Undecidability and 1-types in Intervals of the Computably Enumerable Degrees*

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Abstract

We show that the theory of the partial ordering of the computably enumerable degrees in any given nontrivial interval is undecidable and has uncountably many 1-types.

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0 Introduction

Early results in the study of the structure of the computably enumerable (c.e.) Turing degrees led to Sacks's conjecture [11] that the theory of this structure is decidable. This conjecture turned out to be false. The undecidability of the theory of the partial ordering $\mathcal{R} = \langle \mathbf{R}, \leqslant \rangle$ of the c.e. degrees was first proved by Harrington and Shelah [5], using a very complicated $0'''$ argument.

This proof underwent various changes and attempted simplifications, particularly by Harrington and Slaman, and a considerably simpler approach was later developed by Slaman and Woodin (see [9]). However, none of these proofs could be extended to establish a similar result for the theory of a given interval of $\mathcal{R}$, or even a given ideal.

Ambos-Spies and Shore [2] gave a simpler infinite injury proof of the undecidability of the theory of $\mathcal{R}$, as well as a proof that this theory has uncountably many 1-types (extending a result of

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Ambos-Spies and Soare [1]), and showed that both proofs combine with the permitting technique to yield similar results for any nontrivial ideal of the c.e. degrees. (The fact that the theory of $\mathcal{R}$ has infinitely many 3-types, and so is not $\aleph_0$-categorical, had been established in Lerman, Shore, and Soare [8], by proving the embeddability into the c.e. degrees of all finite lattices with a certain property. In 1989, Ambos-Spies, Lempp and Soare (unpublished) obtained embeddings of the same class of lattices into arbitrary intervals of the c.e. degrees, thereby showing that the theory of any such interval also has infinitely many 3-types and so is not $\aleph_0$-categorical.)

The proofs in [2] required no more in the way of technique than the construction of branching and nonbranching degrees. Thus, in light of Slaman's proof [13] of the density of branching degrees and Fejer's proof [4] of the density of nonbranching degrees, it was hoped that the results could be extended to any nontrivial interval of the c.e. degrees. This is what we do in the present paper.

Our notation is for the most part standard (as in [14]). If W is a c.e. set then we assume we have fixed some enumeration of W and let $W[s]$ denote the part of W enumerated after $s+1$ many steps. However, for any c.e. set X we construct, $X[s]$ will denote the part of X enumerated by some point during stage s of the construction, whose exact location will have to be inferred from the context. Instead of $X[s](x)$ we write $X(x)[s]$. We let $X \upharpoonright m = X \cap \{0, \dots, m-1\}$.

The e^{th} Turing functional with oracle X is denoted by $\Phi_e(X)$, and its value at x by $\Phi_e(X; x)$. We let $\Phi_e(X)[s]$ be the evaluation of $\Phi_e(X[s])$ at some point during stage s, and $\Phi_e(X, x)[s]$ be the value of this evaluation at x. Again, the exact point during stage s to which these notations refer should always be clear from context; when there might be some doubt, we have pointed it out explicitly. The use functions of $\Phi_e(X; x)$ and $\Phi_e(X; x)[s]$ are denoted by $\varphi_e(X; x)$ and $\varphi_e(X; x)[s]$, respectively.

When we write $\Phi_e(X; x)[s] \neq \Phi_e(X; x)[t]$ this is understood to include the possibility that one side of the inequality converges while the other diverges.

When we mention a "fresh large number" in our construction, we mean a number larger than any appearing in the construction up to that point.

We adopt the following conventions, where we have in mind some fixed computable enumeration of the c.e. set X.

1. $s < x \Rightarrow \Phi_e(X; x)[s]\uparrow$.

2. $x < y \wedge \Phi_e(X; x)[s]\downarrow \wedge \Phi_e(X; y)[s]\downarrow \Rightarrow \varphi_e(X; x)[s] < \varphi_e(X; y)[s]$.

3. $s < t \wedge \Phi_e(X; x)[s]\downarrow \wedge \Phi_e(X; x)[t]\downarrow \Rightarrow \varphi_e(X; x)[s] \leqslant \varphi_e(X; x)[t]$.

4. $x < y \wedge \Phi_e(X; x)[s]\uparrow \Rightarrow \Phi_e(X; y)[s]\uparrow$.

5. $X_0 \oplus \cdots \oplus X_{n-1} = \{x \mid x = nk + m,\ k \in X_m,\ m < n\}$.

6. We treat the use of a functional as if it were the largest number actually used in the computation, so that a change in X at or below $\varphi_e(X; x)$ will be taken to destroy the current computation $\Phi_e(X; x)$, and hence will cause us to say that the computation has changed. However, if the oracle of a functional is given as the join of two or more sets, we redefine the use as follows: $\varphi_e(X_0 \oplus \cdots \oplus X_{n-1}; x) = \max\{k \mid nk + m$ is used in the computation $\Phi_e(X_0 \oplus \cdots \oplus X_{n-1}; x)$ for some $m < n\}$, and similarly for $\varphi_e(X_0 \oplus \cdots \oplus X_{n-1}; x)[s]$. (This means that we will act as if a change in any X_i, $i < n$, at or below $\varphi_e(X_0 \oplus \cdots \oplus X_{n-1}; x)$ destroys a

current computation $\Phi_e(X_0 \oplus \cdots \oplus X_{n-1}; x)$. It also means that when we make reference to "imposing a restraint r on $X_0 \oplus \cdots \oplus X_{n-1}$", we mean that we impose a restraint r on each X_i, $i < n$.)

In addition, we make the following a rule of our construction. If any of the strategies described below acts at stages s and t of the construction and not at any stage strictly between s and t, $\Phi_e(X; x)[s]$ converges at the end of the strategy's stage s action, and the computation $\Phi_e(X; x)$ changes between the end of the strategy's stage s action and the beginning of its stage t action, then the strategy treats $\Phi_e(X; x)[t]$ as if it were divergent.

1 The main results

A set of sentences Σ is said to be *strongly undecidable* if there is no computable set R such that $V \cap \Sigma \subseteq R \subseteq \Sigma$, where V is the set of logically valid sentences. Ershov and Taitslin [3] have shown that the set of all sentences in the language $L(\leqslant)$ that are true in all finite partial orderings is strongly undecidable. This implies that, for any structure $\mathcal{S}$ in which all finite partial orderings are elementarily definable with parameters, the first-order theory $\mathrm{Th}(\mathcal{S})$ of $\mathcal{S}$ is undecidable. In particular, the following holds for upper semilattices.

1.1. Proposition. *Let $\mathcal{U} = \langle U, \leqslant_U, \cup \rangle$ be an upper semilattice and let θ be a formula in the language of partial orderings with free variables $x_0, \ldots, x_{k-1}$ and y. Suppose that for any $n \geqslant 1$ and any partial ordering $\leqslant_0$ on $\{0, \ldots, n-1\}$ there are elements $a_0, \ldots, a_{k-1}, b_0, \ldots, b_{n-1}$ and c of U such that for any $i, j < n$,*

$$i \neq j \Rightarrow b_i \neq b_j,$$

$$\{b_0, \ldots, b_{n-1}\} = \big\{ b \in U \mid \mathcal{U} \vDash \theta_{x_0, \ldots, x_{k-1}, y}[a_0, \ldots, a_{k-1}, b] \big\}, \text{ and}$$

$$i \leqslant_0 j \Leftrightarrow b_i \leqslant_U b_j \cup c.$$

Then the first-order theory of $\langle U, \leqslant_U \rangle$ is undecidable.

Let $\mathbf{e} < \mathbf{f}$ be c.e. degrees. In order to apply Proposition 1.1 to the u.s.l. $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant, \cup \big\rangle$, we need an elementary property of this u.s.l. that, for varying parameters $\mathbf{a_0}, \ldots, \mathbf{a_{k-1}} \in [\mathbf{e}, \mathbf{f}]$, defines finite sets of arbitrary size. Following [2], we will get such a property in one parameter $\mathbf{a}$ by considering the branches of branching degrees in $[\mathbf{e}, \mathbf{f}]$.

1.2. Definition. (a) A c.e. degree $\mathbf{a}$ is *branching* if there are c.e. degrees $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a} < \mathbf{b}$, $\mathbf{a} < \mathbf{c}$, and $\mathbf{a} = \mathbf{b} \cap \mathbf{c}$. Otherwise, $\mathbf{a}$ is *nonbranching*.

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be c.e. degrees in $[\mathbf{e}, \mathbf{f}]$ such that $\mathbf{a} < \mathbf{b}$. Then $\mathbf{b}$ is $\mathbf{a}$-*cappable in* $[\mathbf{e}, \mathbf{f}]$ if there is a c.e. degree $\mathbf{c}$ in $[\mathbf{e}, \mathbf{f}]$ such that $\mathbf{a} < \mathbf{c}$ and $\mathbf{a} = \mathbf{b} \cap \mathbf{c}$. The degree $\mathbf{b}$ is *maximal-*$\mathbf{a}$*-cappable in* $[\mathbf{e}, \mathbf{f}]$ if $\mathbf{b}$ is $\mathbf{a}$-cappable in $[\mathbf{e}, \mathbf{f}]$ and no degree $\mathbf{d} > \mathbf{b}$ is $\mathbf{a}$-cappable in $[\mathbf{e}, \mathbf{f}]$.

Note that maximal-$\mathbf{a}$-cappability is definable in the first-order language of partial orderings. Thus the following fact holds.

1.3. Proposition. *Let* $\mathbf{e} < \mathbf{f}$ *be c.e. degrees. There is a first-order formula* θ *in the language of partial orderings with two free variables* x *and* y *such that for any two degrees* $\mathbf{a}, \mathbf{b} \in [\mathbf{e}, \mathbf{f}]$,

$$\big\langle [\mathbf{e}, \mathbf{f}], \leqslant, \cup \big\rangle \vDash \theta_{x,y}[\mathbf{a}, \mathbf{b}] \Leftrightarrow \mathbf{b} \textit{ is maximal-}\mathbf{a}\textit{-cappable in } [\mathbf{e}, \mathbf{f}].$$

We now come to our main technical theorem.

1.4. Theorem. *Let* $\mathbf{e} < \mathbf{f}$ *be c.e. degrees and let* $\leqslant_0$ *be a partial ordering on* $\{0, \ldots, N-1\}$*. There are c.e. degrees* $\mathbf{a}, \mathbf{b_0}, \ldots, \mathbf{b_{N-1}}, \mathbf{c}$ *such that*

$$\mathbf{a}, \mathbf{b_0}, \ldots, \mathbf{b_{N-1}}, \mathbf{c} \in [\mathbf{e}, \mathbf{f}], \tag{1.0}$$

$$\mathbf{b_i} \mid \mathbf{b_j},\ i \neq j, \tag{1.1}$$

$$\mathbf{b_i} \textit{ is } \mathbf{a}\textit{-cappable in } [\mathbf{e}, \mathbf{f}], \tag{1.2}$$

$$\textit{if } \mathbf{d} \textit{ is } \mathbf{a}\textit{-cappable then } \mathbf{d} \leqslant \mathbf{b_i} \textit{ for some } i < N, \textit{ and} \tag{1.3}$$

$$i \leqslant_0 j \textit{ iff } \mathbf{b_i} \leqslant \mathbf{b_j} \cup \mathbf{c}. \tag{1.4}$$

We will prove Theorem 1.4 in Section 2.

Combining Propositions 1.1 and 1.3 with Theorem 1.4 we get the following result.

1.5. Corollary. *For any c.e. degrees* $\mathbf{e} < \mathbf{f}$*, the first-order theory of the partial ordering* $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle$ *of the c.e. degrees between* $\mathbf{e}$ *and* $\mathbf{f}$ *is undecidable.*

By Theorem 1.4, for any two c.e. degrees $\mathbf{e} < \mathbf{f}$, any finite partial ordering can be elementarily defined in $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle$ with two parameters $\mathbf{a}$ and $\mathbf{c}$. This implies that there are continuum many 2-types consistent with $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle$. As the next theorem shows, in certain cases the second parameter $\mathbf{c}$ can be defined from $\mathbf{a}$. We will use this to show that there are in fact continuum many 1-types consistent with $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle$.

1.6. Theorem. *Let* $\leqslant_0$ *be a partial ordering on* $\{0, \ldots, N-1\}$ *with at least three minimal elements. There are c.e. degrees* $\mathbf{a}, \mathbf{b_0}, \ldots, \mathbf{b_{N-1}}, \mathbf{c}$ *satisfying (1.0)–(1.4) and*

$$\mathbf{c} = \bigcup_{i<N} \left(\bigcap_{j \neq i} \mathbf{b_j} \right). \tag{1.5}$$

The proof of Theorem 1.6 will be given in Section 3.

1.7. Corollary. *For any c.e. degrees* $\mathbf{e} < \mathbf{f}$*, the first-order theory of the partial ordering* $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle$ *of the c.e. degrees between* $\mathbf{e}$ *and* $\mathbf{f}$ *has* 2^ω *many 1-types.*

Proof. It suffices to give a sequence $\langle \eta^k \mid k > 0\rangle$ of formulas with one free variable x such that for any nonempty finite set F of natural numbers there is a c.e. degree $\mathbf{a_F} \in [\mathbf{e}, \mathbf{f}]$ such that

$$\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle \vDash (\eta^k)_x[\mathbf{a_F}] \Leftrightarrow k \in F.$$

Fix θ as in Proposition 1.3 and define the formulas $\gamma = \gamma_{x,u}$, $\delta = \delta_{x,v}$, and $\eta^k = (\eta^k)_x$, whose intended meanings are described below, as follows:

$$\gamma \equiv \exists y \Big(\theta \wedge \forall z \big(y \neq z \wedge \theta_{x,y}[x, z] \to u \leqslant z \big) \wedge \forall w \Big[\forall z \big(y \neq z \wedge \theta_{x,y}[x, z] \to w \leqslant z \big) \to w \leqslant u \Big] \Big).$$

$$\delta \equiv \forall u(\gamma \to u \leqslant v) \wedge \forall t \big[\forall u(\gamma \to u \leqslant t) \to v \leqslant t \big].$$

$$\begin{aligned}
\eta^k \quad \equiv \quad & \exists s_0, \ldots, s_k \Big(\theta_{x,y}[x, s_0] \wedge \cdots \wedge \theta_{x,y}[x, s_k] \wedge \\
& \wedge \forall v \big(\delta_{x,v} \to [s_0 \leqslant s_1 \cup v \wedge \cdots \wedge s_{k-1} \leqslant s_k \cup v \wedge s_1 \nleqslant s_0 \cup v \wedge \cdots \wedge s_k \nleqslant s_{k-1} \cup v] \big) \wedge \\
& \wedge \forall s, v \big(\theta_{x,y}[x, s] \wedge s \neq s_0 \wedge \cdots \wedge s \neq s_k \wedge \delta_{x,v} \to \\
& \to [s \nleqslant s_0 \cup v \wedge \cdots \wedge s \nleqslant s_k \cup v \wedge s_0 \nleqslant s \cup v \wedge \cdots \wedge s_k \nleqslant s \cup v] \big) \Big).
\end{aligned}$$

(Where the symbols $\neq$, $\cup$, and $\nleqslant$ should be expressed in terms of $\leqslant$.)

For any partial ordering $\leqslant_0$ on $\{0, \ldots, N-1\}$ with at least three minimal elements and c.e. degrees $\mathbf{a}, \mathbf{b_0}, \ldots, \mathbf{b_{N-1}}, \mathbf{c}$ as in Theorem 1.6,

$$\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle \vDash \theta[\mathbf{a}, \mathbf{g}] \text{ if and only if } \mathbf{g} \in \{\mathbf{b_0}, \ldots, \mathbf{b_{N-1}}\} \tag{1.6}$$

(by (1.2) and (1.3)),

$$\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle \vDash \gamma[\mathbf{a}, \mathbf{h}] \text{ if and only if } \mathbf{h} \in \Big\{ \bigcap_{j \neq i} \mathbf{b_j} \mid i < N \Big\} \tag{1.7}$$

(by (1.6)), and

$$\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle \vDash \delta[\mathbf{a}, \mathbf{i}] \text{ if and only if } \mathbf{i} = \mathbf{c} \tag{1.8}$$

(by (1.5) and (1.7)).

Thus, by (1.4), (1.6), and (1.8), $\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle \vDash \eta^k[\mathbf{a}]$ if and only if $\leqslant_0$ contains a maximal chain of length $k+1$ such that each number not contained in the chain is $\leqslant_0$-incomparable with each member of the chain.

Now let $F = \{m_0, \ldots, m_p\}$ be a nonempty finite set of natural numbers. Say that a partial ordering $\leqslant_0$ on $\{0, \ldots, N-1\}$ is *of chain type* F if it is the disjoint union of maximal $\leqslant_0$-chains such that each maximal chain has length $m_l + 1$ for some $l \leqslant p$, for each $l \leqslant p$ there is a maximal chain of length $m_l + 1$, and members of different maximal chains are $\leqslant_0$-incomparable. Let $\leqslant_0$ be a partial ordering on $\{0, \ldots, N-1\}$ of chain type F with at least three minimal elements, and let $\mathbf{a}$ be as in Theorem 1.6. Then

$$\big\langle [\mathbf{e}, \mathbf{f}], \leqslant \big\rangle \vDash (\eta^k)_x[\mathbf{a}] \Leftrightarrow k \in F.$$

□

Another interesting question is which fragments of the theory of a given interval of the c.e. degrees are undecidable. We will briefly address this question in Section 4.

2 Proof of Theorem 1.4

The case $\mathbf{e} = \mathbf{0}$ is covered by Theorem 1.9 in [2]. Thus we can assume that $\mathbf{e} > \mathbf{0}$. We can further assume that $N > 1$, since the case $N = 1$ follows by letting $\mathbf{a} = \mathbf{b_0} = \mathbf{c}$ be a nonbranching c.e. degree in $[\mathbf{e}, \mathbf{f})$, as constructed in [4].

Let E and F be c.e. sets in $\mathbf{e}$ and $\mathbf{f}$, respectively. We construct sets A, B_i, C_i, and $D_{i,k,l}$ ($i < N; k, l \in \omega$), of which A, B_i, and C_i will be c.e.. These sets will satisfy the following conditions, the first seven of which are the same as conditions (2.0)–(2.6) in [2].

$$i \neq j \Rightarrow C_i \leqslant_{\mathrm{T}} B_j. \tag{2.0}$$

$$C_i \nleqslant_{\mathrm{T}} A. \tag{2.1}$$

$$\text{If } g \text{ is total and } g \leqslant_{\mathrm{T}} A \oplus B_i \text{ for all } i < N \text{ then } g \leqslant_{\mathrm{T}} A. \tag{2.2}$$

$$D_{i,k,l} \leqslant_{\mathrm{T}} W_k \oplus A \text{ and } D_{i,k,l} \leqslant_{\mathrm{T}} W_l \oplus A. \tag{2.3}$$

$$\text{If } W_k \nleqslant_{\mathrm{T}} A \oplus B_i \text{ and } W_l \nleqslant_{\mathrm{T}} A \oplus B_i \text{ then } D_{i,k,l} \nleqslant_{\mathrm{T}} A. \tag{2.4}$$

$$i \leqslant_0 j \Rightarrow B_i \leqslant_{\mathrm{T}} A \oplus B_j \oplus C, \text{ where } C = \bigoplus_{n=0}^{N-1} C_n. \tag{2.5}$$

$$i \nleqslant_0 j \Rightarrow B_i \nleqslant_{\mathrm{T}} A \oplus B_j \oplus C. \tag{2.6}$$

$$E \leqslant_{\mathrm{T}} A. \tag{2.7}$$

$$A \oplus B_i \leqslant_{\mathrm{T}} F. \tag{2.8}$$

Let A, B_i, C_i, and $D_{i,k,l}$ be as above and let $\mathbf{a} = \deg(A)$, $\mathbf{b_i} = \deg(A \oplus B_i)$, $\mathbf{c_i} = \deg(A \oplus C_i)$, and $\mathbf{c} = \deg(A \oplus C)$. We show that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ have the required properties.

We remark that Lachlan [6] has shown that for c.e. degrees $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, $\mathbf{x}$ is the infimum of $\mathbf{y}$ and $\mathbf{z}$ among the c.e. degrees if and only if $\mathbf{x}$ is the infimum of $\mathbf{y}$ and $\mathbf{z}$ among all degrees. This result is necessary for the ensuing because the sets $D_{i,k,l}$ will not necessarily be c.e..

By (2.0), (2.7), and (2.8), $\mathbf{a}, \mathbf{b_0}, \ldots, \mathbf{b_{N-1}}, \mathbf{c}$ lie in $[\mathbf{e}, \mathbf{f}]$. By (2.0) and (2.1), $\mathbf{a} < \mathbf{c_i} \leqslant \mathbf{b_j}$ for $i \neq j < N$, while, by (2.0) and (2.2), $\mathbf{a} = \mathbf{b_i} \cap \mathbf{c_i}$. So $\mathbf{b_i}$ and $\mathbf{b_j}$ are incomparable for $i \neq j$ and $\mathbf{b_i}$ is $\mathbf{a}$-cappable in $[\mathbf{e}, \mathbf{f}]$. To show that (1.3) holds, assume for a contradiction that $\mathbf{g}$ is $\mathbf{a}$-cappable but $\mathbf{g} \nleqslant \mathbf{b_i}$ for all $i < N$. Fix $\mathbf{h} > \mathbf{a}$ such that $\mathbf{a} = \mathbf{g} \cap \mathbf{h}$. Since, by (2.2), $\mathbf{a} = \mathbf{b_0} \cap \cdots \cap \mathbf{b_{N-1}}$ and since $\mathbf{h} > \mathbf{a}$, $\mathbf{h} \nleqslant \mathbf{b_i}$ for at least one $i < N$. So we may fix $i < N$ such that $\mathbf{g} \nleqslant \mathbf{b_i}$ and $\mathbf{h} \nleqslant \mathbf{b_i}$. Then, for any two c.e. sets W_k and W_l in $\mathbf{g}$ and $\mathbf{h}$, respectively, $W_k \nleqslant_{\mathrm{T}} A \oplus B_i$ and $W_l \nleqslant_{\mathrm{T}} A \oplus B_i$. Hence, for $\mathbf{d} = \deg(D_{i,k,l})$, $\mathbf{d} \leqslant \mathbf{g}$ and $\mathbf{d} \leqslant \mathbf{h}$ by (2.3), while by (2.4), $\mathbf{d} \nleqslant \mathbf{a}$. So $\mathbf{a} \neq \mathbf{g} \cap \mathbf{h}$, contrary to our assumption. Thus (1.3) holds. Finally, (1.4) follows immediately from (2.5) and (2.6).

In the following sections we will describe various kinds of strategies. In the eventual tree construction, there will be multiple copies of each strategy. A copy of a strategy X is designated by X^σ, where σ is a sequence coding the strategies of stronger priority than X^σ and their outcomes. A stage of the construction during which X^σ is active will be known as a *σ-stage*. Our construction will be such that each strategy in σ acts during any σ-stage, with Y^α acting before Z^τ for $\alpha \subset \tau \subseteq \sigma$.

For the purpose of keeping the actions of the various strategies from interfering with each other, we will assign infinite disjoint uniformly computable sets P_σ to each possible finite sequence σ of strategies and corresponding outcomes. Each strategy X^σ will work exclusively with numbers in P_σ.

The sections dealing with the satisfaction of (2.0)–(2.2) and (2.8) are a modification of the construction in [13], while that dealing with the satisfaction of (2.3) and (2.4) is based on the construction in [4]. The reader is referred to these papers for more detailed explanations of the ideas behind the strategies described in these sections.

Making the $\mathbf{b}_\mathbf{i}$ a-cappable. We satisfy (2.0) and (2.7) by direct coding. Whenever one of the strategies described below enumerates x into C_i, it will also enumerate $\langle x, i\rangle$ into each B_j, $j \neq i$, and we make sure no other numbers ever enter $B_j^{[i]}$. Similarly, we have a strategy K which will have the strongest priority of all and will act at every stage $s > 0$ by enumerating $2x$ into A for each $x \in E[s] - E[s-1]$, and we make sure this is the only way an even number can enter A.

Since elements of the sets P_σ mentioned above will be enumerated into A, B_i, or C_i by the various strategies described below, the above (together with another direct coding action described later) leads us to require that each P_σ be a subset of the intersection of the odd numbers with $\omega^{[\geqslant 2N]}$.

We break (2.2) into requirements

$$\mathcal{R}_e:\ \Phi_e(A \oplus B_0) = \Phi_e(A \oplus B_1) = \cdots = \Phi_e(A \oplus B_{N-1}) \text{ total } \Rightarrow \Phi_e(A \oplus B_0) \leqslant_{\mathrm{T}} A.$$

That these requirements suffice to satisfy (2.2) follows by an observation of Posner (see IX.1.4 in [14]): If $g \leqslant_{\mathrm{T}} A \oplus B_i$ for all $i < N$ then there are indices e_i such that

$$\Phi_{e_0}(A \oplus B_0) = \Phi_{e_1}(A \oplus B_1) = \cdots = \Phi_{e_{N-1}}(A \oplus B_{N-1}) = g.$$

On the other hand, each C_i will be non-empty, so we can pick $c_0, \ldots, c_{N-1}$, $c_i \in C_i$, and by the coding described above we will have that $\langle c_i, i\rangle \notin B_j$ if and only if $i = j$. Let e be such that $\Phi_e(X; x) = \Phi_{e_i}(X; x)$ for the least $i < N$ such that $2\langle c_i, i\rangle + 1 \notin X$, if such an i exists, and $\Phi_e(X; x)\uparrow$ otherwise. Now

$$\Phi_e(A \oplus B_0) = \Phi_e(A \oplus B_1) = \cdots = \Phi_e(A \oplus B_{N-1}) = g,$$

whence $\mathcal{R}_e$ ensures that if g is total then $g \leqslant_{\mathrm{T}} A$.

Each strategy R_e^σ for satisfying $\mathcal{R}_e$ uses movable markers $\Gamma_\sigma(n)$, $n \in \omega$, which take positions in P_σ. We will denote the position of $\Gamma_\sigma(n)$ at stage s by $\gamma_\sigma(n, s)$. The movement of these markers will be subject to the following rules:

1. Suppose that s is a σ-stage and, at the beginning of R_e^σ's stage s action, $\Phi_e(A\oplus B_0)[s] \upharpoonright n+1 = \Phi_e(A \oplus B_1)[s] \upharpoonright n+1 = \cdots = \Phi_e(A \oplus B_{N-1})[s] \upharpoonright n+1$, $\Phi_e(A \oplus B_i; n)[s]{\downarrow}$ for all $i < N$, and $\Gamma_\sigma(n)$ does not have a position. Then at stage s, $\Gamma_\sigma(n)$ must be assigned a position larger than any number previously mentioned in the construction. Furthermore, this is the only situation in which a Γ-marker is assigned a new position.

2. If s is a σ-stage, $\Gamma_\sigma(n)$ has a position $\gamma_\sigma(n,s)$ assigned at stage t, and for all $i < N$, $\Phi_e(A \oplus B_i; n)[s] \neq \Phi_e(A \oplus B_i; n)[t]$, then at stage s, $\Gamma_\sigma(n)$ must be removed from its position.

3. If $\Gamma_\sigma(n)$ is removed from its position $\gamma_\sigma(n,s)$ at a stage s then so must all $\Gamma_\sigma(m)$, $m > n$, and some number less than or equal to $\gamma_\sigma(n,s)$ must enter A at stage s.

4. Except finitely often, $\Gamma_\sigma(n)$ may not be removed from position $\gamma_\sigma(n,s)$ unless at least one computation $\Phi_e(A \oplus B_i; n)$, $i < N$, has changed since $\Gamma_\sigma(n)$ was assigned position $\gamma_\sigma(n,s)$.

2.1. Lemma. *If there are infinitely many σ-stages and the above rules are obeyed then $\mathcal{R}_e$ is satisfied.*

Proof. Suppose g is total and $g = \Phi_e(A \oplus B_0) = \Phi_e(A \oplus B_1) = \cdots = \Phi_e(A \oplus B_{N-1})$. By rules 1 and 4, $\gamma_\sigma(n) = \lim_s \gamma_\sigma(n,s)$ exists for all n. Let $f(n)$ be the least σ-stage s such that $\gamma_\sigma(n) = \gamma_\sigma(n,s)$ and $\Phi_e(A \oplus B_0; n)[s] = \Phi_e(A \oplus B_1; n)[s] = \cdots = \Phi_e(A \oplus B_{N-1}; n)[s]$. By rule 3, $f \leqslant_{\mathrm{T}} A$. Finally, by rule 2, $g(n) = \Phi_e(A \oplus B_0; n) = \Phi_e(A \oplus B_1; n) = \cdots = \Phi_e(A \oplus B_{N-1}; n) = \Phi_e(A \oplus B_0; n)\big[f(n)\big]$. Thus $g \leqslant_{\mathrm{T}} A$ as required. ☐

Whenever a number enters A or one of the B_i, there is a possibility that action will have to be taken to guarantee that rule 2 is obeyed. Thus we define the R_e^σ *recovery process* as follows:

Search for an x such that $\Gamma_\sigma(x)$ has position $\gamma_\sigma(x,s)$ assigned at stage t and for all $i < N$, $\Phi_e(A \oplus B_i; x)[s] \neq \Phi_e(A \oplus B_i; x)[t]$. If such an x is found then enumerate $\gamma_\sigma(x,s)$ into A, cancel the positions of all $\Gamma_\sigma(y)$, $y \geqslant x$, and repeat the recovery process; otherwise, end the recovery process.

For a sequence σ of strategies, the *σ-R recovery process* consists of iterating the R_e^τ recovery processes for each R_e^τ in σ until each terminates without enumerating any numbers into A.

We make it a feature of our construction that every time a strategy X^σ enumerates a number into A or one of the B_i, it follows this enumeration with the σ-R recovery process. It will be important to distinguish between numbers enumerated into A directly by a given strategy X^σ and numbers that enter A during a recovery process run by X^σ. When we talk about numbers enumerated by X^σ, we mean only those enumerated directly by X^σ.

The action of R_e^σ at a σ-stage s is simple. It first runs the $\sigma^\frown\langle R_e^\sigma\rangle$-$R$ recovery process. Then it assigns fresh large positions to markers as necessary to obey rule 1, making sure that if $j < k$ then $\gamma_\sigma(j,s) < \gamma_\sigma(k,s)$.

Note that we have guaranteed that if s is a σ-stage then for every strategy X^τ, $\tau \supset \sigma$, that acts during stage s, if $\Gamma_\sigma(x)$ has a position assigned at stage t at the beginning of X^τ's stage s action then $\Phi_e(A \oplus B_i; x)[s] = \Phi_e(A \oplus B_i; x)[t]$ for some $i < N$.

We break (2.1) into requirements

$$\mathcal{S}_{i,e}: \ \Phi_e(A) \neq C_i.$$

If there were no requirements of stronger priority, a strategy $S^{\sigma}_{i,e}$ could satisfy $\mathcal{S}_{i,e}$ by the coding/preservation strategy used in the proof of Sacks's Density Theorem [10]. The $\mathcal{R}$-requirements make things more complicated.

In the spirit of the proof of the density theorem, we wish to ensure that

$$\Phi_e(A) = C_i \Rightarrow C_i \leqslant_{\mathrm{T}} E \tag{2.9}$$

and

$$\Phi_e(A) = C_i \Rightarrow F \leqslant_{\mathrm{T}} E \oplus C_i \tag{2.10}$$

by "preserving" enough of A over E and coding enough of F into E-computable locations in C_i. Since $E <_{\mathrm{T}} F$, this would be enough to satisfy $\mathcal{S}_{i,e}$.

The problem is that, in general, the numbers enumerated into A in order to satisfy the rules for markers associated with a given R-strategy will not form an E-computable set. This makes it hard for E to compute $\Phi_e(A)$. (All strategies in this construction other than the R-strategies will enumerate E-computable sets into A, so that making sure E has a handle on the numbers put into A for the sake of the R strategies is really our main problem here.) As we will see, depending on the strategies and corresponding outcomes in σ, it will be the case that for certain of the R-strategies in σ, the numbers put into A in order to satisfy the rules for markers associated with these strategies will be guaranteed to form computable sets. These can safely be ignored in our description of $S^{\sigma}_{i,e}$. Let *Active_strategy*(σ) be the set of R-strategies in σ that $S^{\sigma}_{i,e}$ must respect, that is, those R-strategies that cannot be ignored for the reason mentioned above. (We will eventually give a formal definition of *Active_strategy*(σ) (see page 28).) It will be the case that for $R^{\alpha}_j, R^{\beta}_k \in \mathit{Active\_strategy}(\sigma)$, $\alpha \subset \beta \Leftrightarrow j < k$ (see Lemma 2.24).

The idea for getting around our problem is based on the fact that a marker $\Gamma_\tau(x)$ corresponding to a strategy R^{τ}_j will not be removed from its position except to reflect a change in *all* the computations $\Phi_j(A \oplus B_k; x)$, $k < N$. In particular, the removal of $\Gamma_\tau(x)$ from its position will never be the first reason for a change in $A \oplus B_i$. To exploit this fact, we make the following definition.

2.2. Definition. A number v is a *σ-i-configuration* at stage s if for all $R^{\tau}_j \in \mathit{Active\_strategy}(\sigma)$ and all m, $\gamma_\tau(m,s) < v \Rightarrow \big[\varphi_j(A \oplus B_i; m)[s] < v \wedge \Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[t]$, where t is the stage at which $\Gamma_\tau(m)$ was assigned position $\gamma_\tau(m,s)\big]$.

We say that v is a *permanent* σ-i-configuration if it is a σ-i-configuration at almost all stages.

The following two lemmas are important consequences of Definition 2.2. The first follows immediately from Definition 2.2 and the definition of recovery process.

2.3. Lemma. *Suppose that v is a σ-i-configuration at the beginning of a recovery process run by some strategy (not necessarily $S^{\sigma}_{i,e}$). Suppose further that if $R^{\alpha}_k \notin \mathit{Active\_strategy}(\sigma)$ then no number in P_α is put into $A \upharpoonright v$ by the recovery process. Then no number is put into $A \upharpoonright v$ by the recovery process.*

2.4. Lemma. *Suppose that $S^{\sigma}_{i,e}$ acts infinitely often and can restrain all numbers from entering $A \oplus B_i$ except for the enumeration of finitely many fixed E-computable sets and numbers put in during a recovery process for the purpose of coding the movement of a marker associated with a* *strategy in Active_strategy(σ). Let $f(v)$ be the least stage by which all these E-computable sets have stopped enumerating numbers into $A \restriction v$ and $B_i \restriction v$. If v is a σ-i-configuration at the end of $S^\sigma_{i,e}$'s action at some stage $s > f(v)$ and $S^\sigma_{i,e}$ preserves this configuration, that is, it imposes a restraint v on $A \oplus B_i$ at stage s, then $A \restriction v = A[s] \restriction v$ and $B_i \restriction v = B_i[s] \restriction v$.*

Proof. Recall that, by the conventions of Section 0, placing a restraint v on $A \oplus B_i$ means placing a restraint v on A and placing a restraint v on B_i.

Assume for a contradiction that some number enters $A \restriction v$ or $B_i \restriction v$ after the end of $S^\sigma_{i,e}$'s stage s action. Since $s > f(v)$, there must exist a strategy X^τ and a τ-stage $t \geqslant s$ with the following properties.

1. Either $\tau \supset \sigma$ or $t > s$.

2. No number enters $A \restriction v$ or $B_i \restriction v$ between the end of $S^\sigma_{i,e}$'s stage s action and the beginning of the recovery process run by X^τ at stage t.

3. Some number is put into $A \restriction v$ by the recovery process run by X^τ at stage t.

The first and second properties above imply that v is a σ-i-configuration at the beginning of the recovery process run by X^τ at stage t. Furthermore, the definition of $f(v)$ implies that no number is put into $A \restriction v$ after stage $f(v)$ for the purpose of coding the movement of a Γ-marker corresponding to a strategy that is not in *Active_strategy*(σ). Thus, the third property above contradicts Lemma 2.3. ☐

A σ-i-configuration v at a stage s such that $A \restriction v = A[s] \restriction v$ and $B_i \restriction v = B_i[s] \restriction v$ will be called *correct*. Clearly, all permanent σ-i-configurations will eventually be correct.

Assume for the remainder of this section that the hypotheses of Lemma 2.4 are satisfied. Let f be the function defined in the statement of Lemma 2.4. Since $f \leqslant_{\mathrm{T}} E$, E can enumerate the correct σ-i-configurations preserved by $S^\sigma_{i,e}$.

Now, for each n such that $\Phi_e(A) \restriction n = C_i \restriction n$ and $\Phi_e(A; n)$ converges, $S^\sigma_{i,e}$'s preservation half will attempt to find and preserve a permanent σ-i-configuration greater than $\varphi_e(A; n)$. If $\Phi_e(A) = C_i$ and all such attempts are successful then, by the comments in the previous paragraphs, (2.9) is satisfied. We will see that all of $S^\sigma_{i,e}$'s attempts to find configurations will be successful unless, for some $R^\tau_j \in$ *Active_strategy*(σ), either $\Phi_j(A \oplus B_i)$ is not total or it is not true that $\Phi_j(A \oplus B_0) = \cdots = \Phi_j(A \oplus B_{N-1})$, in which case we will ensure that for some m, $\Gamma_\tau(m)$ does not have a limit position.

If this last possibility holds then $\mathcal{R}^\tau_j$ is satisfied because its antecedent is false. Furthermore, either $\Gamma_\tau(m)$ is moved infinitely often or there is a stage in the construction after which $\Gamma_\tau(m)$ is never assigned a position. It is not hard to see that this implies that the numbers put into A in order to satisfy the rules for markers associated with R^τ_j will form a computable set. Now for $\alpha \supset \sigma$, $R^\tau_j \notin$*Active_strategy*(α), and thus a strategy $S^\alpha_{i,e}$ can safely ignore R^τ_j. In our construction, we will make sure that such a strategy exists, so that eventually some copy of $S_{i,e}$ will be successful in finding the desired configurations and hence, as we shall see, in satisfying $\mathcal{S}_{i,e}$.

We now explain how $S^\sigma_{i,e}$ acts to attempt to find a permanent σ-i-configuration greater than $\varphi_e(A; n)$ for each n such that $\Phi_e(A) \restriction n = C_i \restriction n$ and $\Phi_e(A; n)$ converges. We will first consider the case in which there is only one R-strategy in *Active_strategy*(σ), say R^τ_j.

The basic idea is that $S^\sigma_{i,e}$ waits for a σ-stage s such that $\Phi_e(A)[s] \upharpoonright n = C_i[s] \upharpoonright n$ and $\Phi_e(A;n)[s]$ converges, and then *takes control* of a marker $\Gamma_\tau(m_n)$, where m_n is a fresh large number, and keeps $\gamma_\tau(m_n,t)$ clear of $\varphi_j(A \oplus B_i; m_n)[t]$, $t \geqslant s$, by removing $\Gamma_\tau(m_n)$ from its position every time the computation $\Phi_j(A \oplus B_i; m_n)$ diverges or changes. (In order to satisfy the rules governing $\Gamma_\tau(m_n)$'s movement, if $S^\sigma_{i,e}$ removes $\Gamma_\tau(m_n)$ from its position during stage $t+1$ then it also removes each $\Gamma_\tau(m)$, $m > m_n$, from its position and enumerates $\gamma_\tau(m_n,t)$ into A.) If the computation $\Phi_e(A;n)$ ever changes then $S^\sigma_{i,e}$ *releases control* of $\Gamma_\tau(m)$ for $m \geqslant m_n$.

If $\Phi_j(A \oplus B_i)$ is total then $\Gamma_\tau(m_n)$ will have a limiting position $\gamma_\tau(m_n)$. We would like to claim that $\gamma_\tau(m_n)$ is a permanent σ-i-configuration. Indeed, $\gamma_\tau(m) < \gamma_\tau(m_n) \Rightarrow m < m_n \Rightarrow \varphi_j(A \oplus B_i; m) < \varphi_j(A \oplus B_i; m_n) < \gamma_\tau(m_n)$, and if t is the stage at which $\Gamma_\tau(m_n)$ achieves position $\gamma_\tau(m_n)$ then the computation $\Phi_j(A \oplus B_i; m_n)$ will not change after stage t, which by the conventions in Section 0 implies that for all $m < m_n$ the computation $\Phi_j(A \oplus B_i; m)$ will not change after stage t.

However, this does not guarantee that, for all $m < m_n$, if t is the stage at which $\Gamma_\tau(m)$ achieves its final position then $\Phi_j(A \oplus B_i; m) = \Phi_j(A \oplus B_i; m)[t]$. This is because $\Gamma_\tau(m)$ might achieve its final position much earlier than $\Gamma_\tau(m_n)$. Thus, we need $S^\sigma_{i,e}$ to remove $\Gamma_\tau(m_n)$ from its position whenever there exists an $m < m_n$ such that $\Gamma_\tau(m)$ has a position and the value of $\Phi_j(A \oplus B_i; m)$ is different from what it was when this position was assigned.

Now if $\Gamma_\tau(m_n)$ has a limiting position then this position is a permanent σ-i-configuration. In this case, $S^\sigma_{i,e}$ cancels the positions of markers only finitely often, and hence it respects the rules for the movement of the Γ_τ-markers.

On the other hand, if $\Gamma_\tau(m_n)$ does not have a limiting position then either $\Phi_j(A \oplus B_i)$ is not total or for some m there exist infinitely many s such that $\Gamma_\tau(m)$ has a position at stage s that was assigned at stage t and $\Phi_j(A \oplus B_i; m)[s] \neq \Phi_j(A \oplus B_i; m)[t]$. But it is not hard to see that the latter case cannot happen if $\Phi_j(A \oplus B_0) = \cdots = \Phi_j(A \oplus B_{N-1})$. Thus, if $\Gamma_\tau(m_n)$ does not have a limiting position then $\mathcal{R}_j$ is satisfied because its antecedent is false. Furthermore, either $\Gamma_\tau(m_n)$ has no position from some point on or $S^\sigma_{i,e}$ cancels the position of $\Gamma_\tau(m_n)$ infinitely often. In either case, the numbers put into A for the purpose of coding the movement of Γ_τ-markers form a computable set.

In the general case, in which there are multiple R-strategies in $Active\_strategy(\sigma)$, whenever $S^\sigma_{i,e}$ finds a σ-stage s such that $\Phi_e(A)[s] \upharpoonright n = C_i[s] \upharpoonright n$ and $\Phi_e(A;n)[s]$ converges, instead of taking control of a single marker, it takes control of a marker $\Gamma_\tau(m^\sigma_{n,\tau})$ for each $R^\tau_j \in Active\_strategy(\sigma)$. We would like $S^\sigma_{i,e}$ to keep the position of each of these markers clear of all the uses $\varphi_j(A \oplus B_i; m^\sigma_{n,\tau})$, $R^\tau_j \in Active\_strategy(\sigma)$.

It might seem that $S^\sigma_{i,e}$ could do this simply by moving all the markers under its control whenever necessary. The problem is that, if for some $R^\tau_j \in Active\_strategy(\sigma)$ it turns out that $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})$ is not convergent, then for some other $R^\alpha_k \in Active\_strategy(\sigma)$, $S^\sigma_{i,e}$ might have to remove $\Gamma_\alpha(m^\sigma_{n,\alpha})$ from its position infinitely often without corresponding changes in the computation $\Phi_k(A \oplus B_i; m^\sigma_{n,\alpha})$, thus violating R^α_k's rules. If $\alpha \supset \tau$ this will not be a problem, since we will then guarantee the existence of another copy of R_k following this outcome of $S^\sigma_{i,e}$. Since we will now also remove R^τ_j from the list of active strategies, it will be true that each R_k will have only finitely many copies on a given path. However, we cannot allow R^τ_j to injure R^α_k if $\alpha \subset \tau$, since in this case R^α_k has stronger priority than R^τ_j. If we were to allow such injury to happen, copies of R_k might get injured infinitely often, and hence $\mathcal{R}_k$ might never be satisfied.

Thus, we proceed as follows: At a given σ-stage $t \geqslant s$, if there exists an $R_j^\tau \in$ *Active_strategy*(σ) such that $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})$ diverges, or for some $m < m^\sigma_{n,\tau}$, $\Gamma_\tau(m)$ has a position assigned at some stage t and $\Phi_j(A \oplus B_i; m)[s] \neq \Phi_j(A \oplus B_i; m)[t]$, or $\gamma_\beta(m^\sigma_{n,\beta}, t) \leqslant \varphi_j(A \oplus B_i; m^\sigma_{n,\tau})[t]$ for some $R_k^\beta \in$ *Active_strategy*(σ), then for the greatest such j, $S^\sigma_{i,e}$ releases control of $\Gamma_\alpha(m^\sigma_{n,\alpha})$ and changes the value of $m^\sigma_{n,\alpha}$ for $\alpha \subset \tau$, while for each $\alpha \supseteq \tau$ and each $m \geqslant m^\sigma_{n,\alpha}$, it removes $\Gamma_\alpha(m)$ from its position, enumerating the least among the previous positions of these markers into A.

Now, unless there is an $R_j^\tau \in$ *Active_strategy*(σ) such that $\Phi_j(A \oplus B_i)$ is not total or it is not the case that $\Phi_j(A \oplus B_0) = \cdots = \Phi_j(A \oplus B_{N-1})$, $S^\sigma_{i,e}$ will find a permanent σ-i-configuration greater than $\varphi_e(A; n)$ for each n such that $\Phi_e(A) \upharpoonright n = C_i \upharpoonright n$ and $\Phi_e(A; n)$ converges.

We are still left with the question of how to satisfy (2.10). This can be accomplished as follows: For each n we have a marker $\delta(n)$ with position $\delta(n, s)$ at stage s. $S^\sigma_{i,e}$ moves $\delta(n)$ to a fresh large position each time the n^{th} σ-i-configuration it is preserving changes. If n enters F and $\delta(n)$ has a position then $S^\sigma_{i,e}$ puts this position into C_i.

Now if for all n, $S^\sigma_{i,e}$ eventually finds a permanent σ-i-configuration larger than $\varphi_e(A; n)$ then we can $E \oplus C_i$-computably determine F as follows: Given n, find a stage s such that $S^\sigma_{i,e}$ is preserving a correct σ-i-configuration larger than $\varphi_e(A; n)$ at the beginning of stage s. (As we have seen, E can do this.) Now $n \in F$ if and only if either $n \in F[s]$ or $\delta(n, s) \in C_i$.

On the other hand, if there is an n such that $\Phi_e(A; n) \neq C_i(n)$ or there is no permanent σ-i-configuration larger than $\varphi_e(A; n)$ then $S^\sigma_{i,e}$ codes a computable set into C_i, since for all but finitely many n, $\delta(n)$ is moved from each position it occupies, and each time it is reassigned a position, this position is larger than the stage at which it is assigned.

We now describe in greater detail the action of $S^\sigma_{i,e}$ at a σ-stage s. Let $r(-1, s) = 0$. The preservation half of $S^\sigma_{i,e}$ acts first and proceeds in cycles, beginning with the cycle for 0. The n^{th} cycle operates as follows:

1. If $\Phi_e(A)[s] \upharpoonright n = C_i[s] \upharpoonright n$ and $\Phi_e(A; n)[s]$ converges then go to step 2. Otherwise, cancel the value of $m^\sigma_{n',\tau}$ and the position of $\delta(n')$ for $n' \geqslant n$ and $R_j^\tau \in$ *Active_strategy*(σ); preserve $A \upharpoonright r(n-1, s)$ and $B_i \upharpoonright r(n-1, s)$ and end stage s activity with outcome $\langle d, n, r(n-1, s) \rangle$. (Here d stands for "disagree". If this outcome is repeated infinitely often then $\Phi_e(A) \neq C_i$, so that $\mathcal{S}_{i,e}$ is satisfied.)

2. Assign fresh large values in P_σ to each $m^\sigma_{n,\tau}$, $R_j^\tau \in$ *Active_strategy*(σ), that is not defined.

3. Search for the longest $\tau \subseteq \sigma$, if any, such that $R_j^\tau \in$ *Active_strategy*(σ) and at least one of the following conditions holds.

   (a) $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})[s]\uparrow$.

   (b) For some $m < m^\sigma_{n,\tau}$, $\Gamma_\tau(m)$ has a position assigned at some stage t and $\Phi_j(A \oplus B_i; m)[s] \neq \Phi_j(A \oplus B_i; m)[t]$.

   (c) $\gamma_\beta(m^\sigma_{n,\beta}, s) \leqslant \varphi_j(A \oplus B_i; m^\sigma_{n,\tau})[s]$ for some $R_k^\beta \in$ *Active_strategy*(σ).

   If such a τ exists then proceed as follows. Enumerate $\min\{\gamma_\alpha(m^\sigma_{n,\alpha}, s) \mid R_k^\alpha \in$ *Active_strategy*(σ) and $\alpha \supseteq \tau\}$ into A (if this set is non-empty) and run the σ-R recovery process. For each $R_k^\alpha \in$ *Active_strategy*(σ), if $\alpha \supseteq \tau$ then cancel the position of $\Gamma_\alpha(y)$ for all $y \geqslant m^\sigma_{n,\alpha}$, otherwise cancel the value of $m^\sigma_{n,\alpha}$. For each $x > n$ and each $R^\alpha_k \in \mathit{Active\_strategy}(\sigma)$, cancel the value of $m^\sigma_{x,\alpha}$. For each $x \geqslant n$, cancel the position of $\delta(x)$. Let $r = \max\bigl(r(n-1,s), \varphi_e(A;n)[s]+1\bigr)$. Preserve $A \restriction r$ and $B_i \restriction r$ and end stage s activity with outcome $\langle c, m^\sigma_{n,\tau}, n, j, r\rangle$. (Here c stands for "change". If this outcome is repeated infinitely often then either $\Phi_j(A \oplus B_i)$ is not total or it is not the case that $\Phi_j(A \oplus B_0) = \cdots = \Phi_j(A \oplus B_{N-1})$, so that $\mathcal{R}_j$ is satisfied.)

4. Define
$$r(n,s) = \min\bigl\{\gamma_\tau(m^\sigma_{n,\tau}, s) \mid R^\tau_j \in \mathit{Active\_strategy}(\sigma)\bigr\}.$$
If this set is empty then define
$$r(n,s) = \max\Bigl(\bigl\{\varphi_j(A \oplus B_i; m^\sigma_{n,\tau})[s] \mid R^\tau_j \in \mathit{Active\_strategy}(\sigma)\bigr\} \cup \\ \cup \bigl\{\varphi_e(A;n)[s]+1\bigr\} \cup \bigl\{r(n,t) \mid t < s\bigr\}\Bigr).$$
If $\delta(n)$ does not have a position then assign its new position $\delta(n,s)$ to be a fresh large number in P_σ. Begin the $(n+1)$st cycle.

The coding half of $S^\sigma_{i,e}$ acts as follows. If $\delta(k)$ has a current position then let t be the stage at which it was assigned this position. If $k \in F[s] - F[t]$ then enumerate $\delta(k,t)$ into C_i and $\bigl\langle \delta(k,t), i\bigr\rangle$ into each B_j, $j \neq i$, and run the σ-R recovery process.

Since we have made it a convention that $\Phi_e(A;n)[s]$ diverges for all $n > s$, there are only finitely many cycles in $S^\sigma_{i,e}$'s action at any given stage. Note that if $s < t$ and $r(n,s)$ and $r(n,t)$ are both defined then $r(n,s) \leqslant r(n,t)$.

2.5. Lemma. *Let $n \in \omega$. Suppose there is a σ-stage s satisfying the following conditions.*

1. *The computation $\Phi_e(A;n)$ has stabilized by the beginning of stage s, and so has each computation $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})$, $R^\tau_j \in \mathit{Active\_strategy}(\sigma)$. (Implicit in this is that $m^\sigma_{n,\tau}$ has reached a permanent value for each $R^\tau_j \in \mathit{Active\_strategy}(\sigma)$.)*

2. *Let $R^\tau_j \in \mathit{Active\_strategy}(\sigma)$. For each $m < m^\sigma_{n,\tau}$, if $\Gamma_\tau(m)$ has a position at the beginning of $S^\sigma_{i,e}$'s stage s action that was assigned at stage t then $\Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[t]$.*

3. *The preservation half of $S^\sigma_{i,e}$ reaches step 3 of its n^{th} cycle during its stage s action.*

Then $r(n,s)$ is defined and is a permanent σ-i-configuration greater than $\varphi_e(A;n)$.

Proof. By 1 and 2, the preservation half of $S^\sigma_{i,e}$ reaches step 4 of its n^{th} cycle during its stage s action, and thus $r(n,s)$ is defined. That $r(n,s) > \varphi_e(A;n)$ is obvious from the definition of $r(n,s)$ and the way the $m^\sigma_{n,\tau}$ are assigned values. Now let $R^\tau_j \in \mathit{Active\_strategy}(\sigma)$ and $u \geqslant s$ and suppose that $\gamma_\tau(m,u) < r(n,s)$. Then $m < m^\sigma_{n,\tau}$, so that
$$\varphi_j(A \oplus B_i; m)[u] < \varphi_j(A \oplus B_i; m^\sigma_{n,\tau})[u] = \varphi_j(A \oplus B_i; m^\sigma_{n,\tau})[s] \leqslant r(n,s).$$
Furthermore, at the beginning of $S^\sigma_{i,e}$'s stage s action, if $\Gamma_\tau(m)$ has a position assigned at some stage t then $\Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[t]$. If the computation $\Phi_j(A \oplus B_i; m)$ changes after stage s then so does the computation $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})$, contrary to our assumption. So $\Phi_j(A \oplus B_i; m)[u] = \Phi_j(A \oplus B_i; m)[t]$. ∎

If the hypotheses of Lemma 2.5 hold then we say that $S^\sigma_{i,e}$ *finds a permanent σ-i-configuration greater than* $\varphi_e(A;n)$ and that this configuration has *stabilized* by stage s.

2.6. Lemma. *Let $R^\tau_j \in$ Active_strategy(σ). Suppose that $m^\sigma_{n,\tau}$ has a permanent value for which $\Phi_j(A \oplus B_0) \restriction m^\sigma_{n,\tau} = \cdots = \Phi_j(A \oplus B_{N-1}) \restriction m^\sigma_{n,\tau}$. Then there exists a u such that for each $m < m^\sigma_{n,\tau}$ and each $s > u$, if $\Gamma_\tau(m)$ has a position at stage s that was assigned at stage t then $\Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[t]$.*

Proof. This lemma follows immediately from the fact that $\Gamma_\tau(m)$ is not assigned a position at stage t unless $\Phi_j(A \oplus B_0)[t] \restriction m = \cdots = \Phi_j(A \oplus B_{N-1})[t] \restriction m$. ☐

2.7. Corollary. *If condition 2 of Lemma 2.5 is not satisfied for $R^\tau_j \in$ Active_strategy(σ) at all sufficiently large stages then it is not the case that $\Phi_j(A \oplus B_0) \restriction m^\sigma_{n,\tau} = \cdots = \Phi_j(A \oplus B_{N-1}) \restriction m^\sigma_{n,\tau}$.*

We can now formally describe the possible behaviors of $S^\sigma_{i,e}$. When we say that a Γ-marker has its position canceled by $S^\sigma_{i,e}$ infinitely often, this includes the possibility that the marker never has a position from some point on.

2.8. Lemma. *Suppose that $S^\sigma_{i,e}$ acts infinitely often and can restrain all numbers from entering any $A \oplus B_l$, $l < N$, except for the enumeration of finitely many fixed E-computable sets and numbers put in during a recovery process for the purpose of coding the movement of a marker associated with a strategy in Active_strategy(σ). Suppose further that there is a stage s_0 after which no δ-marker used by the coding half of $S^\sigma_{i,e}$ can have its position canceled except during $S^\sigma_{i,e}$'s action. Then one of the following holds.*

1. *There is an n such that $S^\sigma_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A;n')$ for all $n' < n$ and either $\Phi_e(A;n-1)\downarrow \neq C_i(n-1)$ or $\Phi_e(A;n)\uparrow$.*

   Let s be a stage by which all of these configurations have stabilized, and let r be their supremum. $S^\sigma_{i,e}$'s outcome is infinitely often equal to $\langle d,n,r\rangle$, and it is never of the form $\langle d,n',r'\rangle$, $n' < n$, $\langle c,m,n',j,r'\rangle$, $n' < n$, or $\langle d,n,r'\rangle$, $r' < r$, after stage s.

   $S^\sigma_{i,e}$ cancels the position of any particular marker only finitely often.

2. *There is an n such that $\Phi_e(A) \restriction n = C_i \restriction n$, $\Phi_e(A;n)\downarrow$, and $S^\sigma_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A;n')$ for all $n' < n$ but no permanent σ-i-configuration greater than $\varphi_e(A;n)$.*

   There exist j and τ with the following properties. $R^\tau_j \in$ Active_strategy(σ), $m^\sigma_{n,\tau}$ has a permanent value, and either $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})\uparrow$ or it is not the case that $\Phi_j(A \oplus B_0) \restriction m^\sigma_{n,\tau} = \cdots = \Phi_j(A \oplus B_{N-1}) \restriction m^\sigma_{n,\tau}$. For each k and $\alpha \supset \tau$ such that $R^\alpha_k \in$ Active_strategy(σ), $m^\sigma_{n,\alpha}$ has a permanent value for which $\Phi_k(A \oplus B_i; m^\sigma_{n,\alpha})\downarrow$.

   Let r be the larger of the supremum of the permanent configurations found by $S^\sigma_{i,e}$ and $\varphi_e(A;n)+1$. Let s be a stage by which all of these configurations have stabilized and so have the computation $\Phi_e(A;n)$ and each computation $\Phi_k(A \oplus B_i; m^\sigma_{n,\alpha})$, $R^\alpha_k \in$ Active_strategy(σ), $\alpha \supset \tau$. $S^\sigma_{i,e}$'s outcome is infinitely often equal to $\langle c, m^\sigma_{n,\tau}, n, j, r\rangle$ and is never of the form $\langle d,n',r'\rangle$, $n' \leqslant n$, $\langle c,m',n',j',r'\rangle$, $n' < n$, or $\langle c,m',n,j',r'\rangle$, $j' > j$ or ($j' = j$ and $m' < m^\sigma_{n,\tau}$) or ($j' = j$, $m' = m^\sigma_{n,\tau}$, and $r' < r$), after stage s.

For $\alpha \subset \tau$, $S^\sigma_{i,e}$ cancels the position of any Γ_α-marker only finitely often, while for $\alpha \supseteq \tau$, any Γ_α-marker whose position is canceled by $S^\sigma_{i,e}$ after stage s has its position canceled by it infinitely often.

In either case, $S^\sigma_{i,e}$ enumerates a computable set into each $A \oplus B_l$, $l < N$.

Proof. Assume for a contradiction that $\Phi_e(A) = C_i$ and $S^\sigma_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A;n)$ for all n. Since every permanent σ-i-configuration is eventually correct, $S^\sigma_{i,e}$ finds correct σ-i-configurations greater than $\varphi_e(A;n)$ for all n. Let f be as in Lemma 2.4. Since $f \leqslant_{\mathrm{T}} E$, Lemma 2.4 implies that E can enumerate the correct σ-i-configurations, which means that $C_i = \Phi_e(A) \leqslant_{\mathrm{T}} E$.

On the other hand, we can $E \oplus C_i$-computably determine F as follows. Given n, E-computably find a stage $s > s_0$ such that $S^\sigma_{i,e}$ is preserving a correct σ-i-configuration larger than $\varphi_e(A;n)$ at the beginning of stage s and $C_i[s-1] \upharpoonright n = C_i \upharpoonright n$. At any σ-stage greater than or equal to s, the preservation half of $S^\sigma_{i,e}$ reaches the $(n+1)^{\mathrm{st}}$ cycle of its action. It is easy to check that this implies that the position of $\delta(n)$ is not canceled at any stage greater than or equal to s, which means that if n enters F at any stage greater than or equal to s then the coding half of $S^\sigma_{i,e}$ puts $\delta(n,s)$ into C_i. So $n \in F$ if and only if either $n \in F[s]$ or $\delta(n,s) \in C_i$, and hence $F \leqslant_{\mathrm{T}} E \oplus C_i$.

The previous two paragraphs show that $F \leqslant_{\mathrm{T}} E \oplus C_i \equiv_{\mathrm{T}} E$. Since $F \nleqslant_{\mathrm{T}} E$, this is a contradiction. Thus we have only two possibilities.

Case 1. There is an n such that $S^\sigma_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A;n')$ for all $n' < n$ and either $\Phi_e(A;n-1)\downarrow \neq C_i(n-1)$ or $\Phi_e(A;n)\uparrow$. Let s be a stage by which all of these configurations have stabilized, and let r be their supremum.

In this case, each time $S^\sigma_{i,e}$ acts at a stage $t \geqslant s$, its preservation half reaches its n^{th} cycle, so that $S^\sigma_{i,e}$'s outcome at stage t is not of the forms $\langle d, n', r'\rangle$, $n' < n$, $\langle c, m, n', j, r'\rangle$, $n' < n$, or $\langle d, n, r'\rangle$, $r' < r$. Moreover, it will infinitely often be the case that $S^\sigma_{i,e}$'s preservation half stops at step 1 of its n^{th} cycle (otherwise $\Phi_e(A;n-1) = C_i(n-1)$ and $\Phi_e(A;n)\downarrow$). So infinitely often $S^\sigma_{i,e}$'s outcome is $\langle d, n, r\rangle$.

Now each time $S^\sigma_{i,e}$'s outcome is $\langle d, n, r\rangle$, the values of all $m^\sigma_{n',\tau}$, $n' \geqslant n$, are canceled. When later reassigned, these values will be fresh large numbers. It is not hard to see that this implies that

$$\lim_{s\to\infty} \min\{m \mid S^\sigma_{i,e} \text{ cancels the position of a marker } \Gamma_\tau(m) \text{ at stage } s\} = \infty.$$

Thus $S^\sigma_{i,e}$ cancels the position of any particular marker only finitely often.

Case 2. If the above does not hold then there is an n such that $\Phi_e(A) \upharpoonright n = C_i \upharpoonright n$, $\Phi_e(A;n)\downarrow$, and $S^\sigma_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A;n')$ for all $n' < n$ but no permanent σ-i-configuration greater than $\varphi_e(A;n)$.

In this case, for all but finitely many σ-stages, $S^\sigma_{i,e}$'s preservation half reaches step 3 of its n^{th} cycle. By Lemma 2.5, this means that for some j there is a τ such that $R^\tau_j \in \mathit{Active\_strategy}(\sigma)$ and either $m^\sigma_{n,\tau}$ does not have a permanent value or it does but, for this value, either $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})\uparrow$ or for infinitely many s there exists an $m < m^\sigma_{n,\tau}$ such that $\Gamma_\tau(m)$ has a position at the beginning of $S^\sigma_{i,e}$'s stage s action that was assigned at some stage t and $\Phi_j(A \oplus B_i; m)[s] \neq \Phi_j(A \oplus B_i; m)[t]$. Let j be the largest number with this property.

By the maximality of j, for each k and $\alpha \supset \tau$ such that $R^\alpha_k \in \mathit{Active\_strategy}(\sigma)$, $m^\sigma_{n,\alpha}$ has a permanent value for which $\Phi_k(A \oplus B_i; m^\sigma_{n,\alpha})\downarrow$ and such that for all sufficiently large stages s, if

$m < m^\sigma_{n,\alpha}$ and $\Gamma_\alpha(m)$ has a position at the beginning of $S^\sigma_{i,e}$'s stage s action that was assigned at stage t then $\Phi_k(A \oplus B_i; m)[s] = \Phi_k(A \oplus B_i; m)[t]$.

Since $S^\sigma_{i,e}$'s preservation half reaches step 3 of its n^{th} cycle at all but finitely many σ-stages, the above implies that the value of $m^\sigma_{n,\tau}$ is not canceled infinitely often. Thus $m^\sigma_{n,\tau}$ has a permanent value. So, for this value, either $\Phi_j(A \oplus B_i; m^\sigma_{n,\tau})\uparrow$, or for infinitely many s there exists an $m < m^\sigma_{n,\tau}$ such that $\Gamma_\tau(m)$ has a position at the beginning of $S^\sigma_{i,e}$'s stage s action that was assigned at some stage t and $\Phi_j(A \oplus B_i; m)[s] \neq \Phi_j(A \oplus B_i; m)[t]$. If the latter possibility holds then, by Corollary 2.7, it is not the case that $\Phi_j(A \oplus B_0) \upharpoonright m^\sigma_{n,\tau} = \cdots = \Phi_j(A \oplus B_{N-1}) \upharpoonright m^\sigma_{n,\tau}$.

Let r be the larger of the supremum of the permanent configurations found by $S^\sigma_{i,e}$ and $\varphi_e(A; n)+1$. Let s be a stage by which all of these configurations have stabilized and so have the computation $\Phi_e(A; n)$ and each computation $\Phi_k(A \oplus B_i; m^\sigma_{n,\alpha})$, $R^\alpha_k \in \mathit{Active\_strategy}(\sigma)$, $\alpha \supset \tau$. At any stage after s during which it acts, $S^\sigma_{i,e}$'s preservation half reaches step 3 of its n^{th} cycle, and the search conducted at that step does not stop at any $\alpha \supset \tau$. Thus, $S^\sigma_{i,e}$'s outcome is never of the form $\langle d, n', r'\rangle$, $n' \leqslant n$, $\langle c, m', n', j', r'\rangle$, $n' < n$, or $\langle c, m', n, j', r'\rangle$, $j' > j$ or ($j' = j$ and $m' < m^\sigma_{n,\tau}$) or ($j' = j$, $m' = m^\sigma_{n,\tau}$, and $r' < r$), after stage s, and infinitely often it is equal to $\langle c, m^\sigma_{n,\tau}, n, j, r\rangle$.

Arguing as in the previous case, we can now show that any particular Γ_α-marker, $\alpha \subset \tau$, is removed from its position only finitely often by $S^\sigma_{i,e}$. It is also clear from the description of $S^\sigma_{i,e}$'s action that, after stage s, the only markers $\Gamma_\alpha(x)$, $R^\alpha_k \in \mathit{Active\_strategy}(\sigma)$, $\alpha \supseteq \tau$, whose positions are canceled by $S^\sigma_{i,e}$ are those with $x \geqslant m^\sigma_{n,\alpha}$, and that these have their positions canceled each time $S^\sigma_{i,e}$'s outcome after stage s is $\langle c, m^\sigma_{n,\tau}, n, j, r\rangle$.

In either case, it is easy to see that $S^\sigma_{i,e}$'s preservation half enumerates a computable set into A. As previously remarked, the fact that there exists an n such that either $\Phi_e(A; n) \neq C_i(n)$ or there is no permanent σ-i-configuration larger than $\varphi_e(A; n)$ found by $S^\sigma_{i,e}$ means that $S^\sigma_{i,e}$ codes a computable set into C_i. □

Making the $\mathbf{b_i}$ maximal-a-cappable. We break (2.4) into requirements

$$\mathcal{N}_{i,k,l,e} \;:\; W_k, W_l \not\leqslant_T A \oplus B_i \Rightarrow D_{i,k,l} \neq \Phi_e(A),$$

with corresponding strategies $N_{i,k,l,e}$.

For each $i < N$ and $k, l, x \in \omega$ we have standard markers $\zeta_{i,k,l}(x)$ and $\tilde{\zeta}_{i,k,l}(x)$ which take values $\zeta_{i,k,l}(x, s)$ and $\tilde{\zeta}_{i,k,l}(x, s)$, respectively, at stage s. For each sequence σ of strategies and outcomes, these values are in P_σ for $x \in P_\sigma$. These markers are subject to the following rules: Each time W_k changes below x, $\zeta_{i,k,l}(x)$ is moved; each time W_l changes below x, $\tilde{\zeta}_{i,k,l}(x)$ is moved. If x either enters or leaves $D_{i,k,l}$ at stage s then $\min\big(\zeta_{i,k,l}(x, s), \tilde{\zeta}_{i,k,l}(x, s)\big)$ is put into A. (It will never be the case that $D_{i,k,l}(x)$ changes at a stage s such that $\min\big(\zeta_{i,k,l}(x, s), \tilde{\zeta}_{i,k,l}(x, s)\big) \in A[s-1]$.) This coding guarantees that (2.3) is satisfied.

In its attempt to satisfy $\mathcal{N}_{i,k,l,e}$, a strategy $N^\sigma_{i,k,l,e}$ will launch *attacks* at σ-stages s and through numbers x such that $\Phi_e(A; x)[s] = D_{i,k,l}(x)[s]$ and there exists a σ-i-configuration $r(x)$ greater than $\varphi_e(A; x)[s]$ but less than $\min\big(\zeta_{i,k,l}(x, s), \tilde{\zeta}_{i,k,l}(x, s)\big)$.

The idea is to change $D_{i,k,l}$ at such an x and preserve the corresponding configuration by imposing a restraint $r(x)$ on $A \oplus B_i$, in the hope that we have thus made $D_{i,k,l}$ and $\Phi_e(A)$ permanently different at x. We will have done so if $r(x)$ is correct. In order to respect the rules for $\zeta_{i,k,l}(x)$ and $\tilde{\zeta}_{i,k,l}(x)$,

$N^\sigma_{i,k,l,e}$ will also enumerate $\min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big)$ into A. This is the reason we require $r(x)$ to be less than $\min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big)$.

With an eye toward satisfying (2.8), we will not launch an attack through x unless we have F-permission, that is, unless a number has entered F below x since the last time $N^\sigma_{i,k,l,e}$ was active. This will ensure that if $F \restriction n = F[s] \restriction n$ and s is a σ-stage then no $x < n$ is put into A by $N^\sigma_{i,k,l,e}$ at any stage after s, a fact which we will need later in the proof of Lemma 2.17.

Of course, an attack can be canceled by a change in $A \oplus B_i$ below the attack's associated configuration. We will need to allow for the possibility of multiple simultaneous attacks, as well as multiple consecutive attacks through the same number. (The reason for this will become clear in the proof of Lemma 2.34.) However, in order to keep the restraint due to $N^\sigma_{i,k,l,e}$ finite, we make sure that while an attack through x is in effect, no attack through $y > x$ can be launched. (Recall that, under the use conventions of Section 0, $x < y \wedge \Phi_e(A;x)[s]\downarrow \wedge \Phi_e(A;y)[s]\downarrow \Rightarrow \varphi_e(A;x)[s] < \varphi_e(A;y)[s]$.) We will be able to show that for some copy of $N_{i,k,l,e}$ there is an attack that is never canceled.

We will rely on S-strategies of weaker priority than $N^\sigma_{i,k,l,e}$ to find the necessary correct configurations described above. We will see later that each $N_{i,k,l,e}$ has a copy for which these weaker priority strategies do indeed find arbitrarily large correct configurations. However, even if $N^\sigma_{i,k,l,e}$ is such a copy, we still need to guarantee that there will be enough numbers x such that there is a stage s and a correct σ-i-configuration at stage s that is greater than $\varphi_e(A;x)[s]$ but smaller than $\min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big)$. For this we need to have enough numbers x such that $W_k\big[f(x)\big] \restriction x \neq W_k \restriction x$ and $W_l\big[f(x)\big] \restriction x \neq W_l \restriction x$, where $f(x)$ denotes the least σ-stage at which a correct σ-i-configuration greater than $\varphi_e(A;x)[s]$ exists.

Since being a correct σ-i-configuration at a given stage is an $A \oplus B_i$-computable condition, we will be able to see that this is the case with the help of two applications of the following lemma, which appears as the lemma to Theorem 2 of Chapter 18 in [12]. (See the proof of Lemma 2.34 for details.)

2.9. Lemma. *If X is c.e., $W_m \not\leq_T X$, Y is c.e. in X and infinite, and f is computable in X, then $\big\{y \in Y \mid W_m \restriction y \neq W_m\big[f(y)\big] \restriction y\big\}$ is c.e. in X and infinite.*

For a proof of this lemma, see [4] or [12].

We now describe in greater detail the action of $N^\sigma_{i,k,l,e}$ at a σ-stage s.

1. For each x, if $N^\sigma_{i,k,l,e}$ is under attack through x and $A \oplus B_i[s] \restriction r(x) \neq A \oplus B_i[t] \restriction r(x)$, where t is the last σ-stage before s, then cancel the attack.

2. Search for $x < s$ such that $x \in P_\sigma$ and the following hold.

   (a) $\Phi_e(A;x)[s] = D_{i,k,l}(x)[s]$.

   (b) There exists a σ-i-configuration $v > \varphi_e(A;x)[s]$ such that $q = \min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big) > v$.

   (c) q is greater than all restraints in σ.

   (d) $q \notin A[s]$.

   (e) $F[s] \restriction x \neq F[t] \restriction x$, where t is the last σ-stage before s if one exists, $t = 0$ otherwise.

   (f) $x < y$ for all y such that $N^\sigma_{i,k,l,e}$ was under attack through y at the beginning of stage s.

Choose the least such x (if one exists). If $x \notin D_{i,k,l}[s]$ then put x into $D_{i,k,l}$, otherwise remove x from $D_{i,k,l}$. Put $\min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big)$ into A and run the σ-R recovery process. Declare that $N^{\sigma}_{i,k,l,e}$ is *under attack* through x and define $r(x)$ to be the least v satisfying (b).

3. Let $r = \max\{r(x) \mid N^{\sigma}_{i,k,l,e}$ is under attack through $x\}$, $r = 0$ if $N^{\sigma}_{i,k,l,e}$ is not currently under attack. Preserve $A \upharpoonright r$ and $B_i \upharpoonright r$ and end stage s activity with outcome r.

In the eventual tree construction, $N^{\sigma}_{i,k,l,e}$ will have to respect other restraints beyond the ones in σ, and we will emend condition 2(c) accordingly (see page 29).

2.10. Proposition. *$D_{i,k,l}$ is well-defined and $D_{i,k,l} \leqslant_T W_k \oplus A, W_l \oplus A$, so that (2.3) is satisfied.*

Proof. Fix x and let $q(s) = \min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big)$. If $D_{i,k,l}(x)$ changes at stage s then $q(s)$ is put into A. Since this requires that $q(s)$ not have been previously put into A, and since $q(s)$ can change only finitely often, this means that $D_{i,k,l}(x)$ can change at most finitely often. This gives us the first part of the proposition.

For the second part, we notice that we can $W_k \oplus A$-computably ($W_l \oplus A$-computably) determine $D_{i,k,l}$ as follows: Given x, find the limit n of $\zeta_{i,k,l}(x,s)$ $\big(\tilde{\zeta}_{i,k,l}(x,s)\big)$ as $s \to \infty$. Let s be such that $A[s] \upharpoonright n+1 = A \upharpoonright n+1$. Now $x \in D_{i,k,l}$ if and only if $x \in D_{i,k,l}[s]$. □

2.11. Lemma. *If $N^{\sigma}_{i,k,l,e}$ acts infinitely often then there is an r such that the outcome of $N^{\sigma}_{i,k,l,e}$ is infinitely often equal to r.*

Proof. First suppose there is a stage t such that $N^{\sigma}_{i,k,l,e}$ comes under attack through some x at t and this attack is never canceled. Then the outcome of $N^{\sigma}_{i,k,l,e}$ at any σ-stage $v > t$ is less than or equal to the outcome at t. This is because, by rule (f) above, any attacks on $N^{\sigma}_{i,k,l,e}$ launched after stage t will be through $y < x$, and we have seen that in this case $r(y) < r(x)$.

The other possibility is that all attacks on $N^{\sigma}_{i,k,l,e}$ are eventually canceled. But no attack through x can be launched at a stage after t until *all* attacks through $y < x$ in effect at stage t have been canceled, and only finitely many attacks can be launched through any given number. Since no attack is ever launched at the same stage that another attack through a smaller number is canceled, this means that in this case there will be infinitely many σ-stages during which no attacks on $N^{\sigma}_{i,k,l,e}$ are left uncanceled at the end of $N^{\sigma}_{i,k,l,e}$'s action. At any such stage, $N^{\sigma}_{i,k,l,e}$'s outcome is equal to 0. □

2.12. Lemma. *Suppose that $N^{\sigma}_{i,k,l,e}$ can restrain all numbers from entering $A \oplus B_i$ except for the enumeration of an E-computable set W, numbers put in during a recovery process for the purpose of coding the movement of a marker associated with a strategy in Active_strategy(σ), and numbers enumerated by $N^{\sigma}_{i,k,l,e}$ itself. Suppose further that there is a stage s after which an attack on $N^{\sigma}_{i,k,l,e}$ through x cannot be canceled except by a change in A or B_i below $r(x)$. Then the set V of numbers enumerated into A by $N^{\sigma}_{i,k,l,e}$ is also E-computable.*

Proof. Let $f(m)$ be the least σ-stage after s such that no numbers less than m are put into A or B_i by the enumeration of W during or after stage $f(m)$. By hypothesis, $f \leqslant_{\mathrm{T}} E$.

If some m enters V at a stage $t > f(m)$ then it must be the case that an attack on $N^{\sigma}_{i,k,l,e}$ is launched at stage t through x such that $m = \min\big(\zeta_{i,k,l}(x,t), \tilde{\zeta}_{i,k,l}(x,t)\big)$. But then $t > f(m) \geqslant f\big(r(x)\big)$, so this attack will not be canceled unless an attack on $N^{\sigma}_{i,k,l,e}$ through $y < x$ is later launched, since if no such attack is launched then, by Lemma 2.3, $r(x)$ is a correct σ-i-configuration. In this case, this new attack will not be canceled unless an attack on $N^{\sigma}_{i,k,l,e}$ through $z < y$ is later launched, and so on. Eventually, there will be an attack on $N^{\sigma}_{i,k,l,e}$ that is never canceled, so that, by rule (f) above, V is finite.

If, on the other hand, no m enters V after stage $f(m)$ then $V \leqslant_{\mathrm{T}} f \leqslant_{\mathrm{T}} E$. $\square$

Coding $\leqslant_0$. We satisfy (2.5) by direct coding. Whenever a number x is enumerated into B_i directly, that is, for any purpose other than the coding of a C_k or B_k into B_i, we require that $\langle x, N+i\rangle$ be enumerated into all B_j such that $i <_0 j$. The only strategies that put numbers into B_i directly are the $O_{i,j,e}$ described below, so this does not affect any of the preceding.

We break (2.6) into requirements

$$\mathcal{O}_{i,j,e} \ : \ B_i \neq \Phi_e(A \oplus B_j \oplus C)$$

for $i \not\leqslant_0 j$, with corresponding strategies $O_{i,j,e}$. A copy $O^{\sigma}_{i,j,e}$ of such a strategy will act very much as $S^{\sigma}_{j,e}$ would.

That is, $O^{\sigma}_{i,j,e}$ ensures that

$$\Phi_e(A \oplus B_j \oplus C) = B_i \Rightarrow B_i \leqslant_{\mathrm{T}} E, \tag{2.11}$$

$$\Phi_e(A \oplus B_j \oplus C) = B_i \Rightarrow F \leqslant_{\mathrm{T}} E \oplus B_i. \tag{2.12}$$

via a coding/preservation strategy which differs from the way a strategy $S^{\sigma}_{j,e}$ would act only in that

1. $O^{\sigma}_{i,j,e}$ attempts to find permanent σ-j-configurations greater than n for each n such that $\Phi_e(A \oplus B_j \oplus C) \upharpoonright n = B_i \upharpoonright n$ and $\Phi_e(B_j \oplus A \oplus C; n)$ converges, and

2. instead of coding F into C_i, $O^{\sigma}_{i,j,e}$ codes F into B_i (and hence also into each B_k, $i <_0 k$, in accordance with the coding required to satisfy (2.5)).

Notice that the coding half of $O^{\sigma}_{i,j,e}$ does not put any numbers into $A \oplus B_j$. This is what allows us to use σ-j-configurations in this case.

Formally, $O^{\sigma}_{i,j,e}$ acts as follows at a σ-stage s. Let $r(-1,s) = 0$. The preservation half of $O^{\sigma}_{i,j,e}$ acts first and proceeds in cycles, beginning with the cycle for 0. The n^{th} cycle operates as follows:

1. If $\Phi_e(A \oplus B_j \oplus C)[s] \upharpoonright n = B_i[s] \upharpoonright n$ and $\Phi_e(A \oplus B_j \oplus C; n)[s]$ converges then go to step 2. Otherwise, cancel the value of $m^{\sigma}_{n',\tau}$ and the position of $\delta(n')$ for $n' \geqslant n$ and $R^{\tau}_k \in$ *Active_strategy*(σ); preserve $A \upharpoonright r(n-1,s)$ and $B_j \upharpoonright r(n-1,s)$ and end stage s activity with outcome $\big\langle d, n, r(n-1,s)\big\rangle$.

2. Assign fresh large values in P_σ to each $m^{\sigma}_{n,\tau}$, $R^{\tau}_k \in$ *Active_strategy*(σ), that is not defined.

3. Search for the longest $\tau \subseteq \sigma$, if any, such that $R^{\tau}_k \in$ *Active_strategy*(σ) and at least one of the following conditions holds.

   (a) $\Phi_k(A \oplus B_j; m^{\sigma}_{n,\tau})[s]\!\uparrow$.

(b) For some $m < m^\sigma_{n,\tau}$, $\Gamma_\tau(m)$ has a position assigned at some stage t and $\Phi_k(A\oplus B_j; m)[s] \neq \Phi_k(A \oplus B_j; m)[t]$.

(c) $\gamma_\beta(m^\sigma_{n,\beta}, s) \leqslant \varphi_k(A \oplus B_j; m^\sigma_{n,\tau})[s]$ for some $R^\beta_l \in$ *Active_strategy*(σ).

If such a τ exists then proceed as follows. Enumerate $\min\{\gamma_\alpha(m^\sigma_{n,\alpha}, s) \mid R^\alpha_l \in \textit{Active\_strategy}(\sigma)$ and $\alpha \supseteq \tau\}$ into A (if this set is non-empty) and run the σ-R recovery process. For each $R^\alpha_l \in$ *Active_strategy*(σ), if $\alpha \supseteq \tau$ then cancel the position of $\Gamma_\alpha(y)$ for all $y \geqslant m^\sigma_{n,\alpha}$, otherwise cancel the value of $m^\sigma_{n,\alpha}$. For each $x > n$ and each $R^\alpha_l \in$ *Active_strategy*(σ), cancel the value of $m^\sigma_{x,\alpha}$. For each $x \geqslant n$, cancel the position of $\delta(x)$. Let $r = \max\bigl(r(n-1, s), \varphi_e(A \oplus B_j \oplus C; n)[s]+1\bigr)$. Preserve $A \restriction r$ and $B_j \restriction r$ and end stage s activity with outcome $\langle c, m^\sigma_{n,\tau}, n, k, r\rangle$.

4. Define
$$r(n, s) = \min\bigl\{\gamma_\tau(m^\sigma_{n,\tau}, s) \mid R^\tau_k \in \textit{Active\_strategy}(\sigma)\bigr\}.$$
If this set is empty then define
$$r(n, s) = \max\Bigl(\bigl\{\varphi_k(A \oplus B_j; m^\sigma_{n,\tau})[s] \mid R^\tau_k \in \textit{Active\_strategy}(\sigma)\bigr\}\cup$$
$$\cup\bigl\{\varphi_e(A \oplus B_j \oplus C; n)[s] + 1\bigr\} \cup \bigl\{r(n, t) \mid t < s\bigr\}\Bigr).$$
If $\delta(n)$ does not have a position then assign its new position $\delta(n, s)$ to be a fresh large number in P_σ. Begin the $(n + 1)$st cycle.

The coding half of $O^\sigma_{i,j,e}$ acts as follows. If $\delta(k)$ has a current position then let t be the stage at which it was assigned this position. If $k \in F[s] - F[t]$ then enumerate $\delta(k, t)$ into B_i and $\bigl\langle\delta(k, t), N + i\bigr\rangle$ into each B_k, $i <_0 k$, and finally run the σ-R recovery process.

Corresponding to Lemma 2.8, we have the following lemma.

2.13. Lemma. *Suppose that $O^\sigma_{i,j,e}$ acts infinitely often and can restrain all numbers from entering any $A\oplus B_p$, $p < N$, except for the enumeration of finitely many fixed E-computable sets and numbers put in during a recovery process for the purpose of coding the movement of a marker associated with a strategy in Active_strategy(σ). Suppose further that there is a stage after which no δ-marker used by the coding half of $O^\sigma_{i,j,e}$ can have its position canceled except during $O^\sigma_{i,j,e}$'s action. Then one of the following holds.*

1. *There is an n such that $O^\sigma_{i,j,e}$ finds permanent σ-j-configurations greater than $\varphi_e(A\oplus B_j\oplus C; n')$ for all $n' < n$ and either $\Phi_e(A \oplus B_j \oplus C; n - 1)\downarrow\neq B_i(n - 1)$ or $\Phi_e(A \oplus B_j \oplus C; n)\uparrow$.*

   Let s be a stage by which all of these configurations have stabilized, and let r be their supremum. $O^\sigma_{i,j,e}$'s outcome is infinitely often equal to $\langle d, n, r\rangle$, and it is never of the form $\langle d, n', r'\rangle$, $n' < n$, $\langle c, m, n', l, r'\rangle$, $n' < n$, or $\langle d, n, r'\rangle$, $r' < r$, after stage s.

   $O^\sigma_{i,j,e}$ cancels the position of any particular marker only finitely often.

2. *There is an n such that $\Phi_e(A \oplus B_j \oplus C) \restriction n = B_i \restriction n$, $\Phi_e(A \oplus B_j \oplus C; n)\downarrow$, and $O^\sigma_{i,j,e}$ finds permanent σ-j-configurations greater than $\varphi_e(A \oplus B_j \oplus C; n')$ for all $n' < n$ but no σ-j-configuration greater than $\varphi_e(A \oplus B_j \oplus C; n)$.*

There exist l and τ with the following properties. $R_l^\tau \in$ Active_strategy(σ), $m_{n,\tau}^\sigma$ has a permanent value, and either $\Phi_l(A \oplus B_j; m_{n,\tau}^\sigma)\uparrow$ or it is not the case that $\Phi_l(A \oplus B_0) \upharpoonright m_{n,\tau}^\sigma = \cdots = \Phi_l(A \oplus B_{N-1}) \upharpoonright m_{n,\tau}^\sigma$. For each k and $\alpha \supset \tau$ such that $R_k^\alpha \in$ Active_strategy(σ), $m_{n,\alpha}^\sigma$ has a permanent value for which $\Phi_k(A \oplus B_j; m_{n,\alpha}^\sigma)\downarrow$.

Let r be the larger of the supremum of the permanent configurations found by $O_{i,j,e}^\sigma$ and $\varphi_e(A \oplus B_j \oplus C; n) + 1$. Let s be a stage by which all of these configurations have stabilized and so have the computation $\Phi_e(A \oplus B_j \oplus C; n)$ and each computation $\Phi_k(A \oplus B_j; m_{n,\alpha}^\sigma)$, $R_k^\alpha \in$ Active_strategy(σ), $\alpha \supset \tau$. $O_{i,j,e}^\sigma$'s outcome is infinitely often equal to $\langle c, m_{n,\tau}^\sigma, n, l, r\rangle$ and is never of the form $\langle d, n', r'\rangle$, $n' \leqslant n$, $\langle c, m', n', l', r'\rangle$, $n' < n$, or $\langle c, m', n, l', r'\rangle$, $l' > l$ or ($l' = l$ and $m' < m_{n,\tau}^\sigma$) or ($l' = l$, $m' = m_{n,\tau}^\sigma$, and $r' < r$), after stage s.

For $\alpha \subset \tau$, $O_{i,j,e}^\sigma$ cancels the position of any Γ_α-marker only finitely often, while for $\alpha \supseteq \tau$, any Γ_α-marker whose position is canceled by $O_{i,j,e}^\sigma$ after stage s has its position canceled by it infinitely often.

In either case, $O_{i,j,e}^\sigma$ enumerates a computable set into each $A \oplus B_p$, $p < N$.

The proof of this lemma is completely analogous to that of Lemma 2.8.

Producing F-computable sets. We wish to satisfy condition (2.8). From the previous sections it can be seen that there are four ways for numbers to enter A or some B_i or C_i:

1. A number enters E and this fact is coded into A by K.
2. A number is put into A by some N-strategy as part of an attack on that strategy.
3. A number is put into A in order to code the movement of a marker $\Gamma_\sigma(m)$.
4. A number enters F and this fact is coded into some C_i (B_i) by an S-strategy (O-strategy), and subsequently into B_j, $j \neq i$ ($i <_0 j$).

Clearly, the numbers coded into A by K form an F-computable set.

As mentioned on page 17, no number $n \in P_\sigma$ can be put into A by $N_{i,k,l,e}^\sigma$ as part of the realization of an attack on that strategy after a σ-stage s such that $F[s] \upharpoonright n+1 = F \upharpoonright n+1$. Thus the numbers put into A for this reason also form an F-computable set.

In order to ensure that the numbers entering A as codes for the movement of markers associated with R-strategies form an F-computable set, we will define the concept of a *σ-configuration*, which will be similar to a σ-i-configuration except that it will use $A \oplus B_j$ computations for every $j < N$, instead of only using $A \oplus B_i$ computations. We will then describe, for each k, a strategy M_k^σ which acts to guarantee, for each R_e^τ in σ and each m, that $\Gamma_\tau(m)$ either moves past position k without occupying it, occupies it and is later moved, or there is a permanent σ-configuration that keeps $\Gamma_\tau(m)$ from moving from position k. Since F will be able to enumerate such permanent configurations uniformly, this will give us the desired result. (Whenever a new position is assigned to a marker, this position is larger than any number previously mentioned in the construction. Thus, if $\Gamma_\tau(m)$ does not have a position at stage k then we can consider $\Gamma_\tau(m)$ to have moved past position k.)

This leaves us with the task of ensuring that the numbers entering some C_i or B_i for reason 4 above form an F-computable set. The problem here is the following. Suppose that $F[s] \upharpoonright n = F \upharpoonright n$ and that an S-strategy (O-strategy) X acts at a stage $t > s$. Let $k < n$ and let $\delta(k)$ be the corresponding marker employed by the coding half of X. Let u be the last stage before t at which X acted and suppose that $\delta(k,t) = \delta(k,s) = \delta(k,u)$. If $u < s$ then it is possible that $k \in F[t] - F[u]$, so that $\delta(k,s)$ will be put into some C_i (B_i) by X. This means that, in order to be sure that no numbers less than n will enter any C_i or B_i after stage s, it is not enough to know that $F[s] \upharpoonright n = F \upharpoonright n$. We must also know that none of the S-strategies and O-strategies that still have δ-markers with positions less than n at the end of stage s and that do not act during stage s will ever act after stage s. We will be able to show that this is the case if s is a σ-stage and M_n^σ preserves a *correct* σ-configuration greater than n at stage s (see Definition 2.21). We will also show that, for every $n \in \omega$, some strategy M_n^σ eventually preserves a correct σ-configuration greater than n, and that F can enumerate the correct σ-configurations uniformly. This will be enough to show that the numbers entering some C_i or B_i for reason 4 above form an F-computable set. We will discuss this further in the paragraphs following Corollary 2.20.

The M-strategies will work similarly to the preservation half of the S-strategies. That is, they will take control of certain markers and keep their positions clear of appropriate configurations. Since the M-strategies do not have coding halves, they will be able to use $A \oplus B_i$ computations for every $i < N$. Furthermore, we will see that the activity of such a strategy is finitely bounded, so that it can be allowed to move any marker without danger of violating the rules for its movement.

Before describing the M-strategies, we impose an additional convention on our use functions, namely that if $\Phi_e(A \oplus B_i; n)[s]\downarrow$ through a computation that exists for the first time at stage s then for all $t < s$ and $j < N$ such that $\Phi_e(A \oplus B_j; n)[t]\downarrow$, $\varphi_e(A \oplus B_i; n)[s] \geqslant \varphi_e(A \oplus B_j; n)[t]$. It is easy to check that this new convention does not alter any of the preceding.

For each strategy R_e^σ, if $\gamma_\sigma(n,s)$ is defined then let

$$\mathit{max\_min}(\sigma,n,s) = \min\big\{\varphi_e(A \oplus B_i; n)[s] \mid \Phi_e(A \oplus B_i; n)[s] = \Phi_e(A \oplus B_i; n)[t],\ i < N\big\},$$

where t is the stage at which $\Gamma_\sigma(n)$ was assigned position $\gamma_\sigma(n,s)$. If $\gamma_\sigma(n,s)$ is defined then, due to our new convention on uses, $\mathit{max\_min}(\sigma,n,s)$ is the maximum of the lengths of the shortest computations that have kept the R_e^σ recovery process from moving $\Gamma_\sigma(n)$ during the stages between t and s.

2.14. Lemma. *Suppose that R_e^σ acts infinitely often and $\gamma_\sigma(n,s)$ is defined for infinitely many s. If $\mathit{max\_min}(\sigma,n,s)$ goes to infinity as a function of s then so does $\gamma_\sigma(n,s)$.*

Proof. Assume for a contradiction that $\mathit{max\_min}(\sigma,n,s)$ goes to infinity as a function of s and there exists a t such that $\Gamma_\sigma(n)$ gets assigned a position at stage t and $\gamma_\sigma(n,s) = \gamma_\sigma(n,t)$ for all $s > t$. Let U be the set of all $i < N$ such that $\Phi_e(A \oplus B_i; n)[s] = \Phi_e(A \oplus B_i; n)[t]$ for infinitely many s. Let $u \geqslant t$ be a stage such that for all $i < N$, $i \notin U$, and all $s \geqslant u$, $\Phi_e(A \oplus B_i; n)[s] \neq \Phi_e(A \oplus B_i; n)[t]$.

For each $i \in U$, $\Phi_e(A \oplus B_i; n)$ diverges, since otherwise $\mathit{max\_min}(\sigma,n,s)$ would have a finite limit. Given $x \in \omega$, let $f(x)$ be the first stage greater than or equal to u such that for each $i \in U$ there exists an $s < f(x)$ such that $\varphi_e(A \oplus B_i; n)[s] > 2x$.

Suppose that some x enters E at a stage $s > f(x)$. Then $2x$ enters A at stage s, which means that at the first σ-stage v after s, R_e^σ will treat each computation $\Phi_e(A \oplus B_i; n)[v]$, $i \in U$, as divergent.

Thus, since $v > u$, it will be the case that for all $i < N$, $\Phi_e(A \oplus B_i; n)[v] \neq \Phi_e(A \oplus B_i; n)[t]$, and hence $\Gamma_\sigma(n)$ will have its position canceled at stage v.

Since we have assumed that $\gamma_\sigma(n, s) = \gamma_\sigma(n, t)$ for all $s > t$, we conclude that no number x enters E at a stage after $f(x)$. In other words, $E(x)\big[f(x)\big] = E(x)$ for all $x \in \omega$. However, $f(x)$ is clearly computable, so we have contradicted the noncomputability of E. $\square$

It should be noted that if v is a *σ-i-configuration* at stage s then our new convention on uses implies that for all $R_j^\tau \in \mathit{Active\_strategy}(\sigma)$ and all m, $\gamma_\tau(m, s) < v \Rightarrow \mathit{max\_min}(\tau, m, s) < v$.

We now proceed with the description of M_k^σ.

We first consider the case in which K and R_e^τ are the only strategies in σ. In this case, we define a *σ-configuration* at stage s to be a number v such that $\gamma_\tau(m, s) < v \Rightarrow \mathit{max\_min}(\tau, m, s) < v$.

M_k^σ fixes a number move_σ in advance and removes $\Gamma_\tau(\mathit{move}_\sigma)$ from its position every time the smallest σ-configuration greater than k is larger than the position of $\Gamma_\tau(\mathit{move}_\sigma)$. Otherwise, M_k^σ preserves the smallest σ-configuration greater than k.

Suppose that M_k^σ acts infinitely often. Let U be the set of all numbers $m < \mathit{move}_\sigma$ such that $\gamma_\tau(m, s)$ has a limit $\gamma_\tau(m)$. By Lemma 2.14, for each $m \in U$, $\mathit{max\_min}(\tau, m, s)$ has a limit $\mathit{max\_min}(\tau, m)$.

Let L be the least number greater than k such that for all $m \in U$, $\gamma_\tau(m) < L \Rightarrow \mathit{max\_min}(\tau, m) < L$. Let s be a σ-stage such that $\gamma_\tau(m, s) = \gamma_\tau(m)$ and $\mathit{max\_min}(\tau, m, s) = \mathit{max\_min}(\tau, m)$ for all $m \in U$, and all $\Gamma_\tau(n)$, $n < \mathit{move}_\sigma$, $n \notin U$, have moved past L by stage s. (Such an s exists by Lemma 2.14.) The smallest σ-configuration greater than k at stage s is greater than or equal to L, so if $\gamma_\tau(\mathit{move}_\sigma, s)$ is less than L then $\Gamma_\tau(\mathit{move}_\sigma)$ will be removed from its position by M_k^σ. So for all stages $t > s$, $\gamma_\tau(m, t) < L \Rightarrow m \in U \Rightarrow \mathit{max\_min}(\tau, m, t) < L$, and thus L is a permanent σ-configuration greater than k. Furthermore, it is clear from its definition that L is the smallest such configuration.

All M_k^σ does after stage s is to preserve L, so its action is finite.

In the general case, the definition of σ-configuration is a little more complicated.

2.15. Definition. A number v is a *σ-configuration* at stage s if it satisfies the following conditions.

1. For each $\langle R_e^\tau \rangle \in \sigma$, $\gamma_\tau(m, s) < v \Rightarrow \mathit{max\_min}(\tau, m, s) < v$.
2. v is greater than all restraints in σ.
3. If $\langle X^\tau, c, m, n, l, r \rangle \in \sigma$, where X is one of $S_{i,e}$ or $O_{j,i,e}$, $R_k^\alpha \in \mathit{Active\_strategy}(\tau)$, $k > l$, and the computation $\Phi_k(A \oplus B_i; m_{n,\alpha}^\tau)$ still exists at the end of X^τ's stage s action, then $v > \varphi_k(A \oplus B_i; m_{n,\alpha}^\tau)[s]$.

The third clause of this definition may seem mysterious at this point, but it will become necessary later on when we prove Lemma 2.32.

The following two lemmas are important consequences of Definition 2.15. The first follows immediately from Definition 2.15 and the definition of recovery process.

2.16. Lemma. *Suppose that v is a σ-configuration at the beginning of a recovery process run by some strategy. Suppose further that if $\alpha \not\subseteq \sigma$ then no number in P_α is put into $A \restriction v$ by the recovery process. Then no number is put into $A \restriction v$ by the recovery process.*

We say that a σ-configuration v at stage s is *F and E-correct* if $F[s] \upharpoonright v = F \upharpoonright v$ and $E[s] \upharpoonright v = E \upharpoonright v$.

2.17. Lemma. *Suppose that M_k^σ preserves an F and E-correct σ-configuration v larger than k at some σ-stage s such that for each strategy X^τ, $\tau \subseteq \sigma$, the only strategies that can be the first to put a number less than the restraint due to X^τ into A or any B_i or C_i, $i < N$, after stage s are those mentioned in τ. Then $A[s] \upharpoonright v = A \upharpoonright v$ and, for all $i < N$, $B_i[s] \upharpoonright v = B_i \upharpoonright v$ and $C_i[s] \upharpoonright v = C_i \upharpoonright v$.*

Proof. By Lemma 2.16, it is enough to show that for all strategies X^α in σ, X^α cannot be the first strategy in the construction to put a number into $A \upharpoonright v$ or some $B_i \upharpoonright v$ or $C_i \upharpoonright v$, $i < N$, after stage s. We assume by induction that the lemma is true for each strategy $M_{k'}^\alpha$, $\alpha \subset \sigma$, and remark that it will be the case that for any such strategy, $k' < k$ (see Lemma 2.24).

The case $X = K$ follows immediately from the fact that $E[s] \upharpoonright v = E \upharpoonright v$.

The case $X^\alpha = R_e^\alpha$ follows from Lemma 2.16.

If $X^\alpha = M_{k'}^\alpha$ then, by induction, $M_{k'}^\alpha$ is preserving a permanent α-configuration at stage s. Thus $M_{k'}^\alpha$ has completed its action by stage s.

If $X^\alpha = N_{i,j,l,e}^\alpha$ then the only sets into which $N_{i,j,l,e}^\alpha$ puts numbers are A and $D_{i,j,l}$, and, as mentioned on page 17, the fact that $F[s] \upharpoonright v = F \upharpoonright v$ implies that $N_{i,j,l,e}^\alpha$ does not put any numbers into $A \upharpoonright v$ after stage s.

The other two possibilities can be handled simultaneously. Suppose that X^α is one of $S_{i,e}^\alpha$ or $O_{j,i,e}^\alpha$.

The coding half of X^α does not put numbers into A, and the fact that $F[s] \upharpoonright v = F \upharpoonright v$ implies that it does not put any numbers into $B_p \upharpoonright v$ or $C_p \upharpoonright v$, $p < N$, after stage s. We claim that the preservation half of X^α, which only puts numbers into A, does not put any numbers into $A \upharpoonright v$ after stage s.

Indeed, the configurations preserved by X^α at stage s are all less than v, and hence will not be violated after stage s unless some other strategy creates a change in $A \upharpoonright v$ or $B_i \upharpoonright v$. Thus, unless such a change occurs, none of the markers that still have positions at the end of X^α's stage s action will be removed from their positions by X^α after stage s. Since any positions later assigned to markers that do not have positions at the end of X^α's stage s action will be larger than v, this establishes our claim and completes the proof of the lemma. $\square$

As in the simplified case described above, M_k^σ fixes a number $move_\sigma$ (greater than all restraints in σ) in advance. At a σ-stage s, M_k^σ's action is as follows.

1. Find the smallest σ-configuration $w > k$.

2. If there is no R_e^τ in σ such that $\gamma_\tau(move_\sigma, s) < w$ then let $v = w$ and proceed to step 5.

3. Cancel the position of all markers $\Gamma_\alpha(n)$, $\langle R_{e'}^\alpha \rangle \in \sigma$, $n \geqslant move_\sigma$, enumerate the least of their positions into A, and run the σ-R recovery process.

4. Find the smallest σ-configuration $v > k$.

5. Impose a restraint equal to v on $A \oplus B_i$ and C_i, $i < N$, and end stage s activity with outcome v. (The fact that the restraint is also imposed on C_i will be important in the proof of Lemma 2.32.)

Notice that v, whether defined at step 2 or step 4, is the smallest σ-configuration greater than k and is less than all numbers $\gamma_\tau(move_\sigma, s)$, $R_e^\tau \in \sigma$.

We can now show that, under certain conditions, M_k^σ will eventually preserve an F and E-correct σ-configuration greater than k. Later we will see that each M_k has a copy for which these conditions obtain.

2.18. Lemma. *Suppose that*

1. *M_k^σ acts infinitely often,*

2. *there is a stage t after which the restraints in σ are respected,*

3. *for each $S_{i,e}^\tau$ ($O_{j,i,e}^\tau$) in σ, the hypotheses of Lemma 2.8 (Lemma 2.13) are satisfied, and*

4. *if X is one of $S_{i,e}$ or $O_{j,i,e}$ and $\langle X^\tau, c, m, n, l, r\rangle \in \sigma$ then for some u, X^τ's outcome at a stage after u is never of the form $\langle c, m', n,', l', r'\rangle$, $n' < n$, $\langle d, n', r'\rangle$, $n' \leqslant n$, or $\langle c, m', n, l', r'\rangle$, $l' > l$.*

Then M_k^σ eventually preserves a permanent σ-configuration greater than k.

Proof. If $S_{i,e}^\tau$ ($O_{j,i,e}^\tau$) is as in hypothesis 4, then the second case of Lemma 2.8 (Lemma 2.13) holds, so that if $R_l^\alpha, R_{l'}^\beta \in Active\_strategy(\tau)$ and $\beta \supset \alpha$ then $m_{n,\beta}^\tau$ has a final position for which $\Phi_{l'}(A \oplus B_i; m_{n,\beta}^\tau)$ converges. This implies that there exists a v that satisfies the third part of Definition 2.15 at all sufficiently large σ-stages. We can now argue very much as we did in the special case in which K and R_e^τ were the only strategies in σ.

That is, for each $\langle R_e^\tau\rangle \in \sigma$, let $U(\tau)$ be the set of all numbers $m < move_\sigma$ such that $\gamma_\tau(m, s)$ has a limit $\gamma_\tau(m)$. By Lemma 2.14, for each $m \in U(\tau)$, $max\_min(\tau, m, s)$ has a limit $max\_min(\tau, m)$. Let s_0 be such that for all $\langle R_e^\tau\rangle \in \sigma$ and all $m \in U(\tau)$, $\gamma_\tau(m, s_0) = \gamma_\tau(m)$ and $max\_min(\tau, m, s_0) = max\_min(\tau, m)$.

Let v be the least number that satisfies the second and third parts of Definition 2.15 at all sufficiently large σ-stages and let $s_1 > s_0$ be such that v satisfies the second and third parts of Definition 2.15 at all σ-stages after s_1. Let L be the least number that satisfies each of the following conditions.

1. $L > k$.

2. $L \geqslant v$.

3. For all $\langle R_e^\tau\rangle \in \sigma$ and all $m \in U(\tau)$, $\gamma_\tau(m) < L \Rightarrow max\_min(\tau, m) < L$.

Let $s_2 > s_1$ be such that if $\langle X^\tau, c, m, n, l, r\rangle \in \sigma$, where X is one of $S_{i,e}$ or $O_{j,i,e}$, and $R_k^\alpha \in Active\_strategy(\tau)$, $k > l$, then the computation $\Phi_k(A \oplus B_i; m_{n,\alpha}^\tau)$ has settled down by stage s_2.

By Lemma 2.14, all $\Gamma_\tau(n)$, $\langle R_e^\tau\rangle \in \sigma$, $n < move_\sigma$, $n \notin U(\tau)$, will have moved past L by some σ-stage $s \geqslant s_2$. So if $\gamma_\tau(move_\sigma, s)$ is less than L then $\Gamma_\tau(move_\sigma)$ will be removed from its position by M_k^σ. Thus L is a permanent σ-configuration greater than k. Furthermore, it is clear from its definition that L is the smallest such configuration. ☐

The previous lemma has the following immediate consequences.

2.19. Corollary. *Under the same conditions as in Lemma 2.18, M_k^σ cancels the positions of markers only finitely often.*

2.20. Corollary. *Under the same conditions as in Lemma 2.18, M_k^σ eventually preserves an F and E-correct σ-configuration greater than k.*

We would like to argue that Corollary 2.20, together with Lemma 2.17, gives us (2.8). However, though it is certainly true that F can recognize when a strategy is preserving an F and E-correct configuration, it is not quite the case that if M_k^σ preserves an F and E-correct σ-configuration greater than k at stage s then $A[s] \restriction k = A \restriction k$ and $B_i[s] \restriction k = B_i \restriction k$, $i < N$.

This is because the fact that M_k^σ preserves an F and E-correct configuration does not imply that the hypothesis of Lemma 2.17 is satisfied. It will also be true that this hypothesis cannot be verified by F uniformly. We will discuss this further after we have given the details of the tree of strategies and the construction.

We will then see that we need additional conditions on a configuration, leading to the definition of a correct σ-configuration, for the hypothesis of Lemma 2.17 to hold. Since it will be the case that F can recognize when a strategy is preserving a correct configuration and that every M_k will have a copy that eventually preserves such a configuration, we will be able to satisfy (2.8).

2.21. Definition. A σ-configuration v at stage s is said to be *correct* if it is F and E-correct and satisfies the following condition. Let X be one of $S_{i,e}$ or $O_{j,i,e}$. If $\langle X^\beta, c, m, n, l, r\rangle \in \sigma$ then for all $R_k^\tau \in \mathit{Active\_strategy}(\beta)$, $k > l$, the computation $\Phi_k(A \oplus B_i; m_{n,\tau}^\beta)$ exists at the end of stage s.

2.22. Lemma. *If M_k^σ satisfies the hypotheses of Lemma 2.18 then there is a σ-stage s at which it preserves a correct σ-configuration greater than k.*

Proof. By Corollary 2.20, there is an s_0 such that M_k^σ preserves an F and E-correct σ-configuration greater than k at all stages after s_0.

Now let X be one of $S_{i,e}$ or $O_{j,i,e}$ and suppose that $\langle X^\beta, c, m, n, l, r\rangle \in \sigma$. Arguing as in the proof of Lemma 2.18, there is a t_β such that for all $R_k^\tau \in \mathit{Active\_strategy}(\beta)$, $k > l$, the computation $\Phi_k(A \oplus B_i; m_{n,\tau}^\beta)$ exists at the end of stage t_β and does not change after stage t_β. Let t be the maximum of all t_β, $\langle S_{i,e}^\beta, c, m, n, l, r\rangle \in \sigma$ or $\langle O_{j,i,e}^\beta, c, m, n, l, r\rangle \in \sigma$.

Let s be the least σ-stage greater than s_0 and t. It follows from the definition that M_k^σ preserves a correct σ-configuration greater than k at stage s. $\square$

After presenting our formal construction, we will see that every M_k has a copy satisfying the hypotheses of Lemma 2.18 and will be able to show (Lemma 2.32) that if M_k^σ preserves a correct σ-configuration v at stage s then $A[s] \restriction v = A \restriction v$ and $B_i[s] \restriction v = B_i \restriction v$ for all $i < N$. We will then conclude from Lemma 2.22 that (2.8) is satisfied.

The tree of strategies. The strategies described above are organized along with their possible outcomes into a tree T. We define T by recursion on the length of its nodes. In order to keep the recursion going, we need some auxiliary notions. Let L be a priority list of all the requirements in the construction (including requirements $\mathcal{M}_k$ corresponding to the strategies M_k), such that if $i < j$ then $\mathcal{R}_i$ is listed before $\mathcal{R}_j$ and $\mathcal{M}_i$ is listed before $\mathcal{M}_j$. If $\sigma \in T$ then we make the following definitions.

1. $\mathcal{R}_e$ *is satisfied in* σ if there are $\tau \subseteq \sigma$, i, j, e', m, n, and r such that $\langle S^{\tau}_{i,e'}, c, m, n, e, r\rangle \in \sigma$ or $\langle O^{\tau}_{i,j,e'}, c, m, n, e, r\rangle \in \sigma$.

2. $\mathcal{S}_{i,e}$ *is satisfied in* σ if there are $\tau \subseteq \sigma$, n, and r such that $\langle S^{\tau}_{i,e}, d, n, r\rangle \in \sigma$.

3. $\mathcal{O}_{i,j,e}$ *is satisfied in* σ if there are $\tau \subseteq \sigma$, n, and r such that $\langle O^{\tau}_{i,j,e}, d, n, r\rangle \in \sigma$.

4. R^{τ}_e *is injured in* σ if $\langle R^{\tau}_e\rangle \in \sigma$ and there are α, $j < e$, i, e', k, m, n, and r such that $\tau \subset \alpha \subseteq \sigma$ and either $\langle S^{\alpha}_{i,e'}, c, m, n, j, r\rangle \in \sigma$ or $\langle O^{\alpha}_{i,k,e'}, c, m, n, j, r\rangle \in \sigma$.

5. $N^{\tau}_{i,k,l,e}$ *is injured in* σ if there is an r such that $\langle N^{\tau}_{i,k,l,e}, r\rangle \in \sigma$ and there are $\tau \subset \alpha \subseteq \sigma$, j, i', e', k', m, n, and r' such that $\mathcal{R}_j$ has stronger priority (that is, appears earlier in L) than $\mathcal{N}_{i,k,l,e}$ and either $\langle S^{\alpha}_{i',e'}, c, m, n, j, r'\rangle \in \sigma$ or $\langle O^{\alpha}_{i',k',e'}, c, m, n, j, r'\rangle \in \sigma$.

The root of T is $\langle K^{\lambda}\rangle$, where λ is the empty sequence. Suppose that we have defined "$\sigma \in T$" for all σ of length k. Let $\sigma \in T$ be of length k. We define the immediate successors of σ as follows:

Say that a requirement $\mathcal{X}$ *requires attention at* σ if

1. $\mathcal{X} = \mathcal{R}_e$, $\mathcal{R}_e$ is not satisfied in σ, and every strategy R^{τ}_e in σ is injured in σ,

2. $\mathcal{X} = \mathcal{S}_{i,e}$ and $\mathcal{S}_{i,e}$ is not satisfied in σ,

3. $\mathcal{X} = \mathcal{O}_{i,j,e}$ and $\mathcal{O}_{i,j,e}$ is not satisfied in σ,

4. $\mathcal{X} = \mathcal{N}_{i,k,l,e}$ and every strategy $N^{\tau}_{i,k,l,e}$ in σ is injured in σ, or

5. $\mathcal{X} = \mathcal{M}_k$ and no strategy for $\mathcal{M}_k$ appears in σ.

Let $\mathcal{X}$ be the first strategy in L that requires attention at σ. Let

$$\mathit{Active\_index}(\sigma) = \{j \mid \text{some } R^{\tau}_j \text{ appears in } \sigma \text{ and } \mathcal{R}_j \text{ is not satisfied in } \sigma\}.$$

We have the following cases:

1. $\mathcal{X} = \mathcal{R}_e$. Then the only immediate successor of σ is $\sigma^\frown\langle R^{\sigma}_e\rangle$.

2. $\mathcal{X} = \mathcal{S}_{i,e}$. Then let

$$\mathit{Outcome} = \big\{\langle c, m, n, j, r\rangle\ ,\ \langle d, n, r\rangle \mid j \in \mathit{Active\_index}(\sigma), m, n, j, r \in \omega\big\}.$$

   The immediate successors of σ are the sequences of the form $\sigma^\frown\langle S^{\sigma}_{i,e}, o\rangle$, where $o \in \mathit{Outcome}$.

3. $\mathcal{X} = \mathcal{O}_{i,j,e}$. Then let *Outcome* be as above. The immediate successors of σ are the sequences of the form $\sigma^\frown\langle O^{\sigma}_{i,j,e}, o\rangle$, $o \in \mathit{Outcome}$.

4. $\mathcal{X} = \mathcal{N}_{i,k,l,e}$. Then the immediate successors of σ are the sequences of the form $\sigma^\frown\langle N^{\sigma}_{i,k,l,e}, r\rangle$, $r \in \omega$.

5. $\mathcal{X} = \mathcal{M}_k$. Then the immediate successors of σ are the sequences of the form $\sigma^\frown\langle M^{\sigma}_k, r\rangle$, $r \in \omega$.

There is a notion of α being to the left of β in T derived from the well-ordering of the immediate successors of $\sigma \in T$ generated from the following rules. (This may not appear to be a well-ordering because of clauses 2(c) and 3(c) below, but notice that if $S^{\sigma}_{i,e}$ is the strategy corresponding to σ then σ has successors of the form $\langle S^{\sigma}_{i,e}, c, m, n, k, r\rangle$ for only finitely many k, and similarly for $O^{\sigma}_{i,j,e}$.)

1. The strategy corresponding to σ is R^{σ}_{e}. Then σ has only one immediate successor.

2. The strategy corresponding to σ is $S^{\sigma}_{i,e}$. Then

   (a) $\langle S^{\sigma}_{i,e}, d, n, r\rangle < \langle S^{\sigma}_{i,e}, d, n, r'\rangle$ if $r < r'$;

   (b) $\langle S^{\sigma}_{i,e}, d, n, r\rangle < \langle S^{\sigma}_{i,e}, c, m, n, j, r'\rangle$;

   (c) $\langle S^{\sigma}_{i,e}, c, m, n, j, r\rangle < \langle S^{\sigma}_{i,e}, c, m', n, j', r'\rangle$ if $j > j'$ or if $j = j'$ and $m < m'$ or if $j = j'$, $m = m'$, and $r < r'$;

   (d) $\langle S^{\sigma}_{i,e}, c, m, n, j, r\rangle < \langle S^{\sigma}_{i,e}, d, n+1, r'\rangle$.

3. The strategy corresponding to σ is $O^{\sigma}_{i,j,e}$. Then

   (a) $\langle O^{\sigma}_{i,j,e}, d, n, r\rangle < \langle O^{\sigma}_{i,j,e}, d, n, r'\rangle$ if $r < r'$;

   (b) $\langle O^{\sigma}_{i,j,e}, d, n, r\rangle < \langle O^{\sigma}_{i,j,e}, c, m, n, k, r'\rangle$;

   (c) $\langle O^{\sigma}_{i,j,e}, c, m, n, k, r\rangle < \langle O^{\sigma}_{i,j,e}, c, m', n, k', r'\rangle$ if $k > k'$ or if $k = k'$ and $m < m'$ or if $k = k'$, $m = m'$, and $r < r'$;

   (d) $\langle O^{\sigma}_{i,j,e}, c, m, n, k, r\rangle < \langle O^{\sigma}_{i,j,e}, d, n+1, r'\rangle$.

4. The strategy corresponding to σ is $N^{\sigma}_{i,k,l,e}$. Then $\langle N^{\sigma}_{i,k,l,e}, r\rangle < \langle N^{\sigma}_{i,k,l,e}, r'\rangle$ if $r < r'$.

5. The strategy corresponding to σ is M^{σ}_{k}. Then $\langle M^{\sigma}_{k}, r\rangle < \langle M^{\sigma}_{k}, r'\rangle$ if $r < r'$.

2.23. Definition. For $\alpha, \beta \in T$, say that α is to the left of β, $\alpha <_{\mathrm{L}} \beta$, if there exist $\sigma, \tau, \rho \in T$ such that τ and ρ are immediate successors of σ, $\tau\big(|\sigma|+1\big) < \rho\big(|\sigma|+1\big)$, $\tau \subseteq \alpha$, and $\rho \subseteq \beta$.

Define

$$\mathit{Active\_strategy}(\sigma) = \{R^{\tau}_{e} \mid \tau \subset \sigma,\ e \in \mathit{Active\_index}(\sigma),\ \text{and } R^{\tau}_{e} \text{ is not injured in } \sigma\}.$$

We establish two properties of our tree of strategies that we have been assuming throughout this proof.

2.24. Lemma. *1. If the strategies corresponding to α and β are M^{α}_{j} and M^{β}_{k}, respectively, and $\alpha \subset \beta$ then $j < k$.*

2. If $R^{\alpha}_{j}, R^{\beta}_{k} \in \mathit{Active\_strategy}(\sigma)$ and $\alpha \subset \beta$ then $j < k$.

Proof. The first part of the lemma follows from the fact that if $j < k$ then $\mathcal{M}_j$ appears before $\mathcal{M}_k$ in our priority list.

Now suppose that $R^{\alpha}_{j}, R^{\beta}_{k} \in \mathit{Active\_strategy}(\sigma)$ and $\alpha \subset \beta$. Since $\mathcal{R}_k$ requires attention at β, every R^{τ}_{k} in β is injured in β, and hence in σ. Thus $\tau \subset \beta \Rightarrow R^{\tau}_{k} \notin \mathit{Active\_strategy}(\sigma)$, which implies that $j \neq k$.

Assume for a contradiction that $j > k$. In this case, $\mathcal{R}_k$ appears before $\mathcal{R}_j$ in our priority list, so $\mathcal{R}_k$ does not require attention at α. (Otherwise, the strategy corresponding to α would be R_k^α.) So either $\mathcal{R}_k$ is satisfied in α or there is a strategy R_k^τ in α that is not injured in α. The first case cannot hold, since if $\mathcal{R}_k$ were satisfied in α then it would also be satisfied in β and hence the strategy corresponding to β would not be R_k^β. So the second case holds. Since $\mathcal{R}_k$ requires attention at β, R_k^τ is injured in β. But this means that R_j^α is injured in β, which implies that R_j^α is injured in σ, contradicting the hypothesis that $R_j^\alpha \in \mathit{Active\_strategy}(\sigma)$. This establishes the second part of the lemma. □

The construction. As mentioned earlier, we assign infinite disjoint uniformly computable sets P_σ in the intersection of the odd numbers with $\omega^{[\geqslant 2N]}$ to each $\sigma \in T$.

The construction proceeds in stages. At each stage $s > 0$, a finite path $\sigma[s]$ will be defined and the strategies in $\sigma[s]$ will act. For a given node τ, if $\sigma[s] <_{\mathrm{L}} \tau$ then all actions taken by X^τ, the strategy corresponding to τ, during previous stages are canceled. We say that X^τ has been *initialized*. In particular, any variables X^τ may have defined (such as $m_{n,\alpha}^\tau$ or move_τ) have their values canceled; if later redefined, these variables will be assigned fresh large numbers. If X^τ is an N-strategy, this initialization process includes canceling all attacks on X^τ currently in effect.

On the other hand, if $\tau <_{\mathrm{L}} \sigma[s]$ then any action taken by X^τ during previous stages remains intact, and any uncanceled restraint imposed by it is respected during stage s. In particular, any number through which an N-strategy in $\sigma[s]$ launches an attack during stage s must be greater than any such restraint. Thus, we modify clause (c) of the requirements for a strategy $N_{i,k,l,e}^\sigma$ to launch an attack through x (see page 17). Recall that, in this context, $q = \min\big(\zeta_{i,k,l}(x,s), \tilde{\zeta}_{i,k,l}(x,s)\big)$.

(c) q is greater than all restraints in σ and all uncanceled restraints previously imposed by strategies X^τ with $\tau <_{\mathrm{L}} \sigma$.

At stage 0, let $A[0] = B_i[0] = C_i[0] = D_{i,k,l}[0] = \emptyset$ for $i < N$ and $k, l \in \omega$.

During stage $s > 0$, there are s many substages, beginning with substage 0. Substage m begins with a value for $\sigma = \sigma[s] \restriction m \in T$ and proceeds as follows.

Let X^σ be the strategy that appears as the first coordinate in each of σ's immediate successors. Play X^σ as described in previous sections until it completes its activity with outcome o. Let $\sigma[s](m) = \langle X^\sigma, o\rangle$. Cancel the history of the activity of strategies associated with τ such that $\sigma[s] \restriction m+1 <_{\mathrm{L}} \tau$.

This completes the description of the construction. As usual, we define the true path TP of the construction to be the leftmost path in T visited infinitely often by $\sigma[s]$. We will abuse notation and say that a strategy X^σ is on TP if σ is on TP.

We now wish to show that TP is infinite. We begin with an auxiliary lemma.

2.25. Lemma. *Suppose that $\sigma \in TP$ and all strategies on σ other than the R-strategies enumerate E-computable sets into each $A \oplus B_l$, $l < N$. If R_k^α is on σ and $R_k^\alpha \notin \mathit{Active\_strategy}(\sigma)$ then the numbers put into A for the purpose of coding the movement of the Γ_α markers form a computable set.*

Proof. Let R_k^α on σ be such that $R_k^\alpha \notin$ *Active_strategy*(σ). There exist $\tau \subset \sigma$, $i, j, e, m, n, r \in \omega$, and $k' \leqslant k$ such that either $\langle S_{i,e}^\tau, c, m, n, k', r\rangle \in \sigma$ or $\langle O_{i,j,e}^\tau, c, m, n, k', r\rangle \in \sigma$. Let us assume that $\langle S_{i,e}^\tau, c, m, n, k', r\rangle \in \sigma$, since the proof in the other case is analogous.

We can assume by induction that the lemma holds of τ, which means that the hypotheses of Lemma 2.8 are satisfied. Thus, for all but finitely many m, either the marker $\Gamma_\alpha(m)$ has no position from some point on or $S_{i,e}^\tau$ removes it from its position infinitely often.

Let m be the least number such that $\Gamma_\alpha(m)$ does not have a limiting position, and let s be a stage by which all markers $\Gamma_\alpha(n)$, $n < m$, have reached their limiting positions. For $x \in \omega$, let $f(x)$ be the least stage after s such that all markers $\Gamma_\alpha(n)$, $n \geqslant m$, have moved past x by stage $f(x)$. Clearly, f is computable, and no strategy puts a number into $A \upharpoonright x$ for the purpose of coding the movement of a Γ_α marker after stage $f(x)$. □

2.26. Lemma. *TP is infinite.*

Proof. Let $\sigma \in TP$ and assume by induction that all strategies on σ other than the R-strategies enumerate E-computable sets into each $A \oplus B_l$, $l < N$. Let X^σ be the strategy corresponding to σ.

By the definition of TP, X^σ acts infinitely often and can restrain all numbers from entering any $A \oplus B_l$, $l < N$, except for the finite set of numbers enumerated by strategies to the left of σ, numbers put in by strategies on σ, and numbers put in during a recovery process for the purpose of coding the movement of a marker associated with an R-strategy in σ. Thus, by Lemma 2.25, X^σ can restrain all numbers from entering any $A \oplus B_l$, $l < N$, except for the enumeration of finitely many fixed E-computable sets and numbers put in during a recovery process for the purpose of coding the movement of a marker associated with a strategy in *Active_strategy*(σ). So we conclude the following.

1. If $X^\sigma = R_e^\sigma$ then of course σ has a leftmost successor visited infinitely often.

2. If $X^\sigma = S_{i,e}^\sigma$ then the hypotheses of Lemma 2.8 are satisfied, and thus, by that lemma, σ has a leftmost successor visited infinitely often and $S_{i,e}^\sigma$ enumerates a computable set into each $A \oplus B_l$, $l < N$.

3. If $X^\sigma = O_{i,j,e}^\sigma$ then the hypotheses of Lemma 2.13 are satisfied, and thus, by that lemma, σ has a leftmost successor visited infinitely often and $O_{i,j,e}^\sigma$ enumerates a computable set into each $A \oplus B_l$, $l < N$.

4. If $X^\sigma = N_{i,k,l,e}^\sigma$ then the hypotheses of Lemmas 2.11 and 2.12 are satisfied. Thus, by Lemma 2.11, σ has a leftmost successor visited infinitely often, while by Lemma 2.12, $N_{i,k,l,e}^\sigma$ enumerates an E-computable set into each $A \oplus B_l$, $l < N$.

5. Suppose that $X^\sigma = M_k^\sigma$. The above shows that the first three hypotheses of Lemma 2.18 are satisfied. That the fourth hypothesis is also satisfied follows from the definition of TP. Thus, by Corollary 2.19, M_k^σ's action is finite, which immediately gives us that σ has a leftmost successor visited infinitely often and M_k^σ enumerates a computable set into each $A \oplus B_l$, $l < N$.

Thus, by induction, TP is infinite. □

Examining the various direct codings in our construction, we get the following fact.

2.27. Proposition. *(2.0), (2.5), and (2.7) are satisfied.*

Say that a requirement is satisfied in TP if it is satisfied in some $\sigma \in TP$ and that a strategy is injured in TP if it is injured in some $\sigma \in TP$.

2.28. Lemma. *For each requirement $\mathcal{X}$ there are only finitely many $\sigma \in TP$ at which $\mathcal{X}$ requires attention. Let $e, k, l \in \omega$, $i, j < N$.*

1. *Either $\mathcal{R}_e$ is satisfied in TP or there exists an R_e^σ on TP that is not injured in TP.*
2. *There exists a $\sigma \in TP$ such that $N_{i,k,l,e}^\sigma$ is not injured in TP.*
3. *Both $\mathcal{S}_{i,e}$ and $\mathcal{O}_{i,j,e}$ are satisfied in TP.*
4. *There exists exactly one σ such that M_k^σ is on TP.*

Proof. Let $\mathcal{X}$ be a requirement in our priority list and assume by induction that, for each requirement $\mathcal{Y}$ of stronger priority, there are only finitely many $\sigma \in TP$ at which $\mathcal{Y}$ requires attention. We have three cases.

1. $\mathcal{X} = \mathcal{R}_e$ and $\mathcal{R}_e$ is not satisfied in TP or $\mathcal{X} = \mathcal{N}_{i,k,l,e}$. If X^α is on TP and is injured in $\beta \in TP$ then there exists a $j \in \mathit{Active\_index}(\beta)$ such that $\mathcal{R}_j$ has stronger priority than $\mathcal{X}$ and is satisfied in β. Now for all $\tau \supset \beta$, $j \notin \mathit{Active\_index}(\tau)$. Thus there can only be finitely many α such that X^α is on TP and is injured in TP. Let σ be the shortest string on TP that is longer than all such α and such that no requirement of stronger priority than $\mathcal{X}$ requires attention at σ. (Such a σ exists by the induction hypothesis.) Then the strategy corresponding to σ is X^σ and this strategy is not injured in TP.
2. $\mathcal{X} = \mathcal{S}_{i,e}$ or $\mathcal{X} = \mathcal{O}_{i,j,e}$. If $\langle X^\alpha, c, m, n, j, r\rangle$ is on TP then for some $j \in \omega$ and $\beta \subset \alpha$, $R_j^\beta \in \mathit{Active\_strategy}(\alpha)$. It must be the case that $\mathcal{R}_j$ has stronger priority than $\mathcal{S}_{i,e}$, since if $\mathcal{S}_{i,e}$ had stronger priority than $\mathcal{R}_j$ then the strategy corresponding to β would be $S_{i,e}^\beta$. Now for all $\tau \supset \alpha$, $j \notin \mathit{Active\_index}(\tau)$. So there can only be finitely many α such that for some $m, n, j, r \in \omega$, $\langle S_{i,e}^\alpha, c, m, n, j, r\rangle$ is on TP. Let σ be the shortest string on TP that is longer than all such α and such that no requirement of stronger priority than $\mathcal{S}_{i,e}$ requires attention at σ. Then for some $n, r \in \omega$, $\langle S_{i,e}^\sigma, d, n, r\rangle$ is on TP, and thus $\mathcal{S}_{i,e}$ is satisfied in TP.
3. $\mathcal{X} = \mathcal{M}_k$. Let σ be the shortest string on TP such that no requirement of stronger priority than $\mathcal{M}_k$ requires attention at σ. Then the strategy corresponding to σ is M_k^σ and $\mathcal{M}_k$ does not require attention at any $\tau \supset \sigma$.

In any case, there are only finitely many $\sigma \in TP$ at which $\mathcal{X}$ requires attention. □

2.29. Proposition. *For each $i < N$ and $e \in \omega$, $\mathcal{S}_{i,e}$ is satisfied.*

Proof. By Lemma 2.28, there exist σ, n, and r such that $\langle S_{i,e}^\sigma, d, n, r\rangle$ is on TP. By Lemma 2.8, $S_{i,e}^\sigma$ succeeds in satisfying $\mathcal{S}_{i,e}$. □

A similar argument establishes the following proposition.

2.30. Proposition. *For each $i, j < N$ and $e \in \omega$, if $i \not\leq_0 j$ then $\mathcal{O}_{i,j,e}$ is satisfied.*

2.31. Proposition. *For each $e \in \omega$, $\mathcal{R}_e$ is satisfied.*

Proof. If there exists either $\langle S^\tau_{i,e'}, c, m, n, e, r\rangle$ or $\langle O^\tau_{i,j,e'}, c, m, n, e, r\rangle$ on TP then, by Lemmas 2.8 and 2.13, we are done. Otherwise, by Lemma 2.28, there exists an R^σ_e on TP that is not injured in TP.

In this case, by Lemmas 2.8 and 2.13, each S or O-strategy on TP cancels the position of any particular Γ_σ-marker only finitely often. The same is true of each M-strategy on TP, since by Lemma 2.18 each such strategy eventually preserves a permanent configuration, and thus has finite action. Strategies to the left of TP act only finitely often, while those to the right of TP are initialized infinitely often, and thus each S, O, or M-strategy not on TP also cancels the position of any particular Γ_σ-marker only finitely often. N-strategies and R-strategies do not cancel the position of Γ_σ markers at all, except possibly while running a recovery process. Finally, no strategy X^τ can cancel the position of a marker $\Gamma_\sigma(n)$ if n is less than the least τ-stage, except while running a recovery process. Thus, the rules for the movement of the Γ_σ-markers are respected. So Lemma 2.1 applies and $\mathcal{R}_e$ is satisfied. □

By Lemma 2.22, any M^σ_k on TP will eventually preserve a correct σ-configuration. The following lemma is the last element we need in order to show that (2.8) holds. It is not necessarily true that if M^σ_k preserves an F and E-correct configuration then the construction never again moves to the left of σ. For this to hold, we need the configuration to be correct.

2.32. Lemma. *Suppose that M^σ_k preserves a correct σ-configuration v larger than k at a σ-stage s. Then no number enters $A \restriction v$ or any $B_p \restriction v$, $p < N$, after stage s.*

Proof. By Lemma 2.17, it is enough to show that for all $t > s$, $\sigma[t] \not<_{\mathrm{L}} \sigma$. We proceed by double induction on stages $t > s$ and, during a stage t, along σ. That is, given $t > s$ and $\alpha \subseteq \sigma$, we assume that

1. for all stages u such that $s < u < t$, $\sigma[u] \not<_{\mathrm{L}} \sigma$ and
2. for all $\beta \subset \alpha$, $\sigma[t] \not<_{\mathrm{L}} \beta$,

and show that $\sigma[t] \not<_{\mathrm{L}} \alpha$.

Notice that, by the proof of Lemma 2.17, the inductive hypothesis implies that no number has entered $A \restriction v$ or any $B_p \restriction v$ or $C_p \restriction v$, $p < N$, since the end of M^σ_k's stage s action.

Let $\beta \in T$, X^β, and o be such that $\alpha = \beta^\frown\langle X^\beta, o\rangle$. If $\beta <_{\mathrm{L}} \sigma[t]$ then we are done, so assume that $\beta \subseteq \sigma[t]$.

If $X^\beta = R^\beta_e$ then α is the only immediate successor of β, so there is nothing to show.

If $X^\beta = M^\beta_{k'}$ then, by induction, $M^\beta_{k'}$ is still preserving the β-configuration it was preserving at stage s. Thus its outcome at stage t is o.

If $X^\beta = N^\beta_{i,j,l,e}$ then there are two possibilities. If $o = 0$ then we are done. Otherwise, the restraint o imposed by $N^\beta_{i,j,l,e}$ at stage s has not been violated, so that $N^\beta_{i,j,l,e}$'s outcome at stage t must be equal to o.

The two remaining possibilities are very similar. We do the case $X^\beta = S^\beta_{i,e}$. In this case o has one of the forms $\langle d, n, r\rangle$ or $\langle c, m, n, l, r\rangle$.

If $u < w$ and $r(n-1,u)$ and $r(n-1,w)$ are both defined then $r(n-1,u) \leqslant r(n-1,w)$. Thus if $o = \langle d,n,r \rangle$ then $S^{\beta}_{i,e}$'s outcome at stage t cannot be of the form $\langle d,n,r' \rangle$, $r' < r$, while if $o = \langle c,m,n,l,r \rangle$ then it cannot be of the form $\langle c,m,n,l,r' \rangle$, $r' < r$.

For all $n' < n$, $v > \varphi_e(A;n')[s] \geqslant n'$, $\Phi_e(A)[s] \upharpoonright n' = C_i[s] \upharpoonright n'$, and $\Phi_e(A;n')[s]$ converges. So, for all $n' < n$, $\Phi_e(A)[t] \upharpoonright n' = C_i[t] \upharpoonright n'$ and $\Phi_e(A;n')[t]$ converges. This means that $S^{\beta}_{i,e}$'s outcome at stage t cannot be of the form $\langle d,n',r' \rangle$, $n' < n$.

Now let $n' < n$ and $R^{\tau}_j \in$ *Active_strategy*(β). Clearly, $\Phi_j(A \oplus B_i; m^{\beta}_{n',\tau})[s]$ converges, for all $m < m^{\beta}_{n',\tau}$, if $\Gamma_\tau(m)$ has a position at stage s which was assigned at stage u then $\Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[u]$, and, for all $R^{\tau'}_k \in$ *Active_strategy*(β), if $\gamma_{\tau'}(m^{\beta}_{n',\tau'}, s)$ is defined then $\gamma_{\tau'}(m^{\beta}_{n',\tau'}, s) > \varphi_j(A \oplus B_i; m^{\beta}_{n',\tau})[s]$. It follows from the definition of r and the fact that $v > r$ that these facts still hold with s replaced by t. This means that $S^{\beta}_{i,e}$'s outcome at stage t cannot be of the form $\langle c,m',n',l',r' \rangle$, $n' < n$.

If $o = \langle d,n,r \rangle$ then we are done.

So assume that $o = \langle c,m,n,l,r \rangle$. By the monotonicity of the assignment of values to the $m^{\beta}_{n,\tau}$'s, $S^{\beta}_{i,e}$'s outcome at stage t cannot be of the form $\langle c,m',n,l,r' \rangle$, $m' < m$. We need to show that it also cannot be of either of the forms $\langle d,n,r' \rangle$ or $\langle c,m',n,l',r' \rangle$, $l' > l$.

Since now $r > \varphi_e(A;n)[s]$, $\Phi_e(A)[s] \upharpoonright n = C_i[s] \upharpoonright n$, and $\Phi_e(A;n)[s]$ converges, the same argument as before takes care of the first form.

The fact that $v > \max\{\varphi_k(A \oplus B_i; m^{\beta}_{n,\tau})[s] \mid R^{\tau}_k \in \textit{Active\_strategy}(\beta),\ k > l\}$ means that no numbers less than this maximum have entered A or B_i since the end of stage s. Thus each computation $\Phi_k(A \oplus B_i; m^{\beta}_{n,\tau})$, $R^{\tau}_k \in$ *Active_strategy*(β), $k > l$, has not changed between the end of stage s (when it existed, by Definition 2.21) and stage t. This takes care of the second form. □

2.33. Proposition. *For each $i < N$, $A \oplus B_i \leqslant_T F$.*

Proof. F can tell when M^{σ}_k preserves a correct configuration at a σ-stage s. By Lemma 2.22, such s and M^{σ}_k exist (take any M^{σ}_k on TP), while by Lemma 2.32, for any such s and M^{σ}_k, $A[s] \upharpoonright k = A \upharpoonright k$ and $B_i[s] \upharpoonright k = B_i \upharpoonright k$. □

2.34. Lemma. *If $N^{\sigma}_{i,k,l,e}$ is on TP and is not injured in TP then it satisfies $\mathcal{N}_{i,k,l,e}$.*

Proof. Assume for a contradiction that $W_k, W_l \nleqslant_{\mathrm{T}} A \oplus B_i$ and $D_{i,k,l} = \Phi_e(A)$. Then every attack on $N^{\sigma}_{i,k,l,e}$ is eventually canceled.

Let $x \in \omega$. Since $N^{\sigma}_{i,k,l,e}$ is not injured in TP, some weaker priority strategy $S^{\tau}_{e',i}$ on TP will eventually find and preserve a correct σ-i-configuration greater than $\varphi_e(A;x)$. Indeed, it follows from the fact that *Active_strategy*$(\sigma) \subseteq$ *Active_strategy*(α) for all α such that $\sigma \subseteq \alpha \in TP$ that we can take any $S^{\tau}_{e',i}$, $\sigma \subset \tau \in TP$, that succeeds in satisfying $\mathcal{S}_{e',i}$ and such that $\Phi_{e'}(A) \upharpoonright \varphi_e(A;x)+1 = C_i \upharpoonright \varphi_e(A;x)+1$.

Let s_0 be least stage such that for all $s > s_0$, $\sigma[s] \not<_{\mathrm{L}} \sigma$. Let $s(x)$ be the least stage after s_0 at which a correct σ-i-configuration greater than $\varphi_e(A;x)$ exists and let $v(x)$ be the least such configuration at stage $s(x)$. Notice that $A \oplus B_i$ can compute $s(x)$ and $v(x)$. No attack on $N^{\sigma}_{i,k,l,e}$ through $y \leqslant x$ can be launched during or after stage $s(x)$, for otherwise this attack would never be canceled.

Since $s(x)$ is total and computable in $A \oplus B_i$,

$$U = \{x \in P_\sigma \mid W_k[s(x)] \upharpoonright x \neq W_k \upharpoonright x \ \wedge \ W_l[s(x)] \upharpoonright x \neq W_l \upharpoonright x\}$$

is infinite and computably enumerable in $A \oplus B_i$. (This is by two applications of Lemma 2.9.) Let

$$V = \{x \in U \mid x \text{ is greater than all restraints in } \sigma \text{ and all numbers mentioned at stages } t \text{ with } \sigma[t] <_{\mathrm{L}} \sigma\}.$$

V is infinite and c.e. in $A \oplus B_i$. For $x \in V$, define $w(x)$ to be the least σ-stage after $s(x)$ such that

1. $\min\Big(\zeta_{i,k,l}\big(x, w(x)\big), \tilde{\zeta}_{i,k,l}\big(x, w(x)\big)\Big) \notin A[w(x)]$,
2. $\min\Big(\zeta_{i,k,l}\big(x, w(x)\big), \tilde{\zeta}_{i,k,l}\big(x, w(x)\big)\Big) > v(x)$,
3. $D_{i,k,l}(x)[w(x)] = \Phi_e(A; x)[w(x)]$ and the computation $\Phi_e(A; x)$ has settled down by stage $w(x)$, and
4. all attacks on $N^\sigma_{i,k,l,e}$ launched before stage $s(x)$ have been canceled by the beginning of stage $w(x)$.

(Such a stage exists by the definition of U, which implies that both $\zeta_{i,k,l}(x)$ and $\tilde{\zeta}_{i,k,l}(x)$ move after stage $s(x)$, and the hypothesis that $D_{i,k,l} = \Phi_e(A)$.)

Clearly, w is an $A \oplus B_i$-computable partial function. Now notice that 1, 2, and 3 above continue to hold with any $t > w(x)$ substituted in for $w(x)$. Thus, conditions 2(a), 2(b), and 2(d) on page 17, as well as the modified condition 2(c) on page 29, hold for x at all stages greater than or equal to $w(x)$. The fact, mentioned above, that no attack through $y \leqslant x$ can be launched after stage $s(x)$, combined with 4, implies that for all stages $t \geqslant w(x)$, there is no attack on $N_{i,k,l,e}$ through $y \leqslant x$ in effect at stage t, so that condition 2(f) on page 17 also holds.

This implies that if $F[t+1] \upharpoonright x \neq F[t] \upharpoonright x$ for some $t \geqslant w(x)$ then, at the first σ-stage after t, conditions 2(a), 2(b), and 2(d)–2(f) on page 17, as well as the modified condition 2(c) on page 29, will hold, so that an attack on $N^\sigma_{i,k,l,e}$ through some $y \leqslant x$ will be launched. Since we know this cannot happen, $F[w(x)] \upharpoonright x = F \upharpoonright x$.

As V is infinite and c.e. in $A \oplus B_i$ and $w(x)$ is a partial $A \oplus B_i$-computable function defined for all $x \in V$, this means that $F \leqslant_{\mathrm{T}} A \oplus B_i$. Now let $j < N$ be such that $j \neq i$. By Proposition 2.33, $A \oplus B_j \leqslant_{\mathrm{T}} F \leqslant_{\mathrm{T}} A \oplus B_i$. However, as shown on page 6, it follows from (2.0), (2.1), and (2.2) (which in turn follow from Propositions 2.27, 2.29, and 2.31, respectively) that $\deg(A \oplus B_i)$ and $\deg(A \oplus B_j)$ are incomparable. Thus we have a contradiction. $\square$

By Lemma 2.28, for all $i < N$, $k, l, e \in \omega$, there exists an $N^\sigma_{i,k,l,e}$ satisfying the hypothesis of the previous lemma. Thus we have the following result.

2.35. Proposition. *For all $i < N$, $k, l, e \in \omega$, $\mathcal{N}_{i,k,l,e}$ is satisfied.*

Propositions 2.10, 2.27, 2.29, 2.30, 2.31, 2.33, and 2.35 imply that conditions (2.0)–(2.8) are satisfied. This completes the proof of Theorem 1.4. $\blacksquare$

3 Proof of Theorem 1.6

In this section we prove the following theorem.

1.6. Theorem. *Let $\leqslant_0$ be a partial ordering on $\{0,\ldots,N-1\}$ with at least three minimal elements. There are c.e. degrees $\mathbf{a},\mathbf{b_0},\ldots,\mathbf{b_{N-1}},\mathbf{c}$ satisfying (1.0)–(1.4) and*

$$\mathbf{c} = \bigcup_{i<N}\left(\bigcap_{j\neq i}\mathbf{b_j}\right). \tag{1.5}$$

The proof of Theorem 1.6 is similar to that of Theorem 1.4, so we describe only the necessary changes. As before, we construct sets A, B_i, C_i, and $D_{i,k,l}$ ($i<N; k,l\in\omega$), of which A, B_i, and C_i are c.e., satisfying (2.0)–(2.8). To ensure that (1.5) is satisfied, we have additional requirements

$$\begin{aligned}\hat{\mathcal{R}}_{i,e}:\ f = \Phi_e(A\oplus B_0) = \cdots = \Phi_e(A\oplus B_{i-1}) = \Phi_e(A\oplus B_{i+1}) = \cdots\\ \cdots = \Phi_e(A\oplus B_{N-1})\text{ total }\Rightarrow f\leqslant_{\mathrm{T}} A\oplus C_i.\end{aligned}$$

The requirements $\hat{\mathcal{R}}_{i,e}$ ensure that

$$\mathbf{b_o}\cap\cdots\cap\mathbf{b_{i-1}}\cap\mathbf{b_{i+1}}\cap\cdots\cap\mathbf{b_{N-1}} = \mathbf{c_i},$$

which implies (1.5).

Each strategy $\hat{R}^\sigma_{i,e}$ for satisfying $\hat{\mathcal{R}}_{i,e}$ will work very much like R^σ_e would, but the positions of its markers $\hat{\Gamma}_\sigma(n)$ will be coded into C_i (and hence also into each B_j, $j\neq i$) instead of into A.

More formally, each strategy $\hat{R}^\sigma_{i,e}$ for satisfying $\mathcal{R}_{i,e}$ uses movable markers $\hat{\Gamma}_\sigma(n),\ n\in\omega$, which take positions in P_σ. We denote the position of $\hat{\Gamma}_\sigma(n)$ at stage s by $\hat{\gamma}_\sigma(n,s)$. The movement of these markers are subject to the following rules:

1. Suppose that s is a σ-stage and, at the beginning of $\hat{R}^\sigma_{i,e}$'s stage s action, $\Phi_e(A\oplus B_0)[s]\restriction n+1 = \cdots = \Phi_e(A\oplus B_{i-1})[s]\restriction n+1 = \Phi_e(A\oplus B_{i+1})[s]\restriction n+1 = \cdots = \Phi_e(A\oplus B_{N-1})[s]\restriction n+1$, $\Phi_e(A\oplus B_k;n)[s]\downarrow$ for all $k<N$, $k\neq i$, and $\hat{\Gamma}_\sigma(n)$ does not have a position. Then at stage s, $\hat{\Gamma}_\sigma(n)$ must be assigned a position larger than any number previously mentioned in the construction. Furthermore, this is the only situation in which a $\hat{\Gamma}$-marker is assigned a new position.

2. If s is a σ-stage, $\hat{\Gamma}_\sigma(n)$ has a position $\hat{\gamma}_\sigma(n,s)$ assigned at stage t and for all $k<N$, $k\neq i$, $\Phi_e(A\oplus B_k;n)[s]\neq\Phi_e(A\oplus B_k;n)[t]$, then at stage s, $\hat{\Gamma}_\sigma(n)$ must be removed from its position.

3. If $\hat{\Gamma}_\sigma(n)$ is removed from its position at a stage s then so must all $\hat{\Gamma}_\sigma(m),\ m>n$, and some number less than or equal to $\hat{\gamma}_\sigma(n,s)$ must enter C_i at stage s.

4. Except finitely often, $\hat{\Gamma}_\sigma(n)$ may not be removed from position $\hat{\gamma}_\sigma(n,s)$ unless at least one computation $\Phi_e(A\oplus B_k;n)$, $k<N$, $k\neq i$, has changed since $\hat{\Gamma}_\sigma(n)$ was assigned position $\hat{\gamma}_\sigma(n,s)$.

Arguing as in the proof of Lemma 2.1, we have the following lemma.

3.1. Lemma. *If there are infinitely many σ-stages and the above rules are obeyed then $\hat{\mathcal{R}}_{i,e}$ is satisfied.*

The $\hat{R}^\sigma_{i,e}$ *recovery process* is defined as follows:

Search for an x such that $\hat{\Gamma}_\sigma(x)$ has position $\hat{\gamma}_\sigma(x,s)$ assigned at stage t and for all $j < N$, $j \neq i$, $\Phi_e(A \oplus B_j; x)[s] \neq \Phi_e(A \oplus B_j; x)[t]$. If such an x is found then enumerate $\hat{\gamma}_\sigma(x,s)$ into C_i, and $\langle \hat{\gamma}_\sigma(x,s), i \rangle$ into each B_j, $j \neq i$; cancel the positions of all $\hat{\Gamma}_\sigma(y)$, $y \geqslant x$, and repeat the recovery process. Otherwise, end the recovery process.

For a sequence σ of strategies we redefine the σ-R recovery process to consist of iterating the R^τ_i and $\hat{R}^\alpha_{j,e}$ recovery processes for each R^τ_i and each $\hat{R}^\alpha_{j,e}$ in σ until each such process terminates without enumerating any numbers into A or any C_j, $j < N$.

The action of $\hat{R}^\sigma_{i,e}$ at a σ-stage s now consists of running the $\sigma^\frown\langle \hat{R}^\sigma_{i,e}\rangle$-$R$ recovery process and assigning fresh large positions to markers as necessary to obey rule 1, making sure that if $j < k$ then $\hat{\gamma}_\sigma(j,s) < \hat{\gamma}_\sigma(k,s)$.

Definition 2.2 must be replaced by the following definition.

3.2. Definition. A number v is a *σ-i-configuration* at stage s if

1. for all $R^\tau_j \in$ *Active_strategy*(σ) and all m, $\gamma_\tau(m,s) < v \Rightarrow \big[\varphi_j(A \oplus B_i; m)[s] < v \wedge \Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[t]$, where t is the stage at which $\Gamma_\tau(m)$ was assigned position $\gamma_\tau(m,s)\big]$, and

2. for all $\hat{R}^\tau_{k,j} \in$ *Active_strategy*(σ), $k \neq i$, and all m, $\hat{\gamma}_\tau(m,s) < v \Rightarrow \big[\varphi_j(A \oplus B_i; m)[s] < v \wedge \Phi_j(A \oplus B_i; m)[s] = \Phi_j(A \oplus B_i; m)[t]$, where t is the stage at which $\hat{\Gamma}_\tau(m)$ was assigned position $\hat{\gamma}_\tau(m,s)\big]$.

As before, if v is a *σ-i-configuration* at stage s then the new convention on uses described on page 22 implies that

1. for all $R^\tau_j \in$ *Active_strategy*(σ) and all m, $\gamma_\tau(m,s) < v \Rightarrow$ *max_min*$(\tau, m, s) < v$, and

2. for all $\hat{R}^\tau_{k,j} \in$ *Active_strategy*(σ), $k \neq i$, and all m, $\hat{\gamma}_\tau(m,s) < v \Rightarrow$ *max_min*$(\tau, m, s) < v$.

(*max_min*(σ, n, s) has the same meaning for $\hat{R}^\sigma_{i,e}$ as it would for R^σ_e, namely it is the maximum of the lengths of the shortest computations that have kept $\hat{R}^\sigma_{i,e}$ from moving $\hat{\Gamma}_\sigma(n)$ during the stages between the one at which $\hat{\Gamma}_\sigma(n)$ was assigned position $\hat{\gamma}_\sigma(n,s)$ and s.)

We must also alter the description of the action of $S^\sigma_{i,e}$ to take into account the presence of the $\hat{R}$-strategies. This includes redefining the meaning of the "change" outcomes of $S^\sigma_{i,e}$ so that an outcome of the form $\langle c, m, n, 2j, r\rangle$ will now correspond to a change in a computation associated with a strategy R^τ_j, while an outcome of the form $\langle c, m, n, 2\langle q, j\rangle + 1, r\rangle$ will correspond to a change in a computation associated with a strategy $\hat{R}^\tau_{q,j}$.

This should become clear from the following modified description of the action of the preservation half of $S^\sigma_{i,e}$ at a σ-stage s. (The coding half acts as before.)

Let $r(-1,s) = 0$. The preservation half of $S^\sigma_{i,e}$ proceeds in cycles, beginning with the cycle for 0. The n^{th} cycle operates as follows:

1. If $\Phi_e(A)[s] \upharpoonright n = C_i[s] \upharpoonright n$ and $\Phi_e(A;n)[s]$ converges then go to step 2. Otherwise, cancel the value of $m^{\sigma}_{n',\tau}$ and the position of $\delta(n')$ for each $n' \geqslant n$ and $R^{\tau}_j \in \mathit{Active\_strategy}(\sigma)$ or $\hat{R}^{\tau}_{q,j} \in \mathit{Active\_strategy}(\sigma)$, $q \neq i$; preserve $A \upharpoonright r(n-1,s)$ and $B_i \upharpoonright r(n-1,s)$ and end stage s activity with outcome $\big\langle d, n, r(n-1,s)\big\rangle$.

2. Assign fresh large values in P_σ to each undefined $m^{\sigma}_{n,\tau}$ such that $R^{\tau}_j \in \mathit{Active\_strategy}(\sigma)$ or $\hat{R}^{\tau}_{q,j} \in \mathit{Active\_strategy}(\sigma)$, $q \neq i$.

3. Search for the longest $\tau \subseteq \sigma$, if any, such that either $R^{\tau}_j \in \mathit{Active\_strategy}(\sigma)$ or $\hat{R}^{\tau}_{p,j} \in \mathit{Active\_strategy}(\sigma)$, $p \neq i$, and at least one of the following holds.

   (a) $\Phi_j(A \oplus B_i; m^{\sigma}_{n,\tau})[s]\!\uparrow$.

   (b) For some $m < m^{\sigma}_{n,\tau}$, $\Gamma_\tau(m)$ has a position assigned at some stage t and $\Phi_j(A \oplus B_i; m)[s] \neq \Phi_j(A \oplus B_i; m)[t]$.

   (c) For some $R^{\beta}_k \in \mathit{Active\_strategy}(\sigma)$, $\gamma_\beta(m^{\sigma}_{n,\beta}, s) \leqslant \varphi_j(A \oplus B_i; m^{\sigma}_{n,\tau})[s]$.

   (d) For some $\hat{R}^{\beta}_{q,k} \in \mathit{Active\_strategy}(\sigma)$, $q \neq i$, $\hat{\gamma}_\beta(m^{\sigma}_{n,\beta}, s) \leqslant \varphi_j(A \oplus B_i; m^{\sigma}_{n,\tau})[s]$.

   If such a τ exists then proceed as follows. Enumerate $\min\{\gamma_\alpha(m^{\sigma}_{n,\alpha}, s) \mid R^{\alpha}_k \in \mathit{Active\_strategy}(\sigma)$ and $\alpha \supseteq \tau\}$ into A and for $q \neq i$, enumerate $m = \min\{\hat{\gamma}_\alpha(m^{\sigma}_{n,\alpha}, s) \mid \hat{R}^{\alpha}_{q,k} \in \mathit{Active\_strategy}(\sigma)$ and $\alpha \supseteq \tau\}$ into C_q and $\langle m, q\rangle$ into each B_p, $p \neq q$. Follow this with the σ-R recovery process. For each $R^{\alpha}_k \in \mathit{Active\_strategy}(\sigma)$, if $\alpha \supseteq \tau$ then cancel the position of $\Gamma_\alpha(y)$ for all $y \geqslant m^{\sigma}_{n,\alpha}$, otherwise cancel the value of $m^{\sigma}_{n,\alpha}$. For each $\hat{R}^{\alpha}_{q,k} \in \mathit{Active\_strategy}(\sigma)$, $q \neq i$, if $\alpha \supseteq \tau$ then cancel the position of $\hat{\Gamma}_\alpha(y)$ for all $y \geqslant m^{\sigma}_{n,\alpha}$, otherwise cancel the value of $m^{\sigma}_{n,\alpha}$. For each $x > n$, each $R^{\alpha}_k \in \mathit{Active\_strategy}(\sigma)$, and each $\hat{R}^{\beta}_{q,k} \in \mathit{Active\_strategy}(\sigma)$, $q \neq i$, cancel the value of $m^{\sigma}_{x,\alpha}$ and $m^{\sigma}_{x,\beta}$. For each $x \geqslant n$, cancel the position of $\delta(x)$. Let $r = \max\big(r(n-1,s), \varphi_e(A;n)[s]+1\big)$. Preserve $A \upharpoonright r$ and $B_i \upharpoonright r$.

   If $R^{\tau}_j \in \mathit{Active\_strategy}(\sigma)$ then end stage s activity with outcome $\langle c, m^{\sigma}_{n,\tau}, n, 2j, r\rangle$. If $\hat{R}^{\tau}_{p,j} \in \mathit{Active\_strategy}(\sigma)$ then end stage s activity with outcome $\big\langle c, m^{\sigma}_{n,\tau}, n, 2\langle p, j\rangle + 1, r\big\rangle$.

4. Define

$$r(n,s) = \min\Big(\big\{\gamma_\tau(m^{\sigma}_{n,\tau}, s) \mid R^{\tau}_j \in \mathit{Active\_strategy}(\sigma)\big\} \cup \\ \cup \big\{\hat{\gamma}_\tau(m^{\sigma}_{n,\tau}, s) \mid \hat{R}^{\tau}_{q,j} \in \mathit{Active\_strategy}(\sigma),\ q \neq i\big\}\Big).$$

   If this set is empty then define

$$r(n,s) = \max\Big(\big\{\varphi_j(A \oplus B_i; m^{\sigma}_{n,\tau})[s] \mid R^{\tau}_j \in \mathit{Active\_strategy}(\sigma) \text{ or} \\ \hat{R}^{\tau}_{q,j} \in \mathit{Active\_strategy}(\sigma),\ q \neq i\big\} \cup \big\{\varphi_e(A;n)[s]+1\big\} \cup \big\{r(n,t) \mid t < s\big\}\Big).$$

   If $\delta(n)$ does not have a position then assign its new position $\delta(n,s)$ to be a fresh large number in P_σ. Begin the $(n+1)$st cycle.

The following version of Lemma 2.8 can now be proved in much the same way as before. The key fact to notice is that we do not need to worry about strategies $\hat{R}^{\tau}_{i,k}$, since they do not put numbers into B_i, and hence do not threaten σ-i-configurations.

3.3. Lemma. *Suppose that $S^{\sigma}_{i,e}$ acts infinitely often and can restrain all numbers from entering any $A \oplus B_p$, $p < N$, except for the enumeration of finitely many fixed E-computable sets and numbers put in during a recovery process for the purpose of coding the movement of a marker associated with a strategy in Active_strategy(σ). Suppose further that there is a stage after which no δ-marker used by the coding half of $S^{\sigma}_{i,e}$ can have its position canceled except during $S^{\sigma}_{i,e}$'s action. Then one of the following holds.*

1. *There is an n such that $S^{\sigma}_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A; n')$ for all $n' < n$ and either $\Phi_e(A; n-1)\downarrow \neq C_i(n-1)$ or $\Phi_e(A; n)\uparrow$.*

   Let s be a stage by which all of these configurations have stabilized, and let r be their supremum. $S^{\sigma}_{i,e}$'s outcome is infinitely often equal to $\langle d, n, r\rangle$, and it is never of the form $\langle d, n', r'\rangle$, $n' < n$, $\langle c, m, n', j, r'\rangle$, $n' < n$, or $\langle d, n, r'\rangle$, $r' < r$, after stage s.

   $S^{\sigma}_{i,e}$ cancels the position of any particular marker only finitely often.

2. *There is an n such that $\Phi_e(A) \restriction n = C_i \restriction n$, $\Phi_e(A; n)\downarrow$, and $S^{\sigma}_{i,e}$ finds permanent σ-i-configurations greater than $\varphi_e(A; n')$ for all $n' < n$ but no σ-i-configuration greater than $\varphi_e(A; n)$.*

   There exist j and τ with the following properties.

   (a) *Either $R^{\tau}_j \in$ Active_strategy(σ) or there is a $q \neq i$ such that $\hat{R}^{\tau}_{q,j} \in$ Active_strategy(σ).*

   (b) *$m^{\sigma}_{n,\tau}$ has a permanent value.*

   (c) *Either $\Phi_j(A \oplus B_i; m^{\sigma}_{n,\tau})\uparrow$ or one of the following holds.*

      i. *$R^{\tau}_j \in$ Active_strategy(σ) and it is not the case that $\Phi_j(A \oplus B_0) \restriction m^{\sigma}_{n,\tau} = \cdots = \Phi_j(A \oplus B_{N-1}) \restriction m^{\sigma}_{n,\tau}$.*

      ii. *$\hat{R}^{\tau}_{q,j} \in$ Active_strategy(σ) and it is not the case that $\Phi_j(A \oplus B_0) \restriction m^{\sigma}_{n,\tau} = \cdots = \Phi_j(A \oplus B_{q-1}) \restriction m^{\sigma}_{n,\tau} = \Phi_j(A \oplus B_{q+1}) \restriction m^{\sigma}_{n,\tau} = \cdots = \Phi_j(A \oplus B_{N-1}) \restriction m^{\sigma}_{n,\tau}$.*

   (d) *For each k and $\alpha \supset \tau$ such that $R^{\alpha}_k \in$ Active_strategy(σ) or $R^{\alpha}_{q',k} \in$ Active_strategy(σ), $q' \neq i$, $m^{\sigma}_{n,\alpha}$ has a permanent value for which $\Phi_k(A \oplus B_i; m^{\sigma}_{n,\alpha})\downarrow$.*

   Let r be the larger of the supremum of the permanent configurations found by $S^{\sigma}_{i,e}$ and $\varphi_e(A; n)+1$. Let s be a stage by which all of these configurations have stabilized and so have the computation $\Phi_e(A; n)$ and each computation $\Phi_k(A \oplus B_i; m^{\sigma}_{n,\alpha})$, $R^{\alpha}_k \in$ Active_strategy(σ) or $\hat{R}^{\alpha}_{q',k} \in$ Active_strategy(σ), $q' \neq i$, $\alpha \supset \tau$. If $R^{\tau}_j \in$ Active_strategy(σ) then $S^{\sigma}_{i,e}$'s outcome is infinitely often equal to $\langle c, m^{\sigma}_{n,\tau}, n, 2j, r\rangle$ and is never of the form $\langle d, n', r'\rangle$, $n' \leqslant n$, $\langle c, m', n', j', r'\rangle$, $n' < n$, or $\langle c, m', n, j', r'\rangle$, $j' > 2j$ or ($j' = 2j$ and $m' < m^{\sigma}_{n,\tau}$) or ($j' = 2j$, $m' = m^{\sigma}_{n,\tau}$, and $r' < r$), after stage s. If $\hat{R}^{\tau}_{q,j} \in$ Active_strategy(σ) then $S^{\sigma}_{i,e}$'s outcome is infinitely often equal to $\langle c, m^{\sigma}_{n,\tau}, n, 2\langle q, j\rangle + 1, r\rangle$ and is never of the form $\langle d, n', r'\rangle$, $n' \leqslant n$, $\langle c, m', n', j', r'\rangle$, $n' < n$, or $\langle c, m', n, j', r'\rangle$, $j' > 2\langle q, j\rangle + 1$ or ($j' = 2\langle q, j\rangle + 1$ and $m' < m^{\sigma}_{n,\tau}$) or ($j' = 2\langle q, j\rangle + 1$, $m' = m^{\sigma}_{n,\tau}$, and $r' < r$), after stage s.

For $\alpha \subset \tau$, $S^\sigma_{i,e}$ cancels the position of any Γ_α- or $\hat{\Gamma}_\alpha$-marker only finitely often, while for $\alpha \supseteq \tau$, any Γ_α- or $\hat{\Gamma}_\alpha$-marker whose position is canceled by $S^\sigma_{i,e}$ after stage s has its position canceled by it infinitely often.

In either case, $S^\sigma_{i,e}$ enumerates a computable set into each $A \oplus B_p$, $p < N$.

The only modification that we need to make that requires more than essentially notational work is in the description of the O-strategies.

Whereas a computation $\Phi_e(A \oplus B_j)$ cannot be affected by the action of a strategy $\hat{R}_{j,e'}$, thus allowing us to disregard such strategies when dealing with $S_{j,e}$, the same is not true of a computation $\Phi_e(A \oplus B_j \oplus C)$. So it does not suffice for $O^\sigma_{i,j,e}$ to find permanent σ-j-configurations greater than each n for which $\Phi_e(A \oplus B_j \oplus C) \upharpoonright n = B_i \upharpoonright n$ and $\Phi_e(A \oplus B_j \oplus C; n)$ converges, since these provide no assurance against the action of a strategy $\hat{R}_{j,e'}$.

Thus, we need to make use of our assumption that $\leqslant_0$ has at least three minimal elements to choose, for each $i, j < N$ such that $i \not\leqslant_0 j$, a number $q_{i,j} < N$ such that $j \neq q_{i,j}$ and $i \not\leqslant_0 q_{i,j}$, and make the following definition.

3.4. Definition. A number v is a *σ-j-$q_{i,j}$-configuration* at stage s if it is a σ-j-configuration at s and, in addition, for all $\hat{R}^\tau_{j,k} \in \mathit{Active\_strategy}(\sigma)$, and all m, $\hat{\gamma}_\tau(m,s) < v \Rightarrow \big[\varphi_k(A \oplus B_{q_{i,j}}; m)[s] < v \wedge \Phi_k(A \oplus B_{q_{i,j}}; m)[s] = \Phi_k(A \oplus B_{q_{i,j}}; m)[t]$, where t is the stage at which $\hat{\Gamma}_\tau(m)$ was assigned position $\hat{\gamma}_\tau(m,s)\big]$.

Now $O^\sigma_{i,j,e}$ acts as before, except that it attempts to find permanent σ-j-$q_{i,j}$-configurations greater than each n for which $\Phi_e(A \oplus B_j \oplus C) \upharpoonright n = B_i \upharpoonright n$ and $\Phi_e(B_j \oplus A \oplus C; n)$ converges. We describe formally the action of the preservation half of $O^\sigma_{i,j,e}$. (The coding half acts as before.)

Let s be a σ-stage. Let $r(-1,s) = 0$. The preservation half of $O^\sigma_{i,j,e}$ proceeds in cycles, beginning with the cycle for 0. The n^{th} cycle operates as follows:

1. If $\Phi_e(A \oplus B_j \oplus C)[s] \upharpoonright n = B_i[s] \upharpoonright n$ and $\Phi_e(A \oplus B_j \oplus C; n)[s]$ converges then go to step 2. Otherwise, cancel the value of $m^\sigma_{n',\tau}$ and the position of $\delta(n')$ for $n' \geqslant n$ and $R^\tau_k \in$ *Active_strategy*(σ) or $\hat{R}^\tau_{q,k} \in \mathit{Active\_strategy}(\sigma)$; preserve $A \upharpoonright r(n-1,s)$ and $B_j \upharpoonright r(n-1,s)$ and end stage s activity with outcome $\langle d, n, r(n-1,s)\rangle$.

2. Assign fresh large values in P_σ to each undefined $m^\sigma_{n,\tau}$ such that $R^\tau_k \in \mathit{Active\_strategy}(\sigma)$ or $\hat{R}^\tau_{q,k} \in \mathit{Active\_strategy}(\sigma)$.

3. Search for the longest $\tau \subseteq \sigma$, if any, such that at least one of the following holds.

   (a) $R^\tau_k \in \mathit{Active\_strategy}(\sigma)$ or $\hat{R}^\tau_{p,k} \in \mathit{Active\_strategy}(\sigma)$, $p \neq j$, and one of the following holds.

      i. $\Phi_k(A \oplus B_j; m^\sigma_{n,\tau})[s]\uparrow$.

      ii. For some $m < m^\sigma_{n,\tau}$, $\Gamma_\tau(m)$ has a position assigned at some stage t and $\Phi_k(A \oplus B_j; m)[s] \neq \Phi_k(A \oplus B_j; m)[t]$.

      iii. For some $R^\beta_l \in \mathit{Active\_strategy}(\sigma)$, $\gamma_\beta(m^\sigma_{n,\beta}, s) \leqslant \varphi_k(A \oplus B_j; m^\sigma_{n,\tau})[s]$.

iv. For some $\hat{R}^{\beta}_{q,l} \in \mathit{Active\_strategy}(\sigma)$, $\hat{\gamma}_\beta(m^\sigma_{n,\beta}, s) \leqslant \varphi_k(A \oplus B_j; m^\sigma_{n,\tau})[s]$.

(b) $\hat{R}^{\tau}_{j,k} \in \mathit{Active\_strategy}(\sigma)$ and one of the following holds.

i. $\Phi_k(A \oplus B_{q_{i,j}}; m^\sigma_{n,\tau})[s]\uparrow$.

ii. For some $m < m^\sigma_{n,\tau}$, $\Gamma_\tau(m)$ has a position assigned at some stage t and $\Phi_k(A \oplus B_{q_{i,j}}; m)[s] \neq \Phi_k(A \oplus B_{q_{i,j}}; m)[t]$.

iii. For some $R^{\beta}_{l} \in \mathit{Active\_strategy}(\sigma)$, $\gamma_\beta(m^\sigma_{n,\beta}, s) \leqslant \varphi_k(A \oplus B_{q_{i,j}}; m^\sigma_{n,\tau})[s]$.

iv. For some $\hat{R}^{\beta}_{q,l} \in \mathit{Active\_strategy}(\sigma)$, $\hat{\gamma}_\beta(m^\sigma_{n,\beta}, s) \leqslant \varphi_k(A \oplus B_{q_{i,j}}; m^\sigma_{n,\tau})[s]$.

If such a τ exists then proceed as follows. Enumerate $\min\{\gamma_\alpha(m^\sigma_{n,\alpha}, s) \mid R^\alpha_l \in \mathit{Active\_strategy}(\sigma)$ and $\alpha \supseteq \tau\}$ into A and for each q enumerate $m = \min\{\hat{\gamma}_\alpha(m^\sigma_{n,\alpha}, s) \mid \hat{R}^\alpha_{q,l} \in \mathit{Active\_strategy}(\sigma)$ and $\alpha \supseteq \tau\}$ into C_q and $\langle m, q\rangle$ into each B_p, $p \neq q$. Follow this with the σ-R recovery process. For each $R^\alpha_l \in \mathit{Active\_strategy}(\sigma)$, if $\alpha \supseteq \tau$ then cancel the position of $\Gamma_\alpha(y)$ for all $y \geqslant m^\sigma_{n,\alpha}$, otherwise cancel the value of $m^\sigma_{n,\alpha}$. For each $\hat{R}^\alpha_{q,l} \in \mathit{Active\_strategy}(\sigma)$, if $\alpha \supseteq \tau$ then cancel the position of $\hat{\Gamma}_\alpha(y)$ for all $y \geqslant m^\sigma_{n,\alpha}$, otherwise cancel the value of $m^\sigma_{n,\alpha}$. For each $x > n$, each $R^\alpha_l \in \mathit{Active\_strategy}(\sigma)$, and each $\hat{R}^\beta_{q,l} \in \mathit{Active\_strategy}(\sigma)$, cancel the value of $m^\sigma_{x,\alpha}$ and $m^\sigma_{x,\beta}$. For each $x \geqslant n$, cancel the position of $\delta(x)$. Let $r = \max\big(r(n-1, s), \varphi_e(A \oplus B_j \oplus C; n)[s] + 1\big)$. Preserve $A \upharpoonright r$ and $B_j \upharpoonright r$.

If $R^\tau_k \in \mathit{Active\_strategy}(\sigma)$ then end stage s activity with outcome $\langle c, m^\sigma_{n,\tau}, n, 2k, r\rangle$. If $\hat{R}^\tau_{p,k} \in \mathit{Active\_strategy}(\sigma)$ then end stage s activity with outcome $\big\langle c, m^\sigma_{n,\tau}, n, 2\langle p, k\rangle + 1, r\big\rangle$.

4. Define

$$r(n, s) = \min\Big(\big\{\gamma_\tau(m^\sigma_{n,\tau}, s) \mid R^\tau_k \in \mathit{Active\_strategy}(\sigma)\big\} \cup \cup \big\{\hat{\gamma}_\tau(m^\sigma_{n,\tau}, s) \mid \hat{R}^\tau_{q,k} \in \mathit{Active\_strategy}(\sigma)\big\}\Big).$$

If this set is empty then define

$$r(n, s) = \max\Big(\big\{\varphi_k(A \oplus B_i; m^\sigma_{n,\tau})[s] \mid R^\tau_k \in \mathit{Active\_strategy}(\sigma) \text{ or } \hat{R}^\tau_{q,k} \in \mathit{Active\_strategy}(\sigma),\ q \neq j\big\} \cup \big\{\varphi_k(A \oplus B_{q_{i,j}}; m^\sigma_{n,\tau})[s] \mid \hat{R}^\tau_{j,k} \in \mathit{Active\_strategy}(\sigma)\big\} \cup \cup \big\{\varphi_e(A \oplus B_j \oplus C; n)[s] + 1\big\} \cup \big\{r(n, t) \mid t < s\big\}\Big).$$

If $\delta(n)$ does not have a position then assign its new position $\delta(n, s)$ to be a fresh large number in P_σ. Begin the $(n+1)$st cycle.

The fact that the coding half of $O^\sigma_{i,j,e}$ does not put any numbers into $A \oplus B_j$ or $A \oplus B_{q_{i,j}}$ means that the following version of Lemma 2.13 can be established via essentially the same argument as before.

3.5. Lemma. *Suppose that $O^\sigma_{i,j,e}$ acts infinitely often and can restrain all numbers from entering any $A \oplus B_p$, $p < N$, except for the enumeration of finitely many fixed E-computable sets and numbers* *put in during a recovery process for the purpose of coding the movement of a marker associated with a strategy in Active_strategy(σ). Suppose further that there is a stage after which no δ-marker used by the coding half of $O^{\sigma}_{i,j,e}$ can have its position canceled except during $O^{\sigma}_{i,j,e}$'s action. Then one of the following holds.*

1. *There is an n such that $O^{\sigma}_{i,j,e}$ finds permanent σ-j-$q_{i,j}$-configurations greater than $\varphi_e(A \oplus B_j \oplus C; n')$ for all $n' < n$ and either $\Phi_e(A \oplus B_j \oplus C; n-1)\downarrow \neq B_i(n-1)$ or $\Phi_e(A \oplus B_j \oplus C; n)\uparrow$.*

   Let s be a stage by which all of these configurations have stabilized, and let r be their supremum. $O^{\sigma}_{i,j,e}$'s outcome is infinitely often equal to $\langle d, n, r\rangle$, and it is never of the form $\langle d, n', r'\rangle$, $n' < n$, $\langle c, m, n', l, r'\rangle$, $n' < n$, or $\langle d, n, r'\rangle$, $r' < r$, after stage s.

   $O^{\sigma}_{i,j,e}$ cancels the position of any particular marker only finitely often.

2. *There is an n such that $\Phi_e(A \oplus B_j \oplus C) \upharpoonright n = B_i \upharpoonright n$, $\Phi_e(A \oplus B_j \oplus C; n)\downarrow$, and $O^{\sigma}_{i,j,e}$ finds permanent σ-j-$q_{i,j}$-configurations greater than $\varphi_e(A \oplus B_j \oplus C; n')$ for all $n' < n$ but no σ-j-$q_{i,j}$-configuration greater than $\varphi_e(A \oplus B_j \oplus C; n)$.*

   There exist l and τ satisfying the following conditions.

   (a) *One of the following holds.*

      i. *$R^{\tau}_l \in$ Active_strategy(σ), $m^{\sigma}_{n,\tau}$ has a permanent value, and either $\Phi_l(A \oplus B_j; m^{\sigma}_{n,\tau})\uparrow$ or it is not the case that $\Phi_l(A \oplus B_0) \upharpoonright m^{\sigma}_{n,\tau} = \cdots = \Phi_l(A \oplus B_{N-1}) \upharpoonright m^{\sigma}_{n,\tau}$.*

      ii. *There exists a $p \neq j$ such that $\hat{R}^{\tau}_{p,l} \in$ Active_strategy(σ), $m^{\sigma}_{n,\tau}$ has a permanent value, and either $\Phi_l(A \oplus B_j; m^{\sigma}_{n,\tau})\uparrow$ or it is not the case that $\Phi_l(A \oplus B_0) \upharpoonright m^{\sigma}_{n,\tau} = \cdots = \Phi_l(A \oplus B_{p-1}) \upharpoonright m^{\sigma}_{n,\tau} = \Phi_l(A \oplus B_{p+1}) \upharpoonright m^{\sigma}_{n,\tau} = \cdots = \Phi_l(A \oplus B_{N-1}) \upharpoonright m^{\sigma}_{n,\tau}$.*

      iii. *$\hat{R}^{\tau}_{j,l} \in$ Active_strategy(σ), $m^{\sigma}_{n,\tau}$ has a permanent value, and either $\Phi_l(A \oplus B_{q_{i,j}}; m^{\sigma}_{n,\tau})\uparrow$ or it is not the case that $\Phi_l(A \oplus B_0) \upharpoonright m^{\sigma}_{n,\tau} = \cdots = \Phi_l(A \oplus B_{j-1}) \upharpoonright m^{\sigma}_{n,\tau} = \Phi_l(A \oplus B_{j+1}) \upharpoonright m^{\sigma}_{n,\tau} = \cdots = \Phi_l(A \oplus B_{N-1}) \upharpoonright m^{\sigma}_{n,\tau}$.*

   (b) *For each k and $\alpha \supset \tau$,*

      i. *$R^{\alpha}_k \in$ Active_strategy(σ) $\Rightarrow m^{\sigma}_{n,\alpha}$ has a permanent value for which $\Phi_k(A \oplus B_j; m^{\sigma}_{n,\alpha})$ converges,*

      ii. *$R^{\alpha}_{q',k} \in$ Active_strategy(σ), $q' \neq j \Rightarrow m^{\sigma}_{n,\alpha}$ has a permanent value for which $\Phi_k(A \oplus B_j; m^{\sigma}_{n,\alpha})$ converges, and*

      iii. *$R^{\alpha}_{j,k} \in$ Active_strategy(σ) $\Rightarrow m^{\sigma}_{n,\alpha}$ has a permanent value for which $\Phi_k(A \oplus B_{q_{i,j}}; m^{\sigma}_{n,\alpha})$ converges.*

   Let r be the larger of the supremum of the permanent configurations found by $O^{\sigma}_{i,j,e}$ and $\varphi_e(A \oplus B_j \oplus C; n)+1$. Let s be a stage by which all of these configurations have stabilized and so have the computation $\Phi_e(A \oplus B_j \oplus C; n)$ and each computation $\Phi_k(A \oplus B_j; m^{\sigma}_{n,\alpha})$, $R^{\alpha}_k \in$ Active_strategy(σ) or $\hat{R}^{\alpha}_{q',k} \in$ Active_strategy(σ), $q' \neq j$, $\alpha \supset \tau$, and each computation $\Phi_k(A \oplus B_{q_{i,j}}; m^{\sigma}_{n,\alpha})$, $\hat{R}^{\alpha}_{j,k} \in$ Active_strategy(σ), $\alpha \supset \tau$. If $R^{\tau}_l \in$ Active_strategy(σ) then $O^{\sigma}_{i,j,e}$'s outcome is infinitely often equal to $\langle c, m^{\sigma}_{n,\tau}, n, 2l, r\rangle$ and is never of the form $\langle d, n', r'\rangle$, $n' \leqslant n$, $\langle c, m', n', l', r'\rangle$, $n' < n$, or $\langle c, m', n, l', r'\rangle$, $l' > 2l$ or ($l' = 2l$ and $m' < m^{\sigma}_{n,\tau}$) or ($l' = 2l$, $m' = m^{\sigma}_{n,\tau}$, and $r' < r$), after stage s. If $\hat{R}^{\tau}_{p,l} \in$ Active_strategy(σ) then $S^{\sigma}_{i,e}$'s outcome is infinitely often equal *to $\langle c, m^\sigma_{n,\tau}, n, 2\langle p,l\rangle + 1, r\rangle$ and is never of the form $\langle d, n', r'\rangle$, $n' \leqslant n$, $\langle c, m', n', l', r'\rangle$, $n' < n$, or $\langle c, m', n, l', r'\rangle$, $l' > 2\langle p,l\rangle + 1$ or ($l' = 2\langle p,l\rangle + 1$ and $m' < m^\sigma_{n,\tau}$) or ($l' = 2\langle p,l\rangle + 1$, $m' = m^\sigma_{n,\tau}$, and $r' < r$), after stage s.*

For $\alpha \subset \tau$, $O^\sigma_{i,j,e}$ cancels the position of any Γ_α- or $\hat\Gamma_\alpha$-marker only finitely often, while for $\alpha \supseteq \tau$, any Γ_α- or $\hat\Gamma_\alpha$-marker whose position is canceled by $O^\sigma_{i,j,e}$ after stage s has its position canceled by it infinitely often.

In either case, $O^\sigma_{i,j,e}$ enumerates a computable set into each $A \oplus B_k$, $k < N$.

Definitions 2.15 and 2.21 must be replaced by the following definitions.

3.6. Definition. A number v is a *σ-configuration* at stage s if it satisfies the following conditions.

1. For each $\langle R^\tau_e\rangle \in \sigma$, $\gamma_\tau(m,s) < v \Rightarrow \mathit{max\_min}(\tau, m, s) < v$.
2. For each $\langle \hat R^\tau_{i,e}\rangle \in \sigma$, $\hat\gamma_\tau(m,s) < v \Rightarrow \mathit{max\_min}(\tau, m, s) < v$.
3. v is greater than all restraints in σ.
4. If $\langle X^\tau, c, m, n, l, r\rangle \in \sigma$, where X is one of $S_{i,e}$ or $O_{j,i,e}$, $R^\alpha_k \in \mathit{Active\_strategy}(\tau)$, $2k > l$, and the computation $\Phi_k(A \oplus B_i; m^\tau_{n,\alpha})$ still exists at the end of X^τ's stage s action, then $v > \varphi_k(A \oplus B_i; m^\tau_{n,\alpha})[s]$.
5. If $\langle X^\tau, c, m, n, l, r\rangle \in \sigma$, where X is one of $S_{i,e}$ or $O_{j,i,e}$, $\hat R^\alpha_{q,k} \in \mathit{Active\_strategy}(\tau)$, $2\langle q,k\rangle+1 > l$, $q \neq i$, and the computation $\Phi_k(A \oplus B_i; m^\tau_{n,\alpha})$ still exists at the end of X^τ's stage s action, then $v > \varphi_k(A \oplus B_i; m^\tau_{n,\alpha})[s]$. In addition, if $X = O_{j,i,e}$, $\hat R^\alpha_{i,k} \in \mathit{Active\_strategy}(\tau)$, $2\langle i,k\rangle + 1 > l$, and the computation $\Phi_k(A \oplus B_{q_{j,i}}; m^\tau_{n,\alpha})$ still exists at the end of X^τ's stage s action, then $v > \varphi_k(A \oplus B_{q_{j,i}}; m^\tau_{n,\alpha})[s]$.

3.7. Definition. A σ-configuration v at stage s is said to be *correct* if it is F and E-correct and satisfies the following conditions. Let X be one of $S_{i,e}$ or $O_{j,i,e}$. If $\langle X^\beta, c, m, n, l, r\rangle \in \sigma$ then for all $R^\tau_k \in \mathit{Active\_strategy}(\beta)$, $2k > l$, and all $\hat R^\tau_{q,k} \in \mathit{Active\_strategy}(\beta)$, $2\langle q,k\rangle + 1 > l$, $q \neq i$, the computation $\Phi_k(A \oplus B_i; m^\beta_{n,\tau})$ exists at the end of stage s. In addition, if $X = O_{j,i,e}$ and $\hat R^\alpha_{i,k} \in \mathit{Active\_strategy}(\tau)$, $2\langle i,k\rangle + 1 > l$, then the computation $\Phi_k(A \oplus B_{q_{j,i}}; m^\tau_{n,\alpha})$ still exists at the end of stage s.

As before, M^σ_k fixes a number move_σ in advance. At a σ-stage s, M^σ_k's action is as follows.

1. Find the smallest σ-configuration $w > k$.
2. If there is no R^τ_e in σ such that $\gamma_\tau(\mathit{move}_\sigma, s) < w$ and no $\hat R^\tau_{i,e}$ in σ such that $\hat\gamma_\tau(\mathit{move}_\sigma, s) < v$ then let $v = w$ and proceed to step 7.
3. Cancel the position of all markers $\Gamma_\alpha(n)$, $\langle R^\alpha_{e'}\rangle \in \sigma$, $n \geqslant \mathit{move}_\sigma$, and enumerate the least of their positions into A.

4. For each $i' < N$, cancel the position of all markers $\hat{\Gamma}_\alpha(n)$, $\langle \hat{R}^\alpha_{i',e'} \rangle \in \sigma$, $n \geqslant move_\sigma$, enumerate the least of their positions, m, into $C_{i'}$, and enumerate $\langle m, i' \rangle$ into each B_p, $p \neq i'$.

5. Run the σ-R recovery process.

6. Find the smallest σ-configuration $v > k$.

7. Impose a restraint equal to v on $A \oplus B_p$ and C_p, $p < N$ and end stage s activity with outcome v.

It is now straightforward to modify our previous proofs to establish Lemma 2.18, and hence Corollaries 2.19 and 2.20, as well as Lemmas 2.17, and 2.22.

We add the requirements $\hat{\mathcal{R}}_{i,e}$ to our priority list L in such a way that $\hat{\mathcal{R}}_{i,e}$ is listed after $\mathcal{R}_{\langle i,e \rangle}$ but before $\mathcal{R}_{\langle i,e \rangle + 1}$. If $\sigma \in T$ then we make the following definitions.

1. $\hat{\mathcal{R}}_{q,e}$ *is satisfied in* σ if there are $\tau \subseteq \sigma$, i, j, e', m, n, and r such that $\big\langle S^\tau_{i,e'}, c, m, n, 2\langle q, e \rangle + 1, r \big\rangle \in \sigma$ or $\big\langle O^\tau_{i,j,e'}, c, m, n, 2\langle q, e \rangle + 1, r \big\rangle \in \sigma$.

2. $\hat{R}^\tau_{q,e}$ *is injured in* σ if $\langle \hat{R}^\tau_{q,e} \rangle \in \sigma$ and there are α, $j \leqslant 2\langle q, e \rangle$, i, e', k, m, n, and r such that $\tau \subset \alpha \subseteq \sigma$ and either $\langle S^\alpha_{i,e'}, c, m, n, j, r \rangle \in \sigma$ or $\langle O^\alpha_{i,k,e'}, c, m, n, j, r \rangle \in \sigma$.

We also redefine some of the notions from the previous section as follows.

1. $\mathcal{R}_e$ *is satisfied in* σ if there are $\tau \subseteq \sigma$, i, j, e', m, n, and r such that $\langle S^\tau_{i,e'}, c, m, n, 2e, r \rangle \in \sigma$ or $\langle O^\tau_{i,j,e'}, c, m, n, 2e, r \rangle \in \sigma$.

2. R^τ_e *is injured in* σ if $\langle R^\tau_e \rangle \in \sigma$ and there are α, $j < 2e$, i, e', k, m, n, and r such that $\tau \subset \alpha \subseteq \sigma$ and either $\langle S^\alpha_{i,e'}, c, m, n, j, r \rangle \in \sigma$ or $\langle O^\alpha_{i,k,e'}, c, m, n, j, r \rangle \in \sigma$.

3. $N^\tau_{i,k,l,e}$ *is injured in* σ if there is an r such that $\langle N^\tau_{i,k,l,e}, r \rangle \in \sigma$ and there are $\tau \subset \alpha \subseteq \sigma$, j, i', e', k', m, n, and r' such that either
   (a) $\mathcal{R}_j$ has stronger priority than $\mathcal{N}_{i,k,l,e}$ and either $\langle S^\alpha_{i',e'}, c, m, n, 2j, r' \rangle \in \sigma$ or $\langle O^\alpha_{i',k',e'}, c, m, n, 2j, r' \rangle \in \sigma$, or
   (b) $\hat{\mathcal{R}}_{q,j}$ has stronger priority than $\mathcal{N}_{i,k,l,e}$ and either $\big\langle S^\alpha_{i',e'}, c, m, n, 2\langle q, j \rangle + 1, r' \big\rangle \in \sigma$ or $\big\langle O^\alpha_{i',k',e'}, c, m, n, 2\langle q, j \rangle + 1, r' \big\rangle \in \sigma$.

$\hat{\mathcal{R}}_{i,e}$ requires attention at σ if it is not satisfied in σ and every strategy $\hat{R}^\tau_{i,e}$ in σ is injured in σ.

If $\hat{\mathcal{R}}_{i,e}$ is the first strategy in L that requires attention at σ then the only immediate successor of σ is $\sigma^\frown \langle \hat{R}^\sigma_{i,e} \rangle$.

We redefine

1. $Active\_index(\sigma) = \{2e \mid \text{some } R^\tau_e \text{ appears in } \sigma \text{ and } \mathcal{R}_e \text{ is not satisfied in } \sigma\} \cup$
   $\cup \{2\langle i, e \rangle + 1 \mid \text{some } \hat{R}^\tau_{i,e} \text{ appears in } \sigma \text{ and } \hat{\mathcal{R}}_{i,e} \text{ is not satisfied in } \sigma\}$;

2. $Active\_strategy(\sigma) = \{R^\tau_e \mid \tau \subset \sigma,\ 2e \in Active\_index(\sigma), \text{ and } R^\tau_e \text{ is not injured in } \sigma\} \cup$
   $\cup \{\hat{R}^\tau_{i,e} \mid \tau \subset \sigma,\ 2\langle i, e \rangle + 1 \in Active\_index(\sigma), \text{ and } \hat{R}^\tau_{i,e} \text{ is not injured in } \sigma\}$.

We can now prove Lemmas 2.26 and 2.32 and Proposition 2.33 in much the same way as before.

By Lemma 3.3, Proposition 2.29 still holds, while by Lemma 3.5, the same is true of Proposition 2.30.

Propositions 2.10, 2.27, 2.31, and 2.35 still hold, by essentially the same arguments as before.

Finally, arguing as in the proof of Proposition 2.31, we have the following result.

3.8. Proposition. *For each $i < N$ and each $e \in \omega$, $\hat{\mathcal{R}}_{i,e}$ is satisfied.*

This completes the proof of Theorem 1.6. ■

4 Fragments of the theory

In this final section we address the question of which fragments of the theory of the c.e. degrees in a given interval are undecidable. Let Σ_n (Π_n) be the set of sentences in prenex normal form where the block of quantifiers starts with an existential (universal) quantifier and contains at most $n-1$ alternations of quantifiers. For a structure $\mathcal{S}$ over a language L, the Π_n-theory of $\mathcal{S}$ is defined by

$$\Pi_n\text{-Th}(\mathcal{S}) = \{\theta \in \Pi_n \cap L \mid \mathcal{S} \vDash \theta\}.$$

We will show that the results of Section 1 imply that for any c.e. degrees $\mathbf{e} < \mathbf{f}$, the Π_5-theory of $\langle [\mathbf{e}, \mathbf{f}], \leqslant \rangle$ over the language $L(\leqslant)$ of partial orderings is undecidable. For this we need the following result.

4.1. Proposition. *(Ambos-Spies and Shore [2]) The set of all Σ_2 sentences that are true in every finite partial ordering is strongly undecidable.*

Call a formula $\delta_{x_0,\ldots,x_k,y}$ in the language $L(\leqslant)$ a *coding formula* for a partial ordering $\mathcal{S} = \langle S, \leqslant_S \rangle$ if, for any finite partial ordering $\mathcal{P} = \langle P, \leqslant_P \rangle$, there are elements $a_0, \ldots, a_k$ of S such that the partial ordering

$$\Big\langle \big\{ b \in S \mid \mathcal{S} \vDash \delta_{x_0,\ldots,x_k,y}[a_0, \ldots, a_k, b] \big\}, \leqslant_S \Big\rangle$$

is isomorphic to $\mathcal{P}$.

4.2. Lemma. *(Ambos-Spies and Shore [2]) Let $\mathcal{S} = \langle S, \leqslant_S \rangle$ be a partial ordering and let δ be a Σ_m coding formula for $\mathcal{S}$. Then Π_{m+2}-Th($\mathcal{S}$) is undecidable.*

4.3. Theorem. *For any c.e. degrees $\mathbf{e} < \mathbf{f}$, the Π_5-theory of $\langle [\mathbf{e}, \mathbf{f}], \leqslant \rangle$ is undecidable.*

Proof. By Lemma 4.2, it suffices to show that there is a Σ_3 coding formula δ for $\langle [\mathbf{e}, \mathbf{f}], \leqslant \rangle$. Now, by Theorem 1.4, the formula δ with free variables x_0, x_1, and y expressing that there is a maximal-x_0-cappable degree z such that y is the join of x_1 and z is a coding formula for $\langle [\mathbf{e}, \mathbf{f}], \leqslant \rangle$. Formally, δ can be defined by

$$\begin{aligned}
\delta \;\equiv\; \exists z \Bigg(& \Big[x_0 \leqslant z \wedge \exists v \big(x_0 < v \wedge \forall w (w \leqslant v, z \rightarrow w \leqslant x_0) \big) \Big] \wedge \\
& \wedge \forall u \Bigg(\Big[x_0 \leqslant u \wedge \exists v \big(x_0 < v \wedge \forall w (w \leqslant v, u \rightarrow w \leqslant x_0) \big) \Big] \rightarrow \neg (z < u) \Bigg) \wedge \\
& \wedge \big[x_1, z \leqslant y \wedge \forall s (x_1, z \leqslant s \rightarrow y \leqslant s) \big] \Bigg)
\end{aligned}$$

which, as one can easily check, is equivalent to a Σ_3 formula. ■

We note that, for the structure $\mathcal{R}$ as a whole, Lempp, Nies, and Slaman [7] have shown that the Π_3-theory is undecidable by using a considerably more delicate coding procedure and various computability-theoretic constructions that do not seem available in arbitrary intervals.

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