R}$ *as a lattice preserving* 0.

The next step in the story was also taken by Lachlan who proved that the two basic nondistibutive lattices can be embedded in $\mathcal{R}$. M_5 (or the pentagon) and N_5 (or 1-3-1) pictured in Figures 2.1 and 2.2 below are nondistributive and every nondistributive lattice has one of them embedded in it.

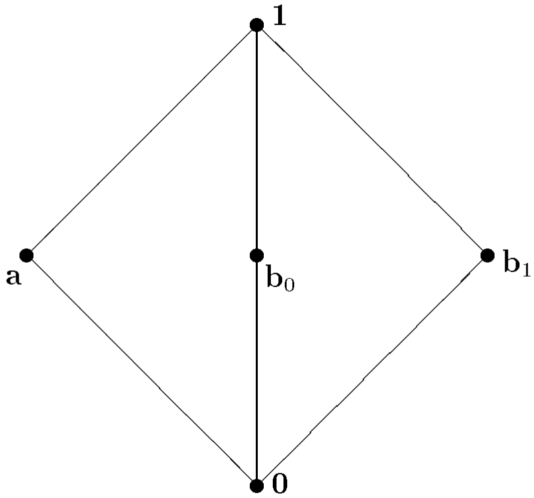

Figure 2.1: The lattice M_5

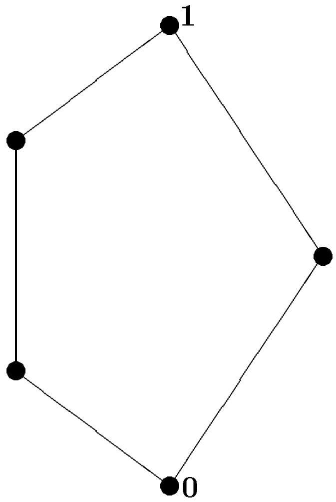

Figure 2.2: The lattice N_5

Theorem 2.10. (Lachlan [1972]) *Both M_5 and N_5 can be embedded (as lattices) in $\mathcal{R}$ preserving* 0.

Now, not every countable lattice can be embedded in any single countable structure. (A sublattice $\mathcal{L}'$ of a lattice $\mathcal{L}$ is generated by a subset A of $\mathcal{L}$ if $\mathcal{L}'$ is the smallest sublattice of $\mathcal{L}$ containing A. It is obvious that any particular countable lattice $\mathcal{L}$ has only countably many sublattices $\mathcal{L}'$ generated by finite subsets of $\mathcal{L}$. On the other hand, there is an uncountable set of lattices $\mathcal{L}$ such that each one is generated by four of its elements. (For an example see Shore [1982].) So the obvious conjecture was simply that every finite lattice can be embedded in $\mathcal{R}$. The true state of affairs, however, proved not to be so simple.

Theorem 2.11. (Lachlan and Soare [1980]) *The lattice S_8 (the diamond on top of 1-3-1) pictured in Figure 2.3 below cannot be embedded in $\mathcal{R}$.*

The failure of the simplest form of a lattice embedding conjecture is the source of much of the difficulty in further progress in deciding the next level of the theory of $\mathcal{R}$. The $\exists$-theory of $\mathcal{R}$ (as a p. o. or usl) is decidable by Theorem 2.2. The next obvious goal is to decide the Σ_2 or $\exists\forall$–theory of $\mathcal{R}$, i. e. find an algorithm for deciding the truth of all sentences of the form $\exists x_1 .... \exists x_n \forall y_1 ... \forall y_m \phi$ with ϕ a quantifier free formula in the language of partial orderings (or, perhaps, even usls). It is routine to verify that, for every finite lattice $\mathcal{L}$, there is an $\exists\forall$ sentence ψ such that $\mathcal{L}$ is embeddable in $\mathcal{R}$ if and only if ψ is true in $\mathcal{R}$. (For example, the emdeddability of 1-3-1 is equivalent to the truth of the following sentence:

$$\exists a_0, a_1 a_2, b, c\{(c < a_0, a_1 a_2 < b) \ \& \ \forall x[(x \geq a_0, a_1 \ \vee \ x \geq a_0, a_2 \vee x \geq a_1, a_2) \\ \rightarrow \ x \geq b] \ \& \ \forall x[(x \leq a_0, a_1 \ \vee \ x \leq a_0, a_2 \vee x \leq a_1, a_2) \rightarrow \ x \leq c]\}.$$

Thus the embedding problem for lattices is a crucial ingredient in any attempt at deciding the $\exists\forall$-theory of $\mathcal{R}$. Unfortunately, the current state of affairs is quite complicated. There are complex necessary conditions for emdeddability as well sufficient ones (Ambos-Spies and Lerman [1986], [1989]). In an attempt to isolate what seemed to be the crucial obstruction to embedding lattices in $\mathcal{R}$, Downey [1990] (see also Weinstein [1988] for a similar notion) proposed the following definition:

Definition 2.12. *Three r. e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are called* a critical triple *if* $\mathbf{a} \vee \mathbf{c} = \mathbf{a} \vee \mathbf{b}$ *and* $\mathbf{b} \wedge \mathbf{c} \leq \mathbf{a}$.

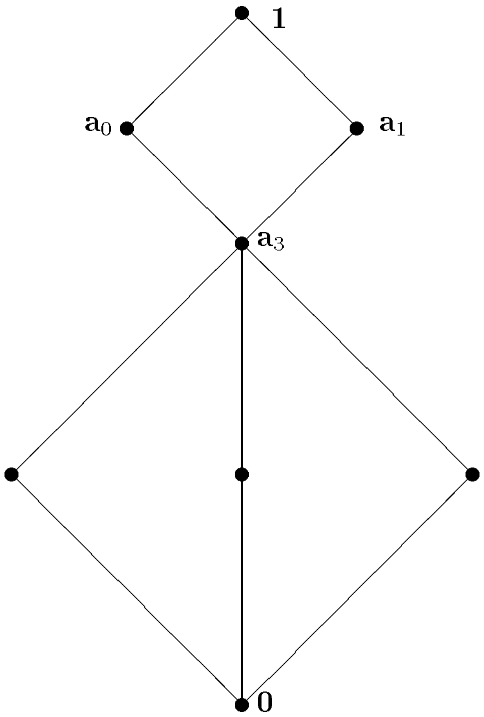

Figure 2.3: The lattice S_8

The nonembedding proofs show that the difficulties in carrying out the embedding construction arise if there are two degrees which "inf down" into a critical triple. This led to the following conjecture.

Conjecture 2.13. *(Downey [1990]) A finite lattice is embeddable in $\mathcal{R}$ if and only if it does not contain a critical triple a, b, c and a pair p, q such that $b \leq p \wedge q \leq b \vee c$.*

Quite recently, Lempp and Lerman [1997] have shown this conjecture to be false by finding a twenty element lattice which has no critical triple but nonetheless cannot be embedded in $\mathcal{R}$. They have suggested, however, that the way they found this counterexample offers some hope of finding a characterization of the finite lattices embeddable in $\mathcal{R}$.

Now, every lattice that is known to be embeddable in $\mathcal{R}$ is known to be embeddable preserving 0. There has also been some work on lattice embedding problems preserving 1 (which is harder than preserving 0) and ones preserving both 0 and 1 (which is much harder and, as can be seen from the nondiamond theorem (Theorem 2.8), known to be much more restricted). For example, Lachlan [1980] and Shoenfield and Soare [1978] showed that the diamond is embeddable preserving 1. Ambos-Spies [1980] extended this result to all countable distributive lattices and Ambos-Spies, Lempp and Lerman [1995] showed that both M_5 and N_5 can be embedded in $\mathcal{R}$ preserving 1. As for preserving both 0 and 1, Ambos-Spies, Lempp and Lerman [1996] characterize the finite distributive latices embeddable in $\mathcal{R}$ preserving both 0 and 1 as those containing a join irreducible noncappable element, i.e. an x which is not the join of elements below it and does not inf down to 0.

Let us reconsider the decision problem for the two quantifier theory of $\mathcal{R}$. Syntactic and algebraic manipulations show (Lerman [1983], 156-158) that deciding the truth of all $\forall\exists$ sentences (in the language of partial orderings) in an usl with 0 such as $\mathcal{R}$ or $\mathcal{D}$ is equivalent to determining for any given collection of extensions of finite partial orderings, $\mathcal{P} \hookrightarrow \mathcal{Q}_i$, $i \leq n$, whether every embedding $f : \mathcal{P} \to \mathcal{R}$ can be extended to an embedding $g : \mathcal{Q}_i \to \mathcal{R}$ for some $i \leq n$. The special case of this question for a single possible extension $\mathcal{Q}$ is called the *extension of embedding problem*. Slaman and Soare have recently solved this problem positively.

Theorem 2.14. *(Slaman and Soare [1995], [1997] the Extension of Embedding Theorem) There is an effective characterization of the pairs $\mathcal{P} \hookrightarrow \mathcal{Q}$ of partial orderings with $0, 1$ such that, for every embedding $f : \mathcal{P} \to \mathcal{R}$, there is an extension g of f to an embedding of $\mathcal{Q}$ into $\mathcal{R}$.*

The idea of the decision procedure is that certain extensions are explicitly ruled out and all remaining ones are then proven extendible. The extension construction is an elaboration of the results and techniques embodied in the density theorem. The nonextension criteria have two parts. The first set of exclusions follows from the embedding of distributive lattices in $\mathcal{R}$. The second, embodies various new results that rely on a level of priority arguments one higher than the infinite injury use for the density theorems and minimal pair arguments. These arguments are now called $\mathbf{0}'''$ constructions. The terminology refers to the difficulty of determining how the requirements are met by the construction. In this terminology, finite and infinite injury arguments are $\mathbf{0}'$ and $\mathbf{0}''$ ones, respectively. These techniques were first introduced by Lachlan to prove that the Sacks Splitting and Density Theorems cannot be combined.

Theorem 2.15. (Lachlan Nonsplitting Theorem, Lachlan [1975]) *There are r. e. degrees* $\mathbf{d} < \mathbf{a}$ *for which there are no r. e. degrees* $\mathbf{b}, \mathbf{c}$ *such that* $\mathbf{d} < \mathbf{b}, \mathbf{c} < \mathbf{a}$ *and* $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$.

Expositions of examples of these methods can be found, for example, in Soare [1987 XV.4], Shore and Slaman [1993] and Slaman [1991a] with other prototypes in Shore [1988] and Downey [1990a]. A typical instance of the facts needed to get Slaman and Soare's nonextension results is the following:

Theorem 2.16. *(Slaman and Soare [1997]) There are incomparable r. e. degrees* $\mathbf{a}, \mathbf{b}$ *such that for every* $\mathbf{z} < \mathbf{a}$, *either* $\mathbf{z} < \mathbf{b}$ *or* $\mathbf{z} \vee \mathbf{b} = \mathbf{0}'$.

The gap between this solution to the extension of embedding problem and deciding the full two quantifier theory is precisely the problem of having more than one possible extension to consider. The reality of this possible problem is epitomized by the nondiamond theorem: Suppose $\mathcal{P}$ is the diamond p. o. with $0, 1$ and incomparable elements a and b. Furthermore, suppose $\mathcal{Q}_0$ puts a nonzero element in below both a and b and $\mathcal{Q}_1$ puts one in which is below 1 but above both a and b. There are then embeddings of $\mathcal{P}$ which cannot be extended to ones of $\mathcal{Q}_0$ by the minimal pair theorem (Theorem 2.6) and there are ones which cannot be extended to $\mathcal{Q}_1$ by the Sacks splitting theorem (Theorem 2.4). On the other hand, the nondiamond theorem (Theorem 2.8) says that every embedding of $\mathcal{P}$ can be extended to either one of $\mathcal{Q}_0$ or one of $\mathcal{Q}_1$.

Now, in other recursion theoretic structures it has been possible to decide the full $\forall\exists$-theory by a combination of embedding results in stronger languages than just partial orderings and a series of extension of embedding results. (Examples include the weak truth-table degrees of the r. e. sets (Ambos-Spies, Fejer, Lempp and Lerman [1996] where the essential ingredient is the characterization of the finite lattices which can be embedded preserving both 0 and 1; the Turing degrees as a whole (Lerman, Shore [1978]; see Lerman [1983], p. 157) and those below $\mathbf{0}'$ (Lerman and Shore [1988]) where lattice embeddings as initial segments and partial order extensions of embeddings suffice; and the lattice of r. e. sets where more elaborate extensions of the language are necessary (Lachlan [1968]).) At times it was thought that it might be possible to similarly separate the lattice embedding and extension of embedding problems in $\mathcal{R}$, solve them individually and so decide the full two quantifier theory. In view of recent developments, this no longer seems a likely scenario. The two problems are more intimately connected than had previously been believed. The roots of this relationship are, on the one hand, the obstructions to extension of embeddings discovered by Slaman and Soare [1995], [1997] and, on the other, an extension of the nondiamond phenomena and the related notion of prompt simplicity introduced by Maass [1982] in the study of the set theoretic structure (and genericity properties) of the r.e. sets but phrased in terms of the dynamics of their enumerations.

Definition 2.17. *A coinfinite r.e. set A is* promptly simple *if there is there is a nondecreasing recursive function p and a recursive one-to-one function f enumerating A (i.e. $A = \operatorname{rg} f$) such that for every infinite r.e. W_e there is an s and an x such that x is enumerated in W at stage s (in some standard uniform enumeration of all the r.e. sets) and is also enumerated in A by stage $p(s)$, i.e. $x = p(n)$ for some $n \leq s$. An r.e. degree $\mathbf{a}$ is* promptly simple *if it contains a promptly simple r.e. set. We let* **PS** *denote the set of promptly simple r.e. degrees.*

It turns out that the property that an r.e. degree is promptly simple can be characterized solely in terms of the ordering of r.e. degrees and has come to play an important role in the study of $\mathcal{R}$. For example, one purely order theoretic result provided by the following theorem is that no join of degrees which are halves of minimal pairs can be $\mathbf{0}'$. It is also connected with another *a priori* apparently extraneous consideration, the relationship between the jump of an r.e. degree and its place in the structure of $\mathcal{R}$. (The jump, $\mathbf{a}'$, of an r.e. degree $\mathbf{a}$ is the degree of $\{e|\phi_e^A(e)\}\downarrow$. It is the analog of $\mathbf{0}'$ for the sets and degrees r.e. in A. The role of this operator in the study of $\mathcal{R}$ will be considered at greater length in §4.)

Theorem 2.18. (Ambos-Spies et al. [1984]) *Let* $\mathbf{M} = \{\mathbf{a} | \exists \mathbf{b}(\mathbf{a} \wedge \mathbf{b} = \mathbf{0})\}$ *be the set of* cappable *r. e. degrees, i. e. those which are halves of minimal pairs; let* $\mathbf{NC}$*, the* noncappable *r. e. degrees be its complement in* $\mathcal{R}$*; and let* $\mathbf{LC} = \{\mathbf{a} | \exists \mathbf{b}(\mathbf{a} \vee \mathbf{b} = \mathbf{0}' \,\&\, \mathbf{b}' = \mathbf{0}'\}$ *be the* low cuppable degrees, *i.e. those which can be cupped (joined) to* $\mathbf{0}'$ *by a low degree* $\mathbf{b}$*, i. e. one such that* $\mathbf{b}' = \mathbf{0}'$*.*
The three classes $\mathbf{PS}, \mathbf{NC}$ *and* $\mathbf{LC}$ *all coincide and together with their complement* $\mathbf{M}$ *partition* $\mathcal{R}$ *as follows:*

i) $\mathbf{M}$ *is a proper ideal in* $\mathcal{R}$*, i. e. it is closed downward in* $\mathcal{R}$ *and if* $\mathbf{a}, \mathbf{b} \in \mathbf{M}$ *then* $\mathbf{a} \vee \mathbf{b} \in \mathbf{M}$ *and* $\mathbf{a} \vee \mathbf{b} < \mathbf{0}'$*.*

ii) $\mathbf{NC}$ *is a strong filter in* $\mathcal{R}$*, i..e. it is closed upward in* $\mathcal{R}$ *and if* $\mathbf{a}, \mathbf{b} \in \mathbf{NC}$ *then there is a* $\mathbf{c} \in \mathbf{NC}$ *with* $\mathbf{c} \leq \mathbf{a}, \mathbf{b}$*.*

As Slaman has pointed out [personal communication], an essential interaction between the lattice embedding problem and the extension of embedding problem arises when this theorem is combined with Theorems 2.16 and the version of the Sacks splitting theorem (Theorem 2.4) [also in 1963b] which says that any given r. e. degree $\mathbf{a}$ can be split into two low r.e. degrees. Thus if $\mathbf{a}$ and $\mathbf{b}$ are as in Theorem 2.16 we can split $\mathbf{a}$ into two low degrees $\mathbf{x}$ and $\mathbf{y}$ by the Sacks splitting theorem. As $\mathbf{a}$ and $\mathbf{b}$ are incomparable, not both $\mathbf{x}$ and $\mathbf{y}$ can be below $\mathbf{b}$. Thus, by the properties guaranteed by Theorem 2.16, one of them joins $\mathbf{b}$ up to $\mathbf{0}'$ and so by Theorem 2.18, $\mathbf{b}$ is not half of a minimal pair. Thus there are unexpected connections between the two problems that can already be expressed by an $\forall\exists$ sentence saying that if $\mathbf{b}$ is half of a minimal pair and $\mathbf{a}$ is incomparable with $\mathbf{b}$ then there is a $\mathbf{y}$ below $\mathbf{a}$ but not below $\mathbf{b}$ such that $\mathbf{b} \vee \mathbf{y} \not= \mathbf{0}'$. Such interactions make framing a specific plan to decide the $\forall\exists$ theory of $\mathcal{R}$ quite difficult. (For a more extensive discussion see Lerman [1996].)

On the other hand, we note that the usual techniques for proving such a theory (or fragment) undecidable cannot work for $\mathcal{R}$. Once one has proven the undecidability of any particular theory (e.g. predicate logic) by coding the computations of Turing machines or the like, one proves other theories undecidable by interpreting a theory that is known to be undecidable in the one of interest. As the general problem of the validity of $\forall\exists$ sentences in purely relational languages is decidable, no standard interpretation along these lines can show that the $\forall\exists$ theory of $\mathcal{R}$ is undecidable. Thus we are simply left with the open problem.

Conjecture 2.19. *The* $\forall\exists$*-theory of* $\mathcal{R}$ *is decidable.*

Should this conjecture prove correct, it will represent the limit of decidability results for $\mathcal{R}$ as Lempp, Nies and Slaman [1977] have shown that its $\forall\exists\forall$-theory is undecidable.

3. Undecidability and Beyond

The minimal pair theorem (Theorem 2.6) refuted Shoenfield's Conjecture and, in retrospect at least, began a long series of developments in other directions. At the time, however, it was still hoped that the r. e. degrees would be "nice" in certain ways. In particular, even Sacks who had conjectured [1963] that there were minimal pairs and that $\mathcal{R}$ is not a lattice, nonetheless continued [1966] to conjecture that the theory of $\mathcal{R}$ is decidable and that there is a strong sense of homogeneity for the notion of r. e. in the sense that "for each (not necessarily r.e.) degree $\mathbf{d}$, the ordering of degrees r. e. in $\mathbf{d}$ and $\geq \mathbf{d}$ is order isomorphic to the r. e. degrees". These conjectures all turned out to be false and some twenty years later a dramatically different view of the structure of the r. e. degrees (as well as of the degrees as a whole) became the prevailing paradigm.

Not only is $\mathcal{R}$ undecidable but its theory is as complicated as possible. It is recursively isomorphic to that of true arithmetic. That is, there are two recursive translations S and T such that S takes sentences ϕ in the language of arithmetic to sentences ϕ^S of partial orderings while T takes sentences ψ in the language of partial orderings to sentences ψ^T of arithmetic. Moreover, the translations preserve truth: $\mathcal{N} \models \phi \leftrightarrow \mathcal{R} \models \phi^S$ and $\mathcal{R} \models \psi \leftrightarrow \mathcal{N} \models \psi^T$. A similar result holds for the theory of all the Turing degrees $\mathcal{D}$. This latter theory is recursively isomorphic to that of true second order arithmetic. Moreover, almost all possible homogeneity conjectures are now known to fail both for relative recursive enumerability as well as relative computability.

At first, such results suggest that there is no hope of understanding or characterizing the structure of $\mathcal{R}$ or $\mathcal{D}$. Later, the opposite view took hold. The idea was that a strong enough proof of the complexity of a structure can characterize it as well as one of its simplicity. Instead of expecting the structure to be decidable and homogeneous, for all degrees to look the same and for there to be many automorphisms, one could look to prove that the theory is as complicated as possible, there are as many different types of degrees as possible (even that no two are alike but rather each is definable) and that the structure has no automorphisms.

In this section, we will describe how far we have been able to travel along this road and what the possible future developments might be like. (See Slaman [1997] for a discussion of these matters in the setting of the general Turing degrees.)

The first hints of the true complexity of $\mathcal{R}$ emerged with the lattice embedding results. These ideas where enough to show that there are many types of degrees in the model theoretic sense. (An n-type over a theory (or a structure) is a set of formulas in n free variables which is consistent with the theory (of the structure). The type is realized in the structure if there is an n-tuple of elements of which every formula in the type is true.) Thus $\mathcal{R}$ is not countably categorical. (A structure M is *countably categorical* if all countable models of its theory are isomorphic.)

Theorem 3.1. (Lerman, Shore and Soare [1984]) *There are countably many 3-types realized in $\mathcal{R}$. More precisely, there are infinitely many distinct finite partial lattices (i. e. infimum is not always defined) each of which is generated (using $\vee$ and $\wedge$ where defined) by three elements that can all be embedded into $\mathcal{R}$ (preserving $\vee$ and $\wedge$ when it is defined).*

Corollary 3.2. *$\mathcal{R}$ is not countably categorical.*

Proof. Each such finite three generated partial lattice embeddable in $\mathcal{R}$ shows that a distinct three type is realizable in $\mathcal{R}$. Thus, by the Ryll-Nardjewski Theorem, $\mathcal{R}$ is not countably categorical.$\Box$

The techniques introduced in this paper were then used to refute Sacks' conjecture on the isomorphism of the r. e. degrees to those r. e. in and above any degree $\mathbf{d}$.

Theorem 3.3. (Shore [1982]) *For each degree $\mathbf{d}$ and each set A which is Π^0_2 in D, there is a partial lattice $\mathcal{P}_A$ generated by four elements which is embeddable in the degrees r. e. in and above $\mathbf{d}$. Moreover, if $\mathcal{P}_A$ is embeddable in an usl $\mathcal{P}$, then A is computable in the jump of any presentation of $\mathcal{P}$(even as a partial order). (So, in particular, there are continuum many four types in the theory of $\mathcal{R}$.)*

Corollary 3.4. *If $\mathbf{d}$ is sufficiently complicated (e.g. $\not\leq \mathbf{0}^{(5)}$), then the degrees r. e. in and above $\mathbf{d}$ are not isomorphic to $\mathcal{R}$.*

Proof. The ordering of Turing reducibility on indices of r. e. sets is definable in arithmetic by a Σ_3 formula. Thus the standard presentation of $\mathcal{R}$ (even as an usl) is recursive in $0^{(4)}$. Consider then $\mathcal{P}_D$ for any D of degree $\mathbf{d}$. $\mathcal{P}_D$ is embeddable in the degrees r. e. in and above $\mathbf{d}$, but not in $\mathcal{R}$. $\square$

(Improvements of this result follow from recent work of Nies, Shore and Slaman [1997] that will be discussed below.)

Corollary 3.5. *$\mathcal{R}$ is not recursively presentable, i. e. it is not isomorphic to any recursive partial ordering.*

Proof. Let C be a Π^0_2 set of degree $\mathbf{0}''$. By the theorem, $\mathcal{P}_C$ is embeddable in $\mathcal{R}$. If $\mathcal{R}$ were recursively presentable, C would be recursive in $\mathbf{0}'$ for a contradiction.$\square$

These results were all proven by lattice embedding and so $\mathbf{0}''$ methods. The real breakthrough came when Harrington began to exploit Lachlan's $\mathbf{0}'''$ methods. A particularly difficult application of these ideas was the first proof of the undecidability of $\mathcal{R}$ by Harrington and Shelah [1982].

Theorem 3.6. (Harrington and Shelah [1982]) *The theory of $\mathcal{R}$ is undecidable.*

The original proof of this theorem has never been published, in part because it was so complicated. Since then simplifications in this proof were made by Harrington and Slaman and a couple of distinctly simpler proofs have been found (Slaman and Woodin [1998] (see Nies, Shore and Slaman [1997]); Ambos-Spies and Shore [1993]) which avoid the need for more than $\mathbf{0}''$ type priority arguments.

All of these proofs of undecidability have the same basic outline. One first constructs a tractable (at least pairwise incomparable) set of degrees which is definable from parameters. The next step is to combine the construction with a procedure for defining a coding of enough partial orderings to get undecidability. To be slightly more precise, each proof first describes a formula $\Phi(x, a, b, c)$ in the language of partial orderings. One then proves that for "enough" partial orderings $\mathcal{P}$, there are r. e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ such that $\langle\{\mathbf{x} \vee \mathbf{d} |\ \Phi(\mathbf{x}, \overrightarrow{a})\}, \leq\rangle \cong \mathcal{P}$:

- Harrington and Shelah [1982]: $\Phi(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv \mathbf{x}$ is maximal among the degrees below $\mathbf{a}$ such that $\mathbf{c} \not\leq \mathbf{b} \vee \mathbf{x}$; "enough" partial orderings $\equiv$ all Δ^0_2 partial orderings.

- Slaman and Woodin [1998]: $\Phi(\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv \mathbf{x}$ is minimal among the degrees below $\mathbf{a}$ such that $\mathbf{c} \leq \mathbf{b} \vee \mathbf{x}$; "enough" partial orderings $\equiv$ all Δ^0_2 partial orderings.

- Ambos-Spies and Shore [1993]: $\Phi(\mathbf{x}, \mathbf{a}) \equiv \mathbf{x}$ is maximal among the degrees above $\mathbf{a}$ such that there is a $\mathbf{y}$ such that $\mathbf{x} \wedge \mathbf{y} = \mathbf{a}$; "enough" partial orderings $\equiv$ all finite partial orderings.

Each of these results suffices to prove the undecidability of $\mathcal{R}$ because each allows us to interpret enough of the theory of partial orderings in $\mathcal{R}$ to exploit the recursively inseparability of that theory. (Recursive inseparability means that there is no recursive set separating the set of theorems of the theory of partial orderings from set of sentences of the language of partial orderings that are false in some finite partial ordering. In fact, the first two codings only require the undecidability of the theory of partial orderings since by the standard Henkin proof of the completeness theorem any sentence that is false in some partial ordering is false in some Δ^0_2 one.) While the first approach requires a difficult application of the $\mathbf{0}'''$ priority technique, the others are less complicated. The original proof of the second uses Boolean combinations of finitely many $0''$ type procedures for each requirement. The third one only requires something like the minimal pair method (branching degree construction) and another type of requirement which is finitary in nature (nonbranching degree). (A degree $\mathbf{a}$ is *branching* if there are degrees $\mathbf{b}, \mathbf{c}$ such that $\mathbf{b} \wedge \mathbf{c} = \mathbf{a}$. Otherwise, it is *nonbranching*.) Recently, Nies, Shore and Slaman [1997] have produced a construction for the second coding scheme that is essentially a minimal pair type argument (albeit a complex one).

Each of these codings of partial orderings automatically supplies us with continuum many different four types (two types for the third coding scheme) in the theory of $\mathcal{R}$. Another early manifestation of the complexity of the r. e. degrees was the proliferation of distinct (one) types of degrees. Along with his construction of a minimal pair, Lachlan [1966] produced a nonzero branching degree. He also constructed degrees which are nonbranching. Another example that we have already seen is being half of a minimal pair. On the other hand, Yates [1966] proved, along with the existence of minimal pairs, that there are r. e. degrees which are *noncappable*, i. e. not halves of minimal pairs. Other examples abound. There are *cuppable* degrees which join (cup) to $\mathbf{0}'$ (by the Sacks splitting theorem, Theorem 2.4) and *noncuppable* ones which don't (Lachlan [1966a]). There are ones which split over every lower degree (any low_2 degree by Shore and Slaman [1990]) and ones which do not (by the Lachlan nonsplitting theorem, Theorem 2.15). There are ones over which $\mathbf{0}'$ splits (any low degree by Robinson [1971]) and ones over which it does not (Harrington; see also Jockusch and Shore [1983]). There are ones below (or above) which we can embed 1-3-1 (Lachlan [1972]) and ones below (or above) which we cannot (Downey [1990], Weinstein [1988], Cholak, Downey and Shore [1997]). The list seemed endless. The proof that it is in fact endless is given by the following theorems:

Theorem 3.7. (Ambos-Spies and Soare [1989]) *There are infinitely many 1-types realized in the r. e. degrees.*

Theorem 3.8. (Ambos-Spies and Shore [1993]) *There are continuum many 1-types over the theory of the r. e. degrees.*

The ultimate version of such results would be have been to show that every r. e. degree realizes a distinct type or even that each one is definable in $\mathcal{R}$. As we shall see, this possibility is also connected to view that the structure of $\mathcal{R}$ is as complicated as possible. The ultimate expression of this new paradigm was the following conjecture:

Conjecture 3.9. Biinterpretability Conjecture for $\mathcal{R}$ *(Harrington; Slaman and Woodin [1998], see Slaman [1991]): There is a definable coding of a standard model of arithmetic in $\mathcal{R}$ for which the relation between degrees* $\mathbf{d}$ *and the codes of sets of degree* $\mathbf{d}$ *is definable. (For $\mathcal{R}$ this is equivalent to there being a definable map taking each r. e. degree* $\mathbf{d}$ *to the code in the standard model for (the index of) a set of degree* $\mathbf{d}$*.)*

This conjecture really expresses the strongest form of the view that the structure the r. e. degrees is as complicated as possible. More than simply saying that the r. e. degrees are complicated, it provides a strong characterization of the structure of $\mathcal{R}$. It would give complete information, for example, about definability in $\mathcal{R}$ (every degree in $\mathcal{R}$ would be definable as would every relation on $\mathcal{R}$ which is definable in arithmetic) and automorphisms for $\mathcal{R}$ (none other than the identity would exist). (To see that these results are corollaries of the conjecture, just use the definable mapping from $\mathcal{R}$ to the standard model of arithmetic. As there are no automorphisms of $\mathcal{N}$, there would then be none of $\mathcal{R}$. As every natural number is definable in $\mathcal{N}$, every degree would be definable in $\mathcal{R}$, etc.)

(There is also an appropriate second order version of the biinterpretability conjecture for $\mathcal{D}$ which would follow from the rigidity of $\mathcal{D}$ (see Slaman [1997]). Surprisingly, the conjecture for $\mathcal{R}$ implies the one for $\mathcal{D}$. Indeed, even more is true.

Theorem 3.10. (Slaman and Woodin [1998]) *If $\mathcal{R}$ is rigid then so is $\mathcal{D}$. In fact, there are finitely many r. e. degrees $\mathbf{a}_1, \ldots, \mathbf{a}_n$ such that if Φ is an automorphism of $\mathcal{R}$ with $\Phi(\mathbf{a}_i) = \mathbf{a}_i$ for each $i \leq n$, then Φ is the identity map.)*

Cooper [1996] has recently announced that there are automorphisms of both $\mathcal{R}$ and $\mathcal{D}$ and so also that the biinterpretability conjectures for both structures fail. Nonetheless, some progress has been made in the direction indicated by the conjecture which we will now describe. As we have mentioned, the theory of the structure $\mathcal{R}$ has already been shown to be as complicated as possible, in particular, it is possible to correctly interpret true arithmetic in the theory of $\mathcal{R}$:

Theorem 3.11. (Harrington and Slaman; Slaman and Woodin; Nies, Shore and Slaman [1997]) *There are recursive translations $S(T)$ taking sentences $\phi(\psi)$ of arithmetic (partial orderings) to sentences ϕ^S, ψ^T of partial orderings (arithmetic) such that $\mathcal{N} \models \phi \leftrightarrow \mathcal{R} \models \phi^S$ ($\mathcal{R} \models \psi \leftrightarrow N \models \psi^T$).*

Each of the various proofs of this theorem begins with one of the codings of partial orderings in $\mathcal{R}$ developed to prove its undecidability. They each provide a translation of the theory of partial orderings into $\mathcal{R}$. As the theory of partial orderings is rich enough to code all of predicate logic, we can view the codings as providing us with models of some finite axiomatization of arithmetic. The real problem, now, is to definably determine the (translations of) sentences true in the models which are standard models of arithmetic (i. e. isomorphic copies of $\mathcal{N}$). The most natural approach to this problem would seem to be to define a standard model or at least a class of models all of which are standard. One would then simply say that a sentence of arithmetic is true (in $\mathcal{N}$) iff the appropriate translation is true in (any of) the definable standard model(s). The first couple of proofs of this theorem (Harrington and Slaman; Slaman and Woodin) did not manage to define standard models and took much more indirect approaches to the theorem. We describe the most recent approach to this problem due to Nies, Shore and Slaman [1997] which not only defines a class of standard models in $\mathcal{R}$ but also (by an identification under a definable equivalence relation) a single definable standard model. It then goes on to establish a weak version of the biinterpretability conjecture that is, however, strong enough to prove many definability results for $\mathcal{R}$.

We begin with Slaman and Woodin's coding and a couple of technical additions:

Theorem 3.12. (Slaman and Woodin) *Given any recursive partial ordering* $\mathcal{P} = \langle\omega, \preceq\rangle$ *there are r.e. degrees* $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{l}$ *and* $\mathbf{g}_i$ *(for* $i \in \omega$*) such that*

1. *the* $\mathbf{g}_i$ *are the minimal degrees* $\mathbf{x} \leq \mathbf{r}$ *such that* $\mathbf{q} \leq \mathbf{x} \vee \mathbf{p}$*;*
2. *for* $i, j \in \omega$*,* $i \preceq j$ *if and only if* $\mathbf{g}_i \oplus \mathbf{l} \leq_T \mathbf{g}_j$*;*
3. $\mathbf{r} \oplus \mathbf{p} \oplus \mathbf{q}$ *is low, i.e.* $(\mathbf{r} \oplus \mathbf{p} \oplus \mathbf{q})' = \mathbf{0}'$
4. *If* $\mathbf{a} > \mathbf{0}$ *is any given r.e. degree, we can also make* $\mathbf{r} < \mathbf{a}$.

(3 and 4 are technical improvements that are needed later.)

We may as well think of the partial ordering $\mathcal{P}$ as coding a model of (a finitely axiomatized version of) arithmetic. The key to definably selecting a set of such models that are all standard is the ability to define comparison maps between (finite) initial segments of certain such models. The idea here is that the standard models are the models $\mathcal{M}$ such that each initial segment of $\mathcal{M}$ can be mapped into an initial segment of every model. The crucial technical lemma needed to define such maps is one that combines Slaman and Woodin coding with cone avoiding and permitting:

Theorem 3.13. (Nies, Shore and Slaman [1997]) *Given any recursive partial ordering* $\mathcal{P} = \langle\omega, \preceq\rangle$ *and low r.e. degrees* $\mathbf{q}_0, \ldots, \mathbf{q}_m$ *there are r.e. degrees* $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{l}$ *and* $\mathbf{g}_i$ *(for* $i \in \omega$*) as in Theorem 3.12 such that if* $\mathbf{g}_{f(i)}$ *is the degree corresponding to the natural number* i *in the model coded by* $\mathcal{P}$*, then* $\mathbf{g}_{f(i)} \leq_T \mathbf{q}_i$ *and* $\mathbf{q}_i \not\leq_T \mathbf{q}_j \Rightarrow g_{f(i)} \not\leq_T \mathbf{q}_j$.

Given two coded low models $\mathcal{M}_1, \mathcal{M}_2$ one can use this theorem to interpolate a third model $\mathcal{M}$ so that one can define isomorphisms between the first n numbers of $\mathcal{M}_1$ and those of $\mathcal{M}$ and between the first n numbers of $\mathcal{M}_2$ and the second n numbers of $\mathcal{M}$. Together with the structure inherent in $\mathcal{M}$, these maps define the desired isomorphism between the first n elements of $\mathcal{M}_1$ and those of $\mathcal{M}_2$.

In this way, one can give a sufficient condition for a model to be standard and a definable scheme for maps between initial segments of such models. Thus Nies, Shore and Slaman are able to define a class of models which are all standard and such that there are definable isomorphisms between the natural numbers of any two models in the class:

A model $\mathcal{M}$ of arithmetic coded (as in Theorem 3.12) by parameters $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{l}$ is *good via* $\mathbf{c}$ if $\mathbf{r} < \mathbf{c}$ and for every $\mathcal{M}'$ with its elements below $\mathbf{c}$ the comparison maps described above define maps for each initial segment of $\mathcal{M}$ to initial segments of $\mathcal{M}'$. $\mathcal{M}$ is *good* if it is good via some $\mathbf{c}$.

Now, every low standard model is good by Theorem 3.13. On the other hand, every good model is standard (as it can be mapped into some standard model). Moreover, given any two good models we can define an isomorphism between them. Thus we can define an equivalence relation on the (codes for) natural numbers in these models and interpretations of the language of arithmetic on these equivalence classes that make the structure so defined a standard model of arithmetic.

Theorem 3.14. (Nies, Shore and Slaman [1997]) *There is a coding scheme interpreting arithmetic in $\mathcal{R}$ such that all the models so defined are standard. Moreover, there is a definable equivalence relation on the parameters coding these models and the degrees coding the natural numbers in these models such that the coding scheme defines a standard model $\mathcal{N}_0$ of arithmetic on the equivalence classes.*

We now have the definable copy $\mathcal{N}_0$ of $\mathcal{N}$ in $\mathcal{R}$ required by the biinterpretability conjecture. We next want to come as close as we can to associating each degree with an index for a set of that degree. The idea is to characterize, to the extent possible, a degree $\mathbf{a}$ by the isomorphism type of $\mathcal{R}(\leq \mathbf{a})$ (the ordering of r.e. degrees below $\mathbf{a}$) relative to certain other parameters.

The first ingredient is a coding scheme for a copy of $\mathcal{N}$ which is Σ_3 in the sense that the (codes for) the natural numbers can be enumerated recursively in $\mathbf{0}''$.

Theorem 3.15. (Nies, Shore and Slaman [1997]) *Given any $\mathbf{a} > \mathbf{0}$ and any promptly simple $\mathbf{u}$, there are degrees $\mathbf{b}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1, \mathbf{p}, \mathbf{q}, \mathbf{r}$ and u. r. e. $\mathbf{g}_i$ (for $i \in \omega$) with $\mathbf{p}, \mathbf{q} < \mathbf{u}$ and all the other degrees below both $\mathbf{a}$ and $\mathbf{u}$ satisfying the following conditions:*

- *the minimal degrees* $\mathbf{x}$, $\mathbf{b} < \mathbf{x} < \mathbf{r}$ *such that* $\mathbf{q} \leq \mathbf{x} \vee \mathbf{p}$ *together with the partial ordering on them defined by* $\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \mathbf{x} \oplus \mathbf{l} \leq \mathbf{y}$ *define a standard model of arithmetic as described above with the* $\mathbf{g}_i$ *as the elements* i*;*
- *for each* $i \in \omega$, $(\mathbf{g}_{2i} \vee \mathbf{e}_1) \wedge \mathbf{f}_1 = \mathbf{g}_{2i+1}$ *and* $(\mathbf{g}_{2i+1} \vee \mathbf{e}_0) \wedge \mathbf{f}_0 = \mathbf{g}_{2i+2}$.

Given the properties specified in this theorem, each $\mathbf{g}_i$ can be defined by an existential formula using the first eight degrees and $\mathbf{g}_0$ as parameters. For example, $\mathbf{g}_1 = (\mathbf{g}_0 \vee \mathbf{e}_1) \wedge \mathbf{f}_1$ and so $\mathbf{g}_1$ is the only degree $\mathbf{x}$ such that $\phi_1(\mathbf{x})$ holds where $\phi_1(\mathbf{x})$ says $\mathbf{x} \leq \mathbf{g}_0 \vee \mathbf{e}_1 \,\&\, \mathbf{x} \leq \mathbf{f}_1 \,\&\, \mathbf{q} \leq \mathbf{x} \vee \mathbf{p}$. Next, $\mathbf{g}_2$ is the only degree $\mathbf{y}$ such that $\exists \mathbf{x}(\phi_1(\mathbf{x}) \,\&\, \mathbf{y} \leq \mathbf{x} \vee \mathbf{e}_0 \,\&\, \mathbf{y} \leq \mathbf{f}_0 \,\&\, \mathbf{q} \leq \mathbf{y} \vee \mathbf{p})$. Similarly, we can define each $\mathbf{g}_i$ by such a formula. As the ordering of Turing reducibility relative to any set B is Σ_3^B and join is recursive on indices, we can make this generating procedure recursive in $\mathbf{0}''$ by choosing $\mathbf{u}$ to be low.

The next ingredient in the desired coding is a procedure that shows that every Σ_3^A set can be coded into such a set of degrees $\mathbf{g}_i$ in a positive way using $\leq$ and $\vee$ by degrees below $\mathbf{a}$. As the ordering on degrees below $\mathbf{a}$ is Σ_3^A (and join is recursive on indices) this would make the set coded Σ_3^A as well (and nothing better is possible).

Theorem 3.16. (Nies, Shore and Slaman [1997]) *If* $\langle \mathbf{g}_i \rangle$ *is a uniformly r.e. antichain in* $\mathcal{R}$, $\oplus \mathbf{g}_i$ *is low and* $\mathbf{a} = \deg(A) \not\leq_T \mathbf{g}_i$ *for each* i, *then, for each* Σ_3^A *set* S, *there are* $\mathbf{c}, \mathbf{d} \leq \mathbf{a}$ *such that* $S = \{i | \mathbf{c} \leq_T \mathbf{g}_i \vee \mathbf{d}\}$.

Together, these results show that precisely the Σ_3^A sets can be coded in this way. As this class of sets determines $\mathbf{a}''$, the isomorphism type of $\mathbf{a}$ in $\mathcal{R}$ determines $\mathbf{a}''$. As the coding scheme is amenable to the comparisons described above between the models of arithmetic, one can translate these codings into ones in the definable standard model and so convert this characterization of $\mathbf{a}''$ to a formula defining, from the degree $\mathbf{a}$, a (code for a) set of degree $\mathbf{a}''$ in the standard model of Theorem 3.14. The following results are then all among the corollaries of this work given in Nies, Shore and Slaman [1997].

Theorem 3.17. *There is a definable map* $f : \mathcal{R} \to \mathcal{N}_0$ *such that, for every* $\mathbf{a}$, $f(\mathbf{a})$ *is (the code for) an index of an r.e. set* W *for which* $W'' \in \mathbf{a}''$.

Corollary 3.18. *The double jump is invariant in* $\mathcal{R}$, *i. e. if* φ *is an automorphism of* $\mathcal{R}$ *then* $\varphi(\mathbf{x})'' = \mathbf{x}''$ *for every* $\mathbf{x} \in \mathcal{R}$.

Corollary 3.19. *Any relation on $\mathcal{R}$ which is definable in arithmetic and invariant under the double jump is definable in $\mathcal{R}$.*

Corollary 3.20. *For each* $\mathbf{c}$ *r.e. in and above* $\mathbf{0}''$ *the set of r.e. degrees* $\mathbf{a}$ *with double jump* $\mathbf{c}$ *is definable in* $\mathcal{R}$.

Corollary 3.21. *The jump classes* $L_{n+1} = \{\mathbf{a} |\, \mathbf{a}^{(n+1)} = \mathbf{0}^{(n+1)}\}$ *and* $H_n = \{\mathbf{a} |\, \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}$ *are definable in* $\mathcal{R}$ *for* $n \geq 1$.

Corollary 3.22. *The jump class* $H_1 = \{\mathbf{a} |\, \mathbf{a}' = \mathbf{0}''\}$ *is definable in* $\mathcal{R}$.

Proof. It follows from the Robinson jump interpolation theorem [1971] that $\mathbf{a}' = \mathbf{0}''$ if and only if for every $\mathbf{c}$ r.e. in and above $\mathbf{0}''$ there is a $\mathbf{b} < \mathbf{a}$ with $\mathbf{b}'' = \mathbf{c}$. Thus the definability of H_1 follows from Theorem 3.17. □

There are various *a priori* plausible schemes for defining even more in $\mathcal{R}$ and of proving results somewhat weaker than the full biinterpretability conjecture. Of particular interest is the question of biinterpretability with parameters. Here the definitions required in the conjecture are allowed to use finitely many individual r. e. degrees as parameters. Of course, this would give the definability results implied by the biinterpretability conjecture using parameters. It would also imply that there are at most countably many automorphisms of $\mathcal{R}$ as the finite set of parameters used in the definitions of the conjecture would constitute an *automorphism base* for $\mathcal{R}$, i. e. any automorphism would be completely determined by its action on this set. Of course, the discovery of any finite automorphism base for $\mathcal{R}$ would also imply that it has at most countably many automorphisms.

Now any set A which generates $\mathcal{R}$ under joins and meets is obviously an automorphism base for $\mathcal{R}$ and there are many known generating sets. Indeed, by the Sack Splitting theorem (Theorem 2.4) any dense set of degrees, such as the branching (i.e. the infima of pairs of incomparable degrees) (Slaman [1991a]) and nonbranching (Fejer [1983]) degrees, generates $\mathcal{R}$ and so is an automorphism base for it. More information on generating sets for $\mathcal{R}$ can be found in Ambos-Spies [1985]. There are however, many known automorphism bases for $\mathcal{R}$ that do not generate it. For example, each jump class H_n and L_n as defined above (in Corollary 3.21) is an automorphism base for $\mathcal{R}$ as is the class **PS** of promptly simple degrees (Ambos-Spies [1993]). Perhaps the most remarkable such result (Ambos-Spies [1993]) is that every downward cone in $\mathcal{R}$, i. e. the degrees below $\mathbf{a}$ for any $\mathbf{a} > \mathbf{0}$, is an automorphism base for $\mathcal{R}$. If this result could be extended to include any upper cone with a promptly simple base (the degrees above some promptly simple $\mathbf{a} < \mathbf{0}'$), then the following theorem would imply that there is a finite automorphism base for $\mathcal{R}$ and so there would be at most countably many automorphisms of $\mathcal{R}$.

Theorem 3.23. (Shore and Slaman) *Given any promptly simple degree* $\mathbf{d}$, *we can define (from parameters) a standard model of arithmetic and a one-one onto map from the integers of this model to the degrees above* $\mathbf{d}$.

However, even before all of the above results on definability were proven, Cooper announced that he could construct an automorphism of $\mathcal{R}$ and indeed one that moves a low degree to a nonlow one so that the class of low degrees is not definable in $\mathcal{R}$ (Cooper [1996]). Clearly the existence of such automorphisms implies that the above definability results for the jump classes are the best possible. Given the construction of such an automorphism, one might be tempted to prove that no individual degree is definable in $\mathcal{R}$ by constructing the appropriate automorphisms. In any case, before moving on to issue of "natural" definability (e.g. without codings of arithmetic), we close this section with three conjectures (listed in order of increasing strength) which are all still weaker than the full biinterpretability conjecture and do not contradict the existence of a nontrivial automorphism of $\mathcal{R}$.

Conjecture 3.24. *There are only countably many automorphisms of* $\mathcal{R}$.

Conjecture 3.25. *There is a finite automorphism base for* $\mathcal{R}$.

Conjecture 3.26. *With parameters,* $\mathcal{R}$ *is biinterpretable with the standard model of arithmetic.*

The last conjecture would also imply that $\mathcal{R}$ is the prime model of its theory and that every relation on $\mathcal{R}$ is definable if and only if it is definable in arithmetic and invariant under automorphisms.

4. Natural Definability

Whatever the ultimate achievements of such investigations in terms of definability in $\mathcal{R}$, the definitions produced will not be "natural" ones in terms of the structure of $\mathcal{R}$. Even if $\mathcal{R}$ were biinterpretable with arithmetic (perhaps with parameters) and every r. e. degree and arithmetic relation on the r. e. degrees is then definable (from finitely many parameters), the definitions produced in this way would all proceed by coding everything in models of arithmetic and then using arithmetic itself to produce the definitions. It would give no insight into the relations between the classes and their role in the structure of $\mathcal{R}$. This investigation is the provenance of another area of long term interest in the study of the r. e. degrees: the relationships between order theoretic properties of degrees in $\mathcal{R}$ and external properties of other sorts. Of interest have been set theoretic properties described in terms of the lattice of r. e. sets; dynamic properties of the enumerations of the r. e. sets; rates of growth of functions recursive in the various r. e. degrees; and relations with definability considerations in arithmetic as expressed by the jump operator. The last of these types of properties is often the one that connects all the others. For the r. e. degrees the relations with the jump operator are generally described in terms of the jump classes mentioned in the definability results of the last section:

Definition 4.1. *An r. e. degree* $\mathbf{a}$ *is* high$_n$ *iff* $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$ *(its* n^{th} *jump is as high as possible). The degree* $\mathbf{a}$ *is* low$_n$ *if* $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$ *(its* n^{th} *jump is as low as possible). If* $n = 1$*, we usually omit the subscript.*

There have been many important result connecting the jump classes with the lattice theoretic structure of the r. e. sets; with approximation procedures for functions recursive in different jumps of the given set; and with the growth rates of functions recursive in the sets themselves for several of these jump classes. For example, an r. e. degree $\mathbf{a}$ is high iff it contains a maximal set in $\mathcal{E}^*$ (the lattice of r. e. sets modulo the ideal of finite sets) iff there is a function f of degree $\mathbf{a}$ which dominates every recursive function iff every function h recursive in $\mathbf{0}''$ is approximable by one $g \leq \mathbf{a}$ in the sense that $h(x) = \lim_s g(x,s)$. Another class with a similar list of equivalent definitions consists of the low$_2$ r. e. degrees: $\mathbf{a}$ is low$_2$ iff every r. e. set of degree $\mathbf{a}$ has a maximal superset iff there is a function f recursive in $\mathbf{0}'$ which dominates every $g \leq \mathbf{a}$ iff every function $h \leq \mathbf{a}''$ is approximable by a recursive in g in the sense that $h(x) = \lim_s \lim_t g(x,s,t)$. (Proofs of these facts about high and low$_2$ r. e. sets and other similar ones can be found in Soare [1987, XI]). These interrelations have played an important role in the study of both the lattice of r. e. sets (Soare [1997]) and the degrees below $\mathbf{0}'$ (Cooper [1997]).

These classes and others have also played an important role in the study of $\mathcal{R}$. We want to mention some of the typical applications of a set being in each of several possible jump classes. The two earliest classes that were seen to have degree theoretic implications were the low degrees and the high ones. The intuition underlying most of these results is that the low degrees resemble the recursive one, $\mathbf{0}'$, and the high degrees resemble the complete one, $\mathbf{0}'$. Thus the degrees above a low degree or below a high one should resemble the r. e. degrees as a whole.

The archetypic applications of lowness are in Robinson [1971]. One example was motivated by the attempt to combine the two most prominent early indications that the r. e. degrees are homogeneous – the splitting and density theorems of Sacks (Theorems 2.3 and 2.4). Robinson proved that it is possible to combine these theorems if the bottom degree is low (and so like $\mathbf{0}$).

Theorem 4.2. (Robinson's Splitting Theorem, Robinson [1971]) *For every pair of r. e. degrees* $\mathbf{a} < \mathbf{b}$ *with* $\mathbf{a}$ *low, there are r. e. degrees* $\mathbf{b}_0$ *and* $\mathbf{b}_1$ *such that* $\mathbf{a} < \mathbf{b}_0, \mathbf{b}_1 < \mathbf{b}$ *and* $\mathbf{b}_0 \vee \mathbf{b}_1 = \mathbf{b}$.

Of course, we now know by Lachlan's nonsplitting theorem (Theorem 2.15) that it is not in general possible to combine these two results. Lowness is used to eliminate the potentially infinitary nature of the preservation requirements in the splitting theorem caused by changes in $\mathbf{a}$ destroying computations by recursively approximating answers to questions about $\mathbf{a}'$ ($\mathbf{a}$ is low iff every function h recursive in $\mathbf{a}'$ is approximable by a recursive g in the sense that $h(x) = \lim_s g(x, s)$ (see Soare [1987, III.3])). By similar arguments, Robinson [1971] also showed, for example, that every countable usl is embeddable in every interval $\mathbf{a} < \mathbf{b}$ of r. e. degrees where $\mathbf{a}$ is low. Since then, many other results have shown that the low r. e. sets are much like the recursive ones. Some of these results have also been extended to the low$_2$ r. e. degrees. Indeed, Shore and Slaman [1990] even prove an extension of embedding theorem for the low$_2$ r. e. degrees that essentially says that any extension not ruled out by embeddings of finite distributive lattices can be carried out. Even the density and splitting theorems can be combined when the top degree is low$_2$ (Harrington and also Bickford and Mills; see Shore and Slaman [1990]) but, in contrast to the situation for low degrees, they cannot be combined just under the assumption that the bottom degree is low$_2$.

At the other end of the scale, there are results showing that various phenomena in the r. e. degrees occur below every high degree. Here the archetypic example is Cooper's theorem:

Theorem 4.3. (Cooper [1974]) *If* $\mathbf{h}$ *is a high r. e. degree then there is a minimal pair* $\mathbf{a}, \mathbf{b}$ *below* $\mathbf{h}$.

A couple of other such results follow.

Theorem 4.4. (Harrington; see Miller [1981]) *Every high degree* $\mathbf{h}$ *has the anticupping property, i. e. there is an r. e.* $\mathbf{b} < \mathbf{h}$ *such that for no r. e.* $\mathbf{c} < \mathbf{h}$ *does* $\mathbf{h} = \mathbf{b} \vee \mathbf{c}$.

Theorem 4.5. (Shore and Slaman [1993]) *For every high r. e. degree* $\mathbf{h}$, *there are r. e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c} < \mathbf{h}$ *such that for every r. e.* $\mathbf{w}$, $\mathbf{0} < \mathbf{w} < \mathbf{a}$, $\mathbf{b} \vee \mathbf{w} \geq \mathbf{c}$.

The last result (together with the extension of embedding results below a low_2 mentioned above) definably separated the high r. e. degrees from the low_2 ones as the above results say that no such triple can exits if $\mathbf{h}$ is low_2. The early approach to exploiting highness as in Cooper [1974], was the characterization by the existence of a function f recursive in the high degree $\mathbf{h}$ which dominates every recursive function. This function was then used to control a permitting procedure in which numbers are allowed into the set A being constructed only when the value of our approximation to f determined by its computation from an r. e. set $H \in \mathbf{h}$ changes. This guarantees that A is recursive in $\mathbf{h}$. The domination properties of f were then used to show that the permitting was compatible with satisfying a positive requirement that wants to put all but finitely many of an infinite recursive set of numbers into A. Another view is presented in Shore and Slaman [1993]. There the idea is that $\mathbf{h}$ is high if and only if every function recursive in $\mathbf{0}''$ can be approximated recursively in $\mathbf{h}$. This characterization is used to, recursively in $\mathbf{h}$, approximate the outcome of infinitary (Π_2) requirements in a priority tree construction in such a way that $\mathbf{h}$ can recover enough of the construction (by enumerating when nodes will never be accessible again) to calculate the set being constructed.

Since this work separating high and low_2 (and before the definition of these classes given in Corollaries 3.22 and 3.21 by coding techniques) there were further developments of the techniques for working with low_2 r. e. degrees. In addition, new techniques have been introduced to exploit the hypothesis that an r. e. degree is nonlow_2. The combination of these procedures has produced a natural order theoretic definition of the low_2 degrees in $\mathcal{R}_{tt}$, the structure of the r. e. truth table degrees (Downey and Shore [1995]): In $\mathcal{R}_{tt}$, the low_2 degrees are precisely those with minimal covers, i.e. the degrees $\mathbf{a}$ such that there is a $\mathbf{b} > \mathbf{a}$ with no degree $\mathbf{c}$ between $\mathbf{a}$ and $\mathbf{b}$. Another result whose proof introduces extensions of these techniques that we hope will eventually lead to a natural definition of this jump class in $\mathcal{R}$ is the following:

Theorem 4.6. (Downey and Shore [1996]) *If* $\mathbf{a}$ *is r. e. and nonlow*$_2$*, then the lattice 1-3-1 can be embedded in the r. e. degrees below* $\mathbf{a}$.

This should be contrasted with the earlier result of Downey [1990] and Weinstein [1988] that there are r. e. degrees $\mathbf{a}$ below which 1-3-1 cannot be embedded. Indeed Cholak and Downey [1993] prove that for any r. e. $\mathbf{a} < \mathbf{b}$ there are r. e. $\mathbf{c}, \mathbf{d}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{d} < \mathbf{b}$ such that 1-3-1 cannot be embedded in $[\mathbf{c}, \mathbf{d}]$. Although a more recent result of Cholak, Downey and Shore [1997] shows that these ideas do not seem to lead to a definition of the low$_2$ degrees in $\mathcal{R}$ in terms of embeddings of 1-3-1, we hope that continued development of these techniques will lead to a positive solution of the following conjecture.

Conjecture 4.7. *The class of low*$_2$ *degrees is naturally definable in* $\mathcal{R}$.

The only externally defined class of r. e. degrees that has actually been defined without the use of codings of arithmetic in $\mathcal{R}$ is that of the promptly simple r. e. degrees which coincide with the noncappable (and low cuppable) ones (Theorem 2.18). Ambos-Spies et al. [1984] also showed that this class coincides with **ENC**, the effectively noncappable r. e. degrees (there is a recursive function on indices witnessing the noncappability) and two classes connected to the lattice of r. e. sets one of which, $\mathbf{SP\bar{H}}$, is definable in $\mathcal{E}^*$. Thus the class is definable in both $\mathcal{R}$ and $\mathcal{E}^*$. Not only is this collection of equivalences a remarkable instance of both uniformity and definability in the r. e. degrees, but it has also played a key role in other results including ones phrased solely in terms of order theoretical properties of $\mathcal{R}$. The first, of course, were the results mentioned in Theorem 2.18 that **NC** and its complement **M** (the class of r. e. degrees which are *halves of minimal pairs*) partition $\mathcal{R}$ into a pair of sets consisting of a complementary ideal and filter. As another example, we point out that it also played a crucial role in Slaman and Woodin's proof that the theory of $\mathcal{R}$ is recursively isomorphic to that of true arithmetic. The idea was that prompt simplicity allowed them to construct codes of models of arithmetic with particular properties (the codes for the natural numbers were taken for a set of pairwise minimal degrees) that enabled them to guarantee the standardness of one of the models so constructed. The equivalence of **PS** and **NC** then made this a definable procedure in $\mathcal{R}$.

We now conclude where we began with Post's problem of an intermediate r.e. degree. The motivation for the question was that all naturally occurring r.e. sets were either recursive or complete. By now the problem has been solved over and over in the sense that we now are well aware of the richness of the structure of the r.e. degrees. Nonetheless, the question of whether there is a natural or definable (in some nice way) intermediate degree remains open. Of course, a negative solution requires a precise definition of "natural" or "nice". One suggestion (Steel [1982]) is that a natural degree should be definable and its definition should relativize to an arbitrary degree (and so, in particular, be defined on degrees independently of the choice of representative). Along these lines an old question of Sacks' [1963] asks whether there is a degree invariant solution to Post's problem, i. e. a computably enumerable operator W such that $A <_T W(A) <_T A'$. (A function $W : 2^\omega \to 2^\omega$ is a *computably enumerable operator* if there is an $e \in \omega$ such that, for each set A, $W(A) = W_e^A$, the e^{th} set computably enumerable in A. Any function $f : 2^\omega \to 2^\omega$ is *degree invariant* if for every A and B, $A \equiv_T B$ implies that $f(A) \equiv_T f(B)$.) Such an operator would clearly be a candidate for a natural solution to Post's problem in this sense.

Lachlan [1975a] proved that if we require the degree invariance to be *uniform* in the sense that there is a function h that takes the (pairs of) indices of reductions between A and B to (pairs of) indices of reductions between $W(A)$ and $W(B)$ then there is no such operator. In fact, Lachlan proved that if $\mathbf{d}$ is the degree of the function h providing the uniformity then either $W(\mathbf{d}) = \mathbf{d}'$ or $W(W(\mathbf{d})) \leq W(\mathbf{d})$. In this way, Lachlan also characterized the jump operator as the only uniformly degree invariant computably enumerable operator which is always strictly greater than the identity *on a cone* (of degrees), i. e. on all $\mathbf{a} \geq_T \mathbf{c}$ for some degree $\mathbf{c}$. Sacks's Problem was greatly generalized by Martin (see Kechris and Moschovakis [1978], p.279) in the setting of the axiom of determinacy and Lachlan's results were generalized and extended to this context by Steel [1982], Slaman and Steel [1988] and Becker [1988]. A recent result by Downey and Shore [1977] refines this work in the context of the r.e. degrees in a way that puts severe restrictions on the types of degrees that could be defined in this way or, indeed, by any definition in the structure $\mathcal{R}$.

Theorem 4.8. (Downey and Shore [1997]) *If W is a degree invariant computably enumerable operator such that $A \leq_T W(A)$ for all A Turing above some fixed* *degree (i.e.* in a cone*), then either* $W(A)'' \equiv_T A''$ *for all* A *in some cone or* $W(A)'' \equiv_T A'''$ *in some cone.*

Theorem 4.9. (Downey and Shore [1997]) *For any formula* $\psi(x)$ *in the language of partial orderings which, for every* $\mathbf{z}$, *defines a degree* $\mathbf{x}$ *(other than* $\mathbf{z}$ *or* $\mathbf{z}'$*) in* $\mathcal{R}^{\mathbf{z}}$ *(the degrees r.e. in and above* $\mathbf{z}$*) must define a low*$_2$ *or high*$_2$ *degree (relative to* $\mathbf{z}$*, i.e.* $\mathbf{x}'' = \mathbf{z}''$ *or* $\mathbf{x}'' = \mathbf{z}'''$*) for all* $\mathbf{z}$ *above some fixed degree* $\mathbf{c}$.

Now this last result does not rule out definitions in $\mathcal{R}$ that make direct reference to specific arithmetic sets in some coded model of arithmetic and so would apply only to $\mathcal{R}$ or at best to $\mathcal{R}^{\mathbf{a}}$ for arithmetic $\mathbf{a}$. For example, the results in Corollary 3.20 on the definability of each class of r.e. degrees with double jump a particular $\mathbf{c}$ r.e. in and above $\mathbf{0}''$ are of this sort. On the other hand, the definitions for the jump classes L_{n+1} and H_n for $n \geq 1$ of Corollaries 3.21 and 3.22 can be stated so as to relativize properly to $\mathcal{R}^{\mathbf{a}}$ for any degree $\mathbf{a}$. However, the theorem perhaps does suggest that, whatever happens with biinterpretability and automorphisms, there may be no uniform degree invariant solution to Post's problem and no "naturally" definable intermediate degree in $\mathcal{R}$.

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