USING TREE AUTOMATA TO INVESTIGATE INTUITIONISTIC PROPOSITIONAL LOGIC

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Intuitionistic logic is an important variant of classical logic, but it is not as well-understood, even in the propositional case. In this thesis, we describe a faithful representation of intuitionistic propositional formulas as tree automata. This representation permits a number of consequences, including a characterization theorem for free Heyting algebras, which are the intutionistic analogue of free Boolean algebras, and a new algorithm for solving equations over intuitionistic propositional logic.

BIOGRAPHICAL SKETCH

Michael O'Connor was born in Arlington, VA on December 23, 1979. He attended Thomas Jefferson High School in Alexandria, VA and the University of Massachusetts at Amherst. In 2002, he graduated from the University of Massachusetts and started graduate school at Cornell University. He received his PhD from Cornell in 2008.

I dedicate this thesis to my parents, Mary Kelly O'Connor and Thomas Michael O'Connor, and to my grandmother, Mary Kelly.

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CHAPTER 1
INTRODUCTION

Intuitionistic logic has been explored for many years as a language for computer science, with a guiding principle being the Brouwer-Heyting-Kolmogorov interpretation, under which intuitionistic proofs of implication are functions and existence proofs require witnesses. Higher-order intuitionistic systems which can express a great deal of mathematics, such as Girard's System F and Martin-Löf's type theory (good references are [9] and [3]), have been developed and implemented by prominent computer scientists such as Constable, Huet and Coquand (see [2] and [1]). With all this development and with the existence of well-established topological, Kripke, and categorical semantics for intuitionistic systems, it may come as a surprise that many fundamental structural properties of intuitionistic propositional calculus have not been developed. By way of contrast, corresponding issues for classical logics have been settled for at least 75 years.

For each $n \in \mathbb{N}$, let $V_n = \{x_0, \ldots, x_{n-1}\}$ and let F_n be the set of propositional sentences in variables V_n. Let $\simeq_i$ and $\simeq_c$ be the intuitionistic and classical logical equivalence relations respectively.

The classical Lindenbaum algebra B_n is defined as $F_n/\simeq_c$. It is the free Boolean algebra on n generators. It is completely understood and can be characterized quite easily. For example, B_n is isomorphic to the algebra of subsets of V_n, $\langle \mathcal{P}\mathcal{P}(V_n), \cup, \cap, \emptyset, \mathcal{P}(V_n)\rangle$, where $\emptyset$ is the bottom element and $\mathcal{P}(V_n)$ is the top element. The set $\mathcal{S} \subseteq \mathcal{P}(V_n)$ corresponds to the formula

$$\bigvee_{S \in \mathcal{S}} \left(\bigwedge_{x_i \in S} x_i \wedge \bigwedge_{x_i \notin S} \neg x_i \right).$$

If we let $\mathrm{Up}(P)$ be the set of upward-closed subsets of a partially ordered set P, then we may also describe the algebra of subsets of V_n as $\langle \mathrm{Up}(P_n), \cup, \cap, \emptyset, P_n\rangle$, where P_n is $\mathcal{P}(V_n)$ given the trivial ordering where any two distinct elements are incomparable.

Of course, this was nothing more than a cosmetic change. However, we have the following: Let the intuitionistic Lindenbaum algebra H_n be $F_n/\simeq_i$. Then it is the case that H_n is (isomorphic to) a subalgebra of the algebra of sets $\langle \mathrm{Up}(\mathcal{U}(n)), \cup, \cap, \emptyset, \mathcal{U}(n)\rangle$, where $\mathcal{U}(n)$ is a partial order defined in Chapter 2 that contains P_n as its maximal elements.

Call an upward-closed subset of $\mathcal{U}(n)$ *definable* if it corresponds to an element of H_n. Characterizing exactly *which* upward-closed subsets of $\mathcal{U}(n)$ are definable would help in understanding the structure of H_n. My answer to this is given in Theorem 1 where I show that an upward-closed subset of $\mathcal{U}(n)$ is definable just in case it is recognizable by a specific type of tree automaton, which I call a prefix-closed intuitionistic-equivalence-respecting automaton (the elements of $\mathcal{U}(n)$ are considered to be labeled trees in a natural way). A prefix-closed tree automaton is one with no transition to an accept state from a sequence containing a non-accept state. The term "intuitionistic-equivalence-respecting" is explained below.

Given a subset S of a partial order P, let $S\updownarrow$ be

$$\{p \in P \mid (\forall q \geq p)\,(\exists r \geq q)\,(\exists s \in S)\, r \geq s\}.$$

In Proposition 6 it is shown as an application of Theorem 1 that all subsets of $\mathcal{U}(n)$ of the form $S\updownarrow$ where $S \subseteq \mathcal{U}(n)$ is finite are definable. From the existence of this large class of definable subsets alone we can show in Proposition 15 that all countable partial orders order-embed into any H_n where $n \geq 2$.

Using Proposition 6 and some combinatorial reasoning about $\mathcal{U}(n)$, we show in Theorem 4 that every H_m lattice-embeds as an interval into H_n, where $m \geq 1$ and $n \geq 2$, a result that was first shown by Mardaev in [13].

We may think of a tree automaton as assigning a state to each tree; namely, the final state of the automaton after processing the tree. We may consider all finite Kripke models to be finite labeled trees in a very simple (essentially obvious) way; thus, we may consider automata which accept sets of Kripke models. The explanation for the term "intuitionistic-equivalence-respecting automaton" is as follows. Call two finite Kripke models (defined in Chapter 2) *intuitionistically equivalent* if they force exactly the same propositional formulas. An intuitionistic-equivalence-respecting automaton is then one with the property that for each of its states, the set of Kripke models assigned to that state is closed under intuitionistic equivalence. Part of the power of this notion comes from the fact that, by Proposition 4, there is a simple combinatorial condition that can be used to determine if two finite Kripke models are intuitionistically equivalent.

By showing that if the set of Kripke models accepted by a tree automaton is closed under intuitionistc-equivalence and under taking upward-closed submodels then it is equivalent to a prefix-closed intuitionistic-equivalence-respecting automaton, it is shown in Theorem 2 that every such set of finite Kripke models is of the form $\{\mathcal{M} \mid \mathcal{M} \Vdash \phi\}$ for some ϕ (the converse is also true and follows from the preliminaries in Chapter 2).

For every finite Kripke model there is a unique minimal finite Kripke model to which it is equivalent which is called its p-morphic reduction. The set $\mathcal{U}(n)$

essentially consists of the reduced Kripke models and the ordering is given by embeddability as an upward-closed submodel.

One of the properties of a intuitionistic-equivalence-respecting tree automaton is that if T_1 and T_2 are two trees to which it assigns the same state, then there is a tree T whose root has T_1 and T_2 as its sole subtrees such that it assigns T the same state as T_1 and T_2. As a consequence of this and the pigeonhole principle, if S is a definable subset of $\mathcal{U}(n)$ with an infinite sequence $\{s_1, s_2, \ldots\} \subseteq S$ of incomparable elements, then there must be some $i \neq j$ and $s \in S$ with $s < s_i, s_j$. This is used in Subsection 4.2.3 to show constructively that H_n is incomplete for $n \geq 2$ (this is shown nonconstructively in [4]) and in Proposition 13 to show that the countable atomless Boolean algebra does not lattice-embed into any H_n.

Call an element a of a lattice L *join-irreducible* if for all b, $c \in L$, if $a = b \vee c$ then $a = b$ or $a = c$. It is known that any element of H_n can be written uniquely as a finite join of pairwise incomparable join-irreducibles, a property which is not shared by, for example, the countable atomless Boolean algebra, which does not have any join-irreducibles (see [10] for a general reference on lattice theory). Therefore if we could characterize the set J_n of join-irreducibles in H_n, we would have a characterization of H_n.

Each intuitionistic-equivalence-respecting automaton comes with a partial ordering on its set of states. In Proposition 8 it is shown that if $\phi \in H_n$ and $\mathcal{T}$ is a intuitionistic-equivalence-respecting tree automaton corresponding to it, then, essentially, ϕ is join-irreducible iff $\mathcal{T}$'s set of accepting states has a minimum element. This fact is used together with Proposition 6 and some facts about the combinatorics of $\mathcal{U}(n)$ to show in Theorem 3 that J_n can be partitioned into $J_{1,n}$, $J_{2,n}$, $J_{3,n}$ and $J_{4,n}$ and each of those suborders can be characterized up to isomorphism. Furthermore, $J_{2,n}$, $J_{3,n}$, and $J_{4,n}$ are independent of n.

In Chapter 7, two results with the same flavor are shown which use Theorem 1 more deeply than any of the previous results. By Proposition 30, if ϕ is a formula with a free variable x and a representing automaton $\mathcal{T}$, then if there is any ψ such that $\phi[\psi/x]$ is a tautology, then there is such a ψ with a representing automaton with no more states than $\mathcal{T}$ has. As a corollary to this, we get that the problem of deciding whether or not equations $\phi(x) = \top$ in intuitionistic logic have solutions is decidable, a fact which was first shown by Rybakov in [16]. By a slight modification of Proposition 30, we get that the embedding of H_n into the full subalgebra of upward-closed subsets of $\mathcal{U}(n)$ is existentially closed.

It was first determined in [15] and [14] that for all n, there are infinitely many intuitionistically inequivalent propositional formulas over n variables. A natural question is then to ask whether various fragments of intuitionistic propositional logic obtained by restricting the connectives allowed are also infinite.

The most difficult case is that where the single connective $\rightarrow$ is allowed, and this case was solved by algebraic means in [8], where it is proven that for all n, the number of intuitionistically inequivalent formulas over n variables using only the connective $\rightarrow$ is finite.

Subsequent proofs of this fact using semantic methods (i.e., Kripke models) are given in [17] and [7]. An excellent analysis of all fragments of intuitionistic logic using the methods of [7] is given in [11].

In Chapter 3, I will present a purely combinatorial proof of this fact (which, in particular, does not rely on the tree automata representation given in Chapter 4) by specifying certain rewrite rules by which one can rewrite any propositional formula over n variables with only the connective $\rightarrow$ into an intuitionistically equivalent one whose length is less than a constant depending only on n.

CHAPTER 2
BACKGROUND AND NOTATION

2.1 Intuitionistic Propositional Logic

Definition 1 (Intuitionistic Propositional Logic)**.** Let $\omega + 1$ be $\{0, 1, 2, \ldots\} \cup \{\omega\}$ as usual. Note that contrary to common practice, the variable n will sometimes be used to range over $\omega + 1$ as well as sometimes ranging over ω.

Let $V_\omega = \{x_0, x_1, \ldots\}$ be a countably infinite set whose elements will be considered to be propositional variables. For $n < \omega$, let $V_n = \{x_0, \ldots, x_{n-1}\}$. The symbols x, y and z will be used as metavariables to range over the V_n which is appropriate in context. Symbols of the form x_i may also be used this way if no confusion will result.

For each $n \in \omega + 1$, let F_n be the set of well-formed propositional formulas defined over V_n. The connectives of propositional logic are taken to be $\vee$, $\wedge$, and $\rightarrow$ together with a constant $\perp$. The negation $\neg\phi$ is defined to be $\phi \rightarrow \perp$.

The set IPC $\subseteq F_\omega$ is the smallest set containing, for all x, y, and $z \in V_\omega$:

1. $x \rightarrow (y \rightarrow x)$
2. $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))$
3. $(x \wedge y) \rightarrow x$
4. $(x \wedge y) \rightarrow y$
5. $x \rightarrow (y \rightarrow (x \wedge y))$
6. $x \rightarrow (x \vee y)$
7. $y \rightarrow (x \vee y)$
8. $(x \rightarrow z) \rightarrow ((y \rightarrow z) \rightarrow ((x \vee y) \rightarrow z))$
9. $\perp \rightarrow x$

and which is closed under the inference rules:

1. Modus Ponens: If $\phi \in$ IPC and $\phi \rightarrow \psi \in$ IPC, then $\psi \in$ IPC.
2. Substitution: If $\phi(\bar{x}) \in$ IPC, then $\phi(\bar{\psi}) \in$ IPC for any formulas $\bar{\psi}$.

We say that ϕ is a *tautology* (or an intuitionistic tautology) if $\phi \in \mathrm{IPC}$ and that ϕ implies ψ (or ϕ intuitionistically implies ψ), if $\phi \to \psi \in \mathrm{IPC}$.

If Γ is a set of formulas such that the conjunction of some finite subset of Γ implies ψ, then we write $\Gamma \vdash \psi$. If $\Gamma = \{\phi\}$, then we may write $\phi \vdash \psi$ for $\{\phi\} \vdash \psi$.

Definition 2 (Heyting Algebras)**.** A *Heyting algebra* is a tuple $\langle H, \vee, \wedge, \to, \bot, \top \rangle$ where H is a set, $\vee$, $\wedge$, and $\to$ are binary operations on H, and $\bot$ and $\top$ are constants in H satisfying the following requirements:

1. $\langle H, \vee, \wedge, \bot, \top \rangle$ is a distributive lattice with 0 and 1.
2. For all $a, b \in H$, $a \to b = \sup\{c \mid c \wedge a \leq b\}$.

As usual, an order relation $\leq$ is defined on H by $a \leq b$ iff $a \wedge b = a$.

Lemma 1 (See [12])**.** *The class of Heyting algebras is a variety.*

Definition 3 (Free Heyting Algebras)**.** The existence of free Heyting algebras is given by Lemma 1. For $n \in \omega + 1$, the free Heyting algebra on n generators is denoted H_n.

Definition 4 (Lindenbaum Algebras)**.** Let $n \in \omega + 1$. Define the equivalence relation $\simeq$ on F_n by $\phi \simeq \psi$ iff $\phi \leftrightarrow \psi \in \mathrm{IPC}$.

The *Lindenbaum algebra of intuitionistic propositional logic on V_n* is defined to be the tuple $\langle H, \vee, \wedge, \to, \bot, \top \rangle$, where $H = F_n/\simeq$ and $[\phi] \vee [\psi] = [\phi \vee \psi]$, etc. It can be verified that these operations are all well-defined.

Lemma 2. *For $n \in \omega + 1$, the Lindenbaum algebra of intuitionistic propositional logic on V_n is a Heyting algebra and is in fact isomorphic to H_n via an isomorphism mapping V_n onto the set of free generators of H_n.*

Proof. Straightforward. $\square$

2.2 Kripke Models

Definition 5 (Kripke Frames and Models)**.** A Kripke frame $\mathcal{F} = (F, \leq)$ is a partial order. Given a Kripke frame $\mathcal{F}$, $\mathrm{Up}(\mathcal{F}) = \{S \subseteq F \mid S \text{ is upward-closed}\}$. Given any subset $S \subseteq F$, we let $S\uparrow$ be the upward-closure of S and $S\downarrow$ be the downward-closure of S.

A Kripke model over some V_n is a pair $\mathcal{M} = (\mathcal{F}, f)$ of a Kripke frame $\mathcal{F} = \mathcal{F}(\mathcal{M})$ and a function $f = f_{\mathcal{M}} \colon V_n \to \mathrm{Up}(\mathcal{F})$. Elements of the underlying set of $\mathcal{F}(\mathcal{M})$ are said to be *nodes* of $\mathcal{M}$. Statements like $\alpha \in \mathcal{M}$, $S \subseteq \mathcal{M}$, etc., mean that α is an element of the underlying set of $\mathcal{F}(\mathcal{M})$, S is a subset of the underlying set of $\mathcal{F}(\mathcal{M})$, etc.

A pointed Kripke model is a pair $(\mathcal{M}, \alpha)$ where α is a node of $\mathcal{M}$.

We define a forcing relation $\Vdash$ between pointed Kripke models defined over V_n and F_n by induction as follows:

For $x \in V_n$, $(\mathcal{M}, \alpha) \Vdash x$ iff $\alpha \in f_{\mathcal{M}}(x)$.

$(\mathcal{M}, \alpha) \nVdash \bot$.

Let ϕ and ψ be arbitrary propositional formulas in F_n.

$(\mathcal{M}, \alpha) \Vdash \phi \wedge \psi$ iff $(\mathcal{M}, \alpha) \Vdash \phi$ and $(\mathcal{M}, \alpha) \Vdash \psi$.

$(\mathcal{M}, \alpha) \Vdash \phi \vee \psi$ iff $(\mathcal{M}, \alpha) \Vdash \phi$ or $(\mathcal{M}, \alpha) \Vdash \psi$.

$(\mathcal{M}, \alpha) \Vdash \phi \to \psi$ iff for all $\beta \geq \alpha$, if $(\mathcal{M}, \beta) \Vdash \phi$ then $(\mathcal{M}, \beta) \Vdash \psi$.

We say that $\mathcal{M} \Vdash \phi$ if for all $\alpha \in \mathcal{M}$, $(\mathcal{M}, \alpha) \Vdash \phi$.

Sometimes if $\mathcal{M}$ is clear from context, $(\mathcal{M}, \alpha) \Vdash \phi$ is written $\alpha \Vdash \phi$.

Given a Kripke model $\mathcal{M}$, we let $\mathrm{Forces}_{\mathcal{M}}$ be the set $\{(\alpha, \phi) \mid (\mathcal{M}, \alpha) \Vdash \phi\}$. This is for notational reasons, so that we may write $\mathrm{Forces}_{\mathcal{M}}(\alpha)$ for $\{\phi \mid \alpha \Vdash \phi\}$ and $\mathrm{Forces}_{\mathcal{M}}^{-1}(\phi)$ for $\{\alpha \mid \alpha \Vdash \phi\}$. If $\mathcal{M}$ is clear from context it may be suppressed.

Note that a Kripke model $\mathcal{M}$ defined over V_n is completely determined by its underlying frame $\mathcal{F}$ together with $\mathrm{Forces}_{\mathcal{M}}(\alpha) \cap V_n$ for all $\alpha \in \mathcal{F}$.

Lemma 3 (See [5]). *A propositional formula $\phi \in F_n$ is an intuitionistic tautology iff for all Kripke models $\mathcal{M}$ defined over V_n, $\mathcal{M} \Vdash \phi$.*

We also have the following stronger result.

Lemma 4 (See [5]). *A propositional formula $\phi \in F_n$ is a intuitionistic tautology iff for all finite Kripke models $\mathcal{M}$, defined over V_n, $\mathcal{M} \Vdash \phi$.*

2.3 $\mathcal{U}(n)$

Definition 6 ($\mathcal{U}(n)$)**.** See [5] and [4].

Given $n \in \mathbb{N}, \mathcal{U}(n)$ is defined to be the minimal Kripke model defined over V_n satisfying the following properties:

1. For every $S \subseteq \mathcal{U}(n)$ and $V \subseteq V_n$, there is at most one node α such that $\text{Forces}(\alpha) \cap V_n = V$ and $\{\alpha\}\uparrow - \{\alpha\} = S\uparrow$. In the case that there is one such node, we say that $\text{node}(S, V)$ exists and denote the unique such node by $\text{node}(S, V)$. In the case that there are no such nodes, we say that $\text{node}(S, V)$ does not exist.
2. For all $V \subseteq V_n$, $\text{node}(\emptyset, V)$ exists.
3. Let $S \subseteq \mathcal{U}_n$ be such that S has at least 2 minimal elements. Let $V \subseteq \bigcap\{\text{Forces}(\alpha) \cap V_n \mid \alpha \in S\}$. Then $\text{node}(S, V)$ exists.
4. Let $S \subseteq \mathcal{U}_n$ be such that $S = \{\beta\}\uparrow$. Let $V \subsetneq \text{Forces}(\beta) \cap V_n$. Then $\text{node}(S, V)$ exists.

If $S \subseteq \mathcal{U}(n)$ has at least 2 minimal elements, we write $\text{node}(S)$ for

$$\text{node}\left(S, \bigcap\{\text{Forces}(\alpha) \cap V_n \mid \alpha \in S\}\right)$$

Definition 7 (L_n^m)**.** Given an $\alpha \in \mathcal{U}(n)$, we let the *level* of α (written $\text{Lev}(\alpha)$) be the maximum m such that there are $\alpha = \alpha_1 < \alpha_2 < \cdots \alpha_m$ in $\mathcal{U}(n)$. We define L_n^m to be the set of elements of $\mathcal{U}(n)$ of level m.

Figure 2.1 depicts $\mathcal{U}(1)$ and Figure 2.2 depicts $\mathcal{U}(2)$. In both cases, the propositional variables a node forces are written inside the node. In Figure 2.2, the eight nodes of the form $\text{node}(\{\alpha\} \cup S, \emptyset)$ where $\alpha = \text{node}(\emptyset, \{x, y\})$ and $S \subseteq \{\text{node}(\emptyset, \emptyset), \text{node}(\emptyset, \{x\}), \text{node}(\emptyset, \{y\})\}$ are missing from L_2^2.

Proposition 1. *[See [5] or [4]] Let ϕ, $\psi \in F_n$. Then ϕ intuitionistically implies ψ iff* $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \subseteq \text{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$.

Definition 8 (Definable Subsets of $\mathcal{U}(n)$)**.** Let $S \subseteq \mathcal{U}(n)$. If there is a $\phi \in F_n$ such that $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi) = S$ then we say that S is definable and that ϕ defines S.

If S is definable, then we write ϕ_S for a formula which defines S. (Note that by Proposition 1 all formulas defining a set are logically equivalent.) If $\alpha \in \mathcal{U}(n)$, then we write ϕ_α for $\phi_{\{\alpha\}\uparrow}$ and ϕ'_α for $\phi_{\mathcal{U}(n)-\{\alpha\}\downarrow}$, both of which exist by Proposition 2.

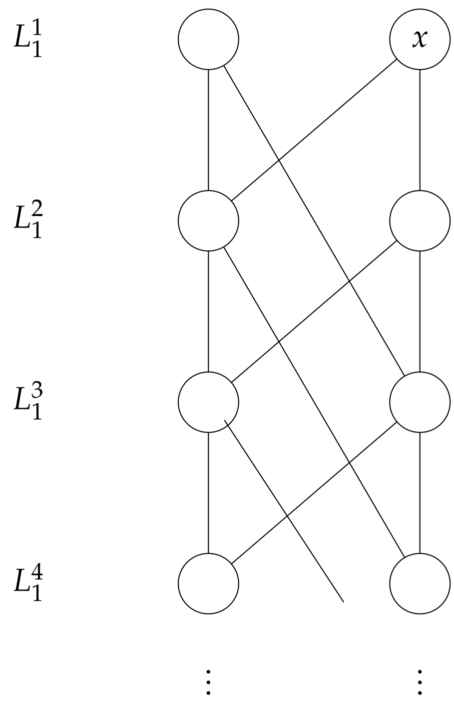

Figure 2.1: $\mathcal{U}(1)$

Lemma 5. *Let $\mathcal{S}$ be the collection of definable subsets of $\mathcal{U}(n)$. Then $\langle \mathcal{S}, \subseteq \rangle$ is isomorphic to $\langle H_n, \leq \rangle$.*

Proof. Straightforward from Proposition 1. □

Proposition 2. *For all $\alpha \in \mathcal{U}(n)$, $\{\alpha\}\uparrow$ and $\mathcal{U}(n) - \{\alpha\}\downarrow$ are definable.*

This proposition will be proven in Subsection 4.2.2.

Corollary 1. *All finite upward-closed subsets of $\mathcal{U}(n)$ are definable.*

2.4 P-morphisms

Definition 9 (P-morphisms)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be Kripke models. A function $f\colon \mathcal{M} \to \mathcal{N}$ is a p-morphism if:

- For all variables x and nodes a of $\mathcal{M}$, $a \Vdash x$ iff $f(a) \Vdash x$.

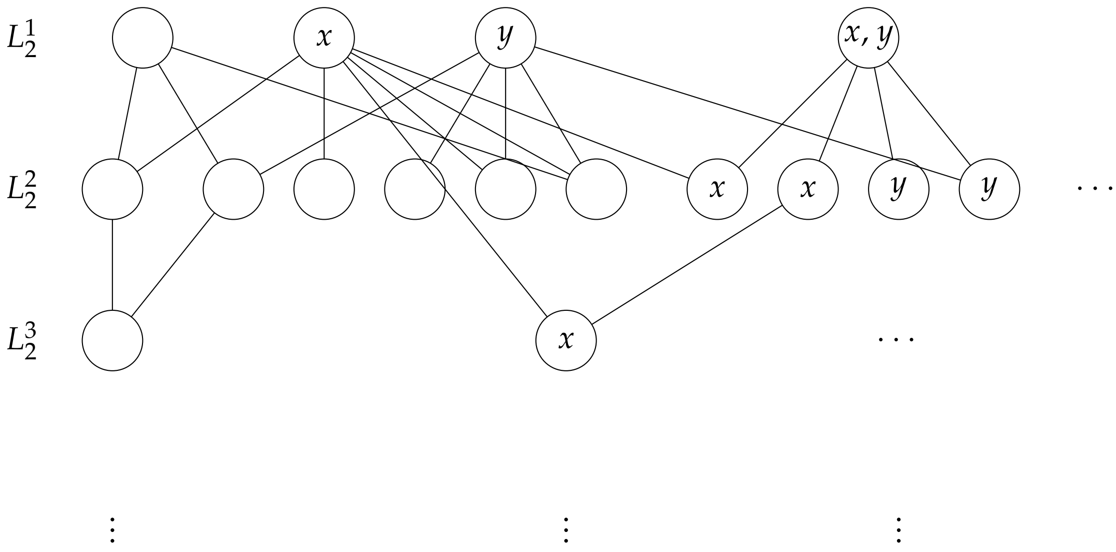

Figure 2.2: $\mathcal{U}(2)$

- Whenever $a \geq b$, $f(a) \geq f(b)$.
- Whenever $a' \geq f(b)$, there is an a such that $f(a) = a'$ and $a \geq b$.

If there is a surjective p-morphism from $\mathcal{M}$ to $\mathcal{N}$, then $\mathcal{N}$ is called a *p-morphic reduction* of $\mathcal{M}$.

If a Kripke model has no p-morphic reductions besides itself, then it is called *p-morphically reduced*.

Definition 10 (Intuitionistic Equivalence)**.** Two Kripke models $\mathcal{M}$ and $\mathcal{N}$ are called *intuitionistically equivalent* if for all ϕ, $\mathcal{M} \Vdash \phi$ iff $\mathcal{N} \Vdash \phi$.

Lemma 6 (see [4])**.** *All finite Kripke models $\mathcal{M}$ have a unique p-morphic reduction which is p-morphically reduced. This is called the p-morphic reduced form of $\mathcal{M}$.*

Proposition 3 (see [5])**.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are finite. Then $\mathcal{M}$ and $\mathcal{N}$ are intuitionistically equivalent iff they have a common p-morphic reduction.*

Definition 11 (Reductions and Expansions)**.** Let $\mathcal{M}$ be a Kripke model defined over V_n, and let α and β be nodes of $\mathcal{M}$ such that $\text{Forces}_{\mathcal{M}}(\alpha) \cap V_n = \text{Forces}_{\mathcal{M}}(\beta) \cap V_n$.

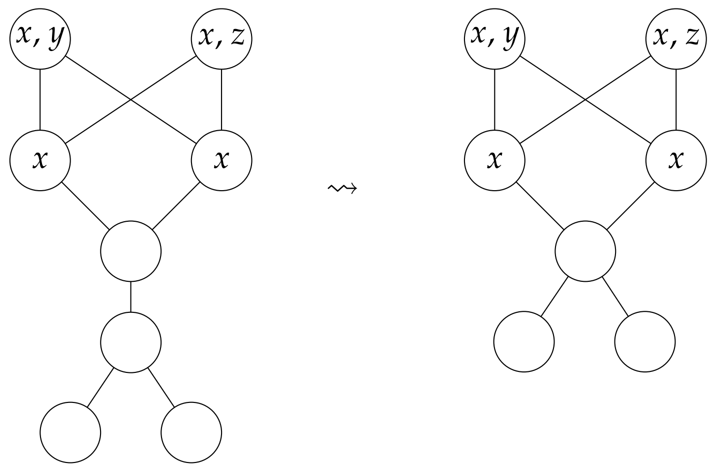

Figure 2.3: A Type I reduction

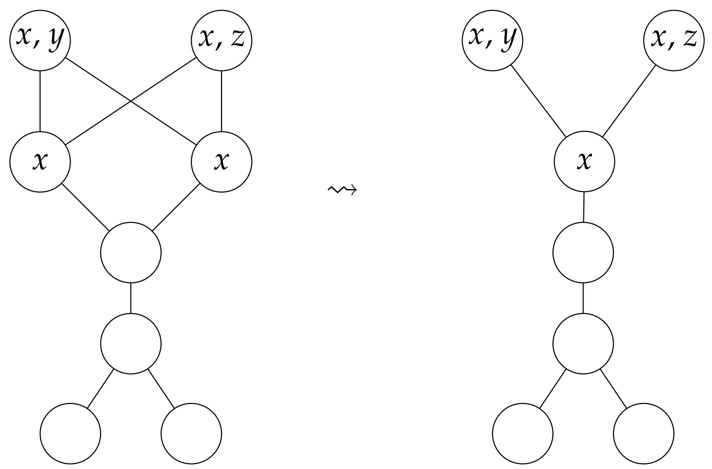

Figure 2.4: A Type II reduction

- If β is α's sole successor, the Kripke model $\mathcal{M}'$ obtained by identifying α and β is called a Type I reduction of $\mathcal{M}$. The canonical map from $\mathcal{M}$ to $\mathcal{M}'$ is a p-morphism.
- If α and β have exactly the same (proper) successors, then the Kripke model $\mathcal{M}'$ obtained by identifying α and β is called a Type II reduction of $\mathcal{M}$. The canonical map from $\mathcal{M}$ to $\mathcal{M}'$ is a p-morphism.
- In either case, $\mathcal{M}$ is called a Type I or Type II expansion of $\mathcal{M}'$.

Proposition 4 (see [5] and [4])**.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are finite. Then $\mathcal{M}$ and $\mathcal{N}$ are intuitionistically equivalent just in case there is a sequence of Type I and II reductions taking $\mathcal{M}$ to $\mathcal{N}$.*

2.5 Trees

Definition 12 (Alphabets and Ground Terms)**.** Let $\mathcal{A}$ be a set of symbols, which we will call an alphabet. We define $T(\mathcal{A})$ to be the smallest set of sequences of symbols from $\mathcal{A}$ such that if $t_i \in T(\mathcal{F})$ for $1 \leq i \leq m$ and $a \in \mathcal{A}$, then $a(t_1, \ldots, t_m) \in T(\mathcal{F})$. (Here "$a(t_1, \ldots, t_m)$" is a purely syntactic sequence of symbols, not an actual function application.)

We call $T(\mathcal{A})$ the set of ground terms over the (unranked) alphabet $\mathcal{A}$.

Definition 13 (Trees)**.** For us, a "tree" is always a finite labeled tree.

Specifically, a tree over a set $\mathcal{A}$ is a finite partial order $(P, \leq)$ to each element of which is associated a member of $\mathcal{A}$ and which has the property that for all $p \in P$, $p{\downarrow}$ is a linear order.

Note that the set of ground terms over $\mathcal{F}$ and the set of tree over $\mathcal{F}$ are essentially the same; we therefore denote them both by $T(\mathcal{F})$.

Definition 14 (Kripke Trees)**.** A Kripke tree is a Kripke model $\mathcal{M}$ over some V_n such that underlying partial order P of $\mathcal{M}$ together with the association $p \mapsto \mathrm{Forces}(p) \cap V_n$ is an element of $T(\mathcal{P}(V_n))$.

We let the alphabet that a Kripke tree is over be $\mathcal{A}_n$.

Definition 15 (Kripke Alphabet)**.** For $n \in \omega$, let $\mathcal{A}_n = \mathcal{P}(V_n)$, considered as an alphabet. We may also call $\mathcal{A}_n$ the Kripke alphabet over V_n.

Lemma 7 (See [5])**.** *A propositional formula $\phi \in F_n$ is a intuitionistic tautology iff for all finite Kripke models $\mathcal{M}$, defined over V_n, $\mathcal{M} \Vdash \phi$.*

Proof. This follows from the fact that all finite Kripke models are p-morphic to Kripke trees. This can be easily proved using only Type II expansions. □

Given $\phi \in F_n$, we let $K(\phi)$ be the set of Kripke trees $\mathcal{M}$ defined over V_n which force ϕ.

The following lemma will be used in Chapter 4.

Lemma 8. *Let $\mathcal{M}$ be a finite Kripke tree. Let $\alpha \leq \beta \leq \gamma$ be nodes of $\mathcal{M}$. If $\alpha{\uparrow}$ is intuitionistically equivalentb to $\gamma_u p$, then so is $\beta{\uparrow}$.*

Proof. Suppose $\beta \Vdash \phi$. Then, since $\beta \leq \gamma$, $\gamma \Vdash \phi$.

Conversely, if $\beta \nVdash \phi$, then, since $\alpha \leq \gamma$, $\alpha \nVdash \phi$.

$\square$

CHAPTER 3
IMPLICATIVE FRAGMENT

Definition 16 (Implicative Fragment of H_n)**.** The implicative fragment of H_n is the smallest set containing all n free generators and closed under $\to$.

The primary result of this chapter is a combinatorial proof of the fact that for $n \in \omega$, the implicative fragment of H_n is finite.

The content of the claim that the proof is "combinatorial" is that nothing about Heyting algebras or Kripke models will be used: the plan of the proof is to produce certain rewrite rules which will allow one to rewrite any implicational formula in n variables to one which is logically equivalent and of length $\leq l$, where l depends only on n.

3.1 The Left-Associated Case

The left-associated case is an instructive subcase.

Definition 17 (Left-Associated Formulas)**.** The set of left-associated implicative formulas L_n over the variables V_n is the smallest set containing V_n and such that if $\phi \in L_n$ and $x \in V_n$, then $\phi \to x \in L_n$.

For this section only, I will omit parentheses in displaying formulas, since the parenthesization is unambiguous.

Definition 18. If ϕ is a formula of the form

$$x_{m-1} \to \cdots \to x_0$$

then m is called the length of ϕ. For $0 \leq i \leq m-1$, we define ϕ_i to be x_i and $\bar{\phi}_i$ to be $x_{m-1} \to \cdots \to x_i$.

Proposition 5. *There are only finitely many left-associated formulas up to logical equivalence.*

Proof. This is based on two steps:

Lemma 9. *For any $\phi \in F_n$ and $x \in V_n$, $\phi \to x \to x \to x$ is logically equivalent to $\phi \to x$.*

Proof. This is simply a standard proof in intuitionistic logic. □

Lemma 10. *Suppose x, y, and x_i for $0 \leq i \leq m-1$ are propositional variables. Further, suppose $0 < j < k \leq m-1$ are such that $x_0 = x$, $x_j = y$, and $x_k = x$. Let ψ be any formula. Then*

$$\psi \to y \to x_{m-1} \to \cdots \to x_0$$

is equivalent to

$$\psi \to x \to x_{m-1} \to \cdots \to x_0$$

Lemmas 9 and 10 imply that every left-associated formula is logically equivalent to one of length $\leq 3n^3 + 2$ (thus giving the proposition) as follows: Consider the rewrite rules on propositions suggested by Lemmas 9 and 10:

$$\psi \to x \to x \to x \to y_1 \to \cdots \to y_k \rightsquigarrow \psi \to x \to y_1 \to \cdots \to y_k.$$

and

$$\phi \to y \to x_m \to \cdots \to x_1 \rightsquigarrow \phi \to x \to x_m \to \cdots \to x_1,$$

where, in the second case, the hypotheses of Lemma 10 hold. It is easy to see that this system of rewrite rules is strongly normalizing. Furthermore, given any proposition ϕ, its normal form ρ under these rewrite rules will have length $\leq 3n^3 + 2$: I claim that each triple $(\rho_{3k}, \rho_{3k+1}, \rho_{3k+2})$ is distinct (as in Definition 18, ρ_{3k} refers to the $3k$th propositional variable from the right in ρ, et.: Suppose not and that

$$(\rho_{3k}, \rho_{3k+1}, \rho_{3k+2}) = (\rho_{3k'}, \rho_{3k'+1}, \rho_{3k'+2})$$

for $k < k'$. By our first rewrite rule, we know that no triple has all three elements the same, so without loss of generality, say $\rho_{3k} \neq \rho_{3k+1}$. But then we could apply our second rewrite rule, as we have $3k < 3k+1 < 3k' < 3k'+1$ and $\rho_{3k} = \rho_{3k'}$, $\rho_{3k+1} = \rho_{3k'+1}$, $\rho_{3k} \neq \rho_{3k+1}$.

There are only n^3 possible distinct triples. Since the length of ρ is at most 2 over a multiple of 3, this gives a maximum length of $3n^3 + 2$.

Proof of Lemma 10. Let ϕ be

$$\psi \to y \to x_{m-1} \to \cdots \to x_0$$

and let ψ be

$$\psi \to x \to x_{m-1} \to \cdots \to x_0.$$

We will use the following sublemma:

Sublemma 1. *Let ϕ and ϕ be formulas.*

Let Δ be a set of formulas, $\Gamma_0(\phi, \phi') = \{\phi_0, \phi'_1\}$, $\Gamma_{2i}(\phi, \phi') = \Gamma_{2i-1}(\phi, \phi') \cup \{\phi'_{2i+1}\}$ for all i, and $\Gamma_{2i+1}(\phi, \phi') = \Gamma_{2i} \cup \{\phi_{2i+2}\}$.

Then if i is even and $\Delta, \Gamma_i(\phi, \phi') \vdash \phi'_i$, then $\Delta, \Gamma_{i-1}(\phi, \phi') \vdash \phi'_{i-1}$ and if i is odd and $\Delta, \Gamma_i(\phi, \phi') \vdash \phi_i$, $\Delta, \Gamma_{i-1}(\phi, \phi') \vdash \phi_{i-1}$.

Proof. Suppose i is odd and $\Delta, \Gamma_i(\phi, \phi') \vdash \phi_i$. Then, as $\Gamma_i(\phi, \phi') = \Gamma_{i-1}(\phi, \phi') \cup \{\bar{\phi}_{i+1}\}$, $\Delta, \Gamma_{i-1}(\phi, \phi') \vdash \bar{\phi}_{i+1} \to \phi_i$. Since $\bar{\phi}_{i+1} \to \phi_i = \bar{\phi}_i$, $\Delta, \Gamma_{i-1}(\phi, \phi') \vdash \bar{\phi}_i$.

Then because $\bar{\phi}_{i-1} \in \Gamma_{i-1}(\phi, \phi')$, and $\bar{\phi}_{i-1} = \bar{\phi}_i \to \phi_{i-1}$, $\Delta, \Gamma_i(\phi, \phi') \vdash \phi_{i-1}$.

The case where i is even is similar. □

Sublemma 2. $\phi, \bar{\phi}'_1, y \vdash x$

Proof. Clearly, $\{y\}, \Gamma_j(\phi, \phi') \vdash \phi_j = y$. By repeated application of Sublemma 1, and the fact that $\phi_i = \phi'_i$ for $i \leq j$, $\{y\}, \Gamma_0(\phi, \phi') \vdash \phi_0 = x$. Since $\Gamma_0 = \{\phi, \bar{\phi}'_1\}$, we are done. □

Sublemma 3. $\phi', \bar{\phi}_1, y, \vdash x$

Proof. Same as Sublemma 2, but use $\Gamma_j(\phi', \phi)$ instead of $\Gamma_j(\phi, \phi')$ □

Sublemma 4. $\bar{\phi}_j, \bar{\phi}'_{j+1}, x \vdash y$

Proof. Clearly, $\{x\}, \Gamma_k(\phi, \phi') \vdash \phi_k = x$. By repeated application of Sublemma 1 and the fact that $\phi_i = \phi'_i$ for $i \leq k$, $\{x\}, \Gamma_j(\phi, \phi') \vdash \phi_j = y$. □

Sublemma 5. $\bar{\phi}'_j, \bar{\phi}_{j+1}, x \vdash y$

Proof. Same as Sublemma 4, but use $\Gamma_k(\phi', \phi)$ instead of $\Gamma_k(\phi, \phi')$. □

□

To show that ϕ implies ϕ': Suppose that m is odd. By Sublemma 1, we will be done if we can show that $\Gamma_m(\phi, \phi') \vdash \phi_m = y$, as then $\Gamma_0(\phi, \phi') = \{\phi, \bar{\phi}'_1\} \vdash x$ and $\phi \vdash \bar{\phi}'_1 \to x = \phi'$.

Since m is odd, $\Gamma_m(\phi, \phi')$ contains $\bar{\phi}_{m+1}$ and $\bar{\phi}'{}_m$, so $\Gamma_m(\phi, \phi') \vdash \bar{\phi}_m = x$. Therefore, $\Gamma_m(\phi, \phi') \vdash y$ by Sublemma 2.

The case where m is even is the same except that we must now apply Sublemma 4 if j is even and 5 if j is odd.

To show that ϕ' implies ϕ: We reason as above, using $\Gamma_m(\phi', \phi)$ in place of $\Gamma_m(\phi, \phi')$. If m is odd, apply Sublemma 3, if m is even: apply Sublemma 5 if j is even and Sublemma 4 if j is odd. □

3.2 The General Case

3.2.1 Preliminaries on Trees

It will be convenient to associate formulas with labeled trees. First, I fix some notation involving trees.

Definition 19 (Trees and Related Basic Concepts)**.** A *tree* $T = (|T|, \leq)$ is a finite set $|T|$ together with an order relation $\leq$ on it such that there is an $a \in |T|$ such that $a \leq b$ for all $b \in |T|$ and such that for each $b \in |T|$, $\{c \in |T| \mid c \leq b\}$ is linearly ordered by $\leq$. Such an a is called the *root* of T. (Outside of this paragraph, the variable a is not intended to denote a root of a tree. Also, we will suppress the distinction between T and $|T|$.)

If $a < b$ and there is no c such that $a < c < b$ then b is called a *child node* of a.

If a is a node of T, then the tree $(\{b \mid b \leq a\}, \leq')$ where $\leq'$ is the restriction of $\leq$ to $\{b \mid b \leq a\}$ is called the *subtree of T below a*.

If b is a child node of a, then the subtree of T below b is called a *child tree* of a.

A sequence of nodes $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is called a *branch* of T if α_1 is the root node of T and α_{i+1} is a child node of α_i for $1 \leq i \leq n-1$. (α_n does not have to be a leaf.)

A labeled tree T with labels in a set S is a tree T together with a function l_T from the nodes of T to S. All of the above definitions carry over in an obvious way to labeled trees.

Definition 20 (Association of labeled trees to formulas)**.** Given a formula ϕ, define $T(\phi)$ as follows: If $\phi = x$, then let $T(\phi)$ be a single node labeled with x.

If ϕ is $(\phi_0 \to \cdots \to (\phi_n \to x) \cdots)$, then let $T(\phi)$ have root node labeled with x with child trees $T(\phi_0), \dots, T(\phi_n)$.

Since $\phi \to (\psi \to x)$ is equivalent to $\psi \to (\phi \to x)$, it is easily shown that if $T(\phi) = T(\psi)$, then ϕ and ψ are logically equivalent.

For example, $T(((x_1 \to x_2) \to x_3) \to ((x_4 \to (x_5 \to x_6)) \to x_7)$ is as in Figure 3.1.

From now on, I will drop the T and simply identify ϕ and $T(\phi)$, when no confusion will result.

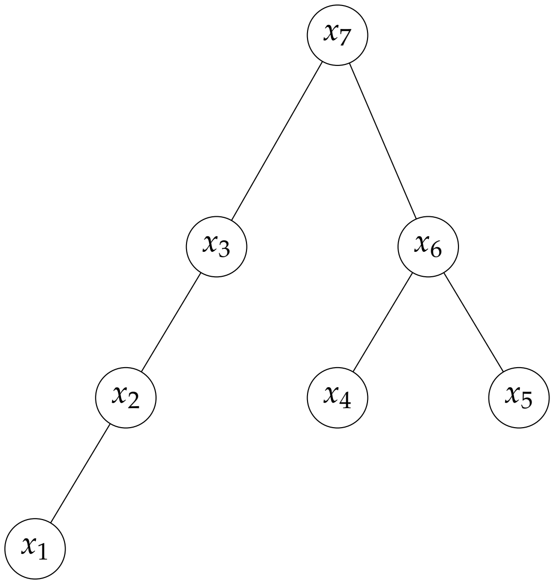

Figure 3.1: $T(((x_1 \to x_2) \to x_3) \to ((x_4 \to (x_5 \to x_6)) \to x_7)$

3.2.2 Plan of the Proof

The proof will use analogues of Lemmas 9 and 10 in a roughly similar manner.

We will show the following:

Lemma 11 (Analogue of Lemma 9)**.** *There is a number M, depending only on n, with the following property: Suppose that T is a labeled tree and x is a propositional variable. Let*

$$f(T) = \max\{m \mid \textit{there is a branch } (\alpha_0, \ldots, \alpha_{m-1}) \textit{ through } T \textit{ and } \forall i\, l(\alpha_i) = x\}$$

Suppose $f(T) \geq M$. Then there is a T' logically equivalent to T such that $f(T') < f(T)$.

Lemma 12 (Analogue of Lemma 10)**.** *Let T be a labeled tree and $(\alpha_0, \ldots, \alpha_m)$ be a branch through T. Suppose that $0 < j < k \leq m - 1$ are such that $l(\alpha_0) = x$, $l(\alpha_j) = y$, $l(\alpha_k) = x$, and $l(\alpha_m) = y$.*

Let T' be the labeled tree whose underlying tree is the same as the underlying tree of T and with label function given by

$$l_{T'}(\alpha) = \begin{cases} l_T(\alpha) & \alpha \neq \alpha_m \\ x & \alpha = \alpha_m \end{cases}$$

Then T and T' are logically equivalent.

I'll now indicate how to finish the proof of the main theorem given these two lemmas.

Definition 21 (Label Changes)**.** Let $(\alpha_0, \alpha_2, \ldots, \alpha_{m-1})$ be a branch through a tree T. A pair (α_i, α_{i+1}) is called a change of labels if $l(\alpha_i) \neq l(\alpha_{i+1})$.

Corollary 2 (Of Lemma 12)**.** *Every tree T is equivalent to some T' such that no branch of T' changes labels more than $n(n-1)$ times (where n is the number of propositional variables).*

Proof. By using Lemma 12 to suggest a rewrite rule in an analogous manner to the way we used Lemma 10 to suggest a rewrite rule, we can show that each tree T is equivalent to a tree T' such that for all branches $(\alpha_0, \ldots, \alpha_{m-1})$ through T' there are no two label changes (α_i, α_{i+1}) and (α_j, α_{j+1}) such that $j > i+1$, $l(\alpha_i) = l(\alpha_j)$ and $l(\alpha_{i+1}) = l(\alpha_{j+1})$.

Since there are only $n(n-1)$ possible distinct types of label changes, the statement follows. □

Corollary 3 (Of Lemma 11)**.** *Let T be a tree and x be a label such that all nodes of T that aren't leaves are labeled with x. Then T is logically equivalent to T' where T' has the same property and has height at most $M+1$.*

Corollary 4 (Of Corollary 3)**.** *Fix a label x and let $C = \{T \mid \text{all non-leaves of } T \text{ are labeled } x\}$. Then the number of equivalence classes under logical equivalence of C is finite.*

Proof. Given that if the root of a tree has two isomorphic child trees, then one can be eliminated while preserving logical equivalence, it is easy to prove that the number of equivalence classes of trees of some fixed height m is finite. The statement then follows immediately from Corollary 3. □

Definition 22 (Top Component of a Tree)**.** The *top component* $c(T)$ of a tree T is defined as follows: We let $c(T)$ be the smallest set such that the root node of T is in $c(T)$, and if $\alpha \in c(T)$ and β is a child node of α and $l(\alpha) = l(\beta)$, then $\beta \in c(T)$.

Definition 23 (Type of a Tree)**.** Call a tree of type 0 if all of its nodes have the same label.

Call a tree T of type $\leq n+1$ if for every node $\alpha \in c(T)$, all of α's child trees whose roots are not in $c(T)$ are of type $\leq n$.

Call a tree T of type n if it is of type $\leq n$ but not $\leq n-1$.

Lemma 13. *For each n there are only finitely many equivalence classes of trees of type n.*

Proof. This is clear for $n = 0$.

Since by induction we know that there are only finitely many trees of type n, we can apply Corollary 4 (replacing trees of type n by a set of fresh labels) to show that there only finitely many trees of type $n + 1$. $\square$

However, it is relatively easy to see that a tree T has type $\geq n$ just in case it has a branch which changes labels n times. Since we know from Corollary 2 that there is an R such that each T is equivalent to a T' such that no branch changes labels more than R times, we are done.

3.2.3 Analogue of Sublemma 1

The following sublemma will be used in the proofs of Lemmas 11 and 12.

Sublemma 6. *Let W be a labeled tree and let $(\beta_1, \ldots, \beta_a)$ be a branch through it. Let W' be a labeled tree obtained from W by removing the child trees of β_a, putting in new child trees of it, and possibly changing the label of β_a. Call $(\beta'_1, \ldots, \beta'_a)$ the corresponding branch in W'.*

Let W_i be the subtree of W below β_i and let W'_i be the subtree of W' below β'_i. For $2 \leq i \leq a$, let $W_{i,0} = W_i$ and let $\{W_{i,1}, \ldots, W_{i,g_i}\}$ enumerate the child trees of β_{i-1} other than W_i. Make a similar definition for $W'_{i,j}$, so that if $j > 0$, and $1 \leq i \leq a$, $T_{i,j} = T'_{i,j}$.

Let Γ_0 be a set of formulas. Let $\Gamma_1 = \Gamma_0 \cup \{W_1, W'_{2,0} \ldots, W'_{2,g_2}\}$ and let

$$\Gamma_m = \begin{cases} \Gamma_{m-1} \cup \{W_{m+1,0}, \ldots, W_{m+1,g(m+1)}\} & m \text{ is even} \\ \Gamma_{m-1} \cup \{W'_{m+1,0}, \ldots, W'_{m+1,g(m+1)}\} & m \text{ is odd} \end{cases}$$

for $2 \leq m \leq a$.

Then, for $1 \leq m \leq a$, if m is even and $\Gamma_m \vdash l(\beta_m)$, $\Gamma_{m-1} \vdash l(\beta_{m-1})$ and if m is odd and $\Gamma_m \vdash l(\beta'_m)$, $\Gamma_{m-1} \vdash l(\beta'_{m-1})$.

Proof. If m is odd and $\Gamma_m \vdash l(\beta'_m)$, then we have

$$\Gamma_{m-1} \vdash W'_{m+1,0} \rightarrow (W'_{m+1,1} \rightarrow (\cdots W'_{m+1,g(m+1)} \rightarrow l(\beta'_m) \cdots))$$

so $\Gamma_{m-1} \vdash W'_m$. Since $W'_{m-1} \in \Gamma_{m-1}$ and $W_{m,i} = W'_{m,i} \in \Gamma_{m-1}$ for $i > 0$, $\Gamma_{m-1} \vdash l(\beta'_{m-1})$.

Similarly, if m is even and $\Gamma_m \vdash l(\beta_m)$, $\Gamma_{m-1} \vdash l(\beta_{m-1})$. □

3.2.4 Proof of Lemma 11

Lemma 14. *Let T be a labeled tree, and let $(\alpha_1, \alpha_2, \ldots, \alpha_{2n+1})$ be a branch of T such that there is some variable (say x) such that each of α_1, α_{2n} and α_{2n+1} are labeled with x. If for each child tree of α_{2n+1} whose root is not labeled by x there is a child tree of α_1 that is isomorphic (as a labeled tree) then T is logically equivalent to T' where T' is defined as follows:*

Let there be i child nodes of α_{2n+1} labeled with x and let $\{V_{j,1}, \ldots, V_{j,g_j}\}$ enumerate the child trees of the jth such child node of α_{2n+1}. Let $\{U_1, \ldots, U_q\}$ enumerate the child trees of α_{2n} apart from the one with root α_{2n+1}.

Let $|T'| = A \cup B \cup C \cup D$ where A is all nodes of T not below (and not equal to) α_{2n}, B is i copies of α_{2n}, call them $\alpha_{2n,1} \ldots, \alpha_{2n,i}$, C is i copies of each of $U_1, \ldots, U_q$ and D is a copy of each $V_{j,k}$. The ordering on A is inherited from T. (Also let α'_b be the corresponding node of α_b in T for $1 \leq b \leq 2n-1$.) Each copy of α_{2n} in B is a child node of the copy of α_{2n-1} in A. Each copy of α_{2n} in B has a copy from D of each U_r as a child tree. The jth copy of α_{2n} has each $V_{j,k}$ as a child tree.

Proof. Let T_m be the subtree of T below α_m for $1 \leq m \leq 2n+1$. Let T'_m be the subtree of T' below α'_m for $1 \leq m \leq 2n-1$. Define $T_{m,j}$ for $2 \leq m \leq 2n+2$ and $T'_{m,j}$ for $2 \leq m \leq 2n$ as has been done previously.

To show that T implies T': We must show that $T_1, T'_{2,0}, \ldots, T'_{2,k} \vdash x$. Apply Sublemma 6 with $\Gamma_0 = \emptyset$, $W = T$, $W' = T'$, $\beta_i = \alpha_i$, $\beta'_i = \alpha'_i$ and $a = 2n-1$. We will be done if we can show that $\Gamma_{2n-1} \vdash l(\alpha_{2n-1}) = l(\alpha'_{2n-1})$.

Γ_{2n-1} contains each $T'_{2n,c}$ and it contains T_{2n-1}. T_{2n-1} has hypotheses $T_{2n,i}$ for all i and conclusion $l(\alpha_{2n-1})$. The formulas $T_{2n,i} = T'_{2n,i}$ for $i > 0$ are in Γ_{2n-1}. So we want to show that $\Gamma_{2n-1} \vdash T_{2n}$.

The formula T_{2n} has hypotheses $T_{2n+1,i}$ for all i and conclusion x. So consider $\Gamma_{2n-1} \cup \{T_{2n+1,i}\}$. The formula T_{2n+1} has hypotheses W for all of the child trees of α_{2n+1}. Those subtrees whose root is not labeled by x are in Γ_{2n-1} because they are of the form $T_{2,i}$ for some i. The other hypotheses have the form $Z \to x$ for a tree Z such that there is an $\alpha'_{2n,i}$ with child trees Z and isomorphic copies of all the $T_{2n+1,j}$. As $l(\alpha'_{2n,i})$, $\Gamma_{2n-1} \vdash Z \to x$. Thus $\Gamma_{2n-1} \cup \{T_{2n+1,i}\} \vdash x$ and we are done.

To show that T' implies T: We must show that $T'_1, T_{2,0}, \ldots, T_{2,k} \vdash x$. Apply Lemma 1 with $\Gamma_0 = \emptyset$, $W = T'$, $W' = T$, $\beta_i = \alpha'_i$, $\beta'_i = \alpha_i$ and $a = 2n-1$. We will be done if we can show that $\Gamma_{2n-1} \vdash l(\alpha'_{2n-1}) = l(\alpha_{2n-1})$.

Γ_{2n-1} contains T'_{2n-1}. T'_{2n-1} has hypotheses $T'_{2n,i}$ for all i and conclusion $l(\alpha_{2n-1})$. So take some particular $T'_{2n,i}$ and we will show that $\Gamma_{2n-1} \vdash T'_{2n,i}$. The conclusion of $T'_{2n,i}$ is x and the hypotheses are W for W a child tree of α_{2n} and Z for Z a child tree of a child node that is labeled x of α_{2n+1}. Let these hypotheses be $Z_1, \ldots, Z_d$ and $W_1, \ldots, W_d$. We must thus show that

$$\Gamma_{2n-1} \cup \{Z_1, \ldots, Z_d, W_1, \ldots, W_d\} \vdash x$$

Γ_{2n-1} contains T_{2n} which has conclusion x and hypotheses $Z_1, \ldots, Z_d$ and the hypothesis T_{2n+1}. So we are now reduced to showing that

$$\Gamma_{2n-1} \cup \{Z_1, \ldots, Z_d, W_1, \ldots, W_d\} \vdash T_{2n+1}$$

The hypotheses of T_{2n+1} include a hypothesis of the form $Z_1 \to (\cdots (Z_d \to x) \cdots)$, so we have achieved our goal. Tracing back through the steps, we see that we are done. $\square$

3.2.5 Proof of Lemma 12

Let l be the function which takes nodes to labels for both T and T' and let α'_i be the node in T' corresponding to α_i in T. Thus, $l(\alpha'_i) = l(\alpha_i)$ unless $i = n$. For $i > 1$, let $g(i)$ be one less than the number of subtrees of α_{i-1} (equivalently, α'_{i-1}).

Let T_i be the subtree of T below α_i and T'_i be the subtree of T' below α'_i. For $i > 1$, let $T_{i,0} = T_i$ and let $\{T_{i,1}, \ldots, T_{i,g(i)}\}$ enumerate the child trees of α_{i-1} apart from T_i. Define $T'_{i,m}$ similarly. Note that every child tree of α_n is the same as a child tree of α'_n. Extending g, let $\{T_{n+1,0}, \ldots, T_{n+1,g(n+1)}$ be an enumeration of the child trees of α_n and let $T'_{n+1,i} = T_{n+1,i}$. Note that if $m > 0$, $T_{i,m} = T'_{i,m}$.

Note that we have by definition that

$$T_m = T_{m+1,0} \to (T_{m+1,1} \to (\cdots T_{m+1,g(m+1)} \to l(\alpha_m) \cdots))$$

and similarly for T'.

Sublemma 7. $T_1, T'_{2,0}, \ldots, T'_{2,g(2)}, y \vdash x$

Proof. Use Sublemma 6 with $\Gamma_0 = \{y\}$, $W = T$, $W' = T'$, $\beta_i = \alpha_i$, $\beta'_i = \alpha'_i$, and $a = n$. Since $y \in \Gamma_j$, $\Gamma_j \vdash y = l(\alpha_j) = l(\alpha'_j)$. So by repeated application of Sublemma 6, $\Gamma_1 \vdash x$ and we are done. $\square$

Sublemma 8. $T'_1, T_{2,0}, \ldots, T_{2,g(2)}, y \vdash x$

Proof. Use Sublemma 6 as above, but with $W = T'$ and $W' = T$ (and making the other corresponding switches). $\square$

Sublemma 9. $T_j, T'_{j+1,0}, \ldots, T'_{j+1,g(j+1)}, x \vdash y$.

Proof. Use Sublemma 6, taking $\Gamma_0 = \{x\}$, $W = T_j$, $W' = T'_j$, $\beta_i = \alpha_{j+i-1}$, $\beta'_i = \alpha'_{j+i-1}$ and $a = n - j + 1$. $\square$

Sublemma 10. $T'_j, T_{j+1,0}, \ldots, T_{j+1,g(j+1)}, x \vdash y$.

Proof. Use Sublemma 6 as in Sublemma 9 but with $W = T'_j$ and $W' = T_j$ (and making the other corresponding switches). $\square$

To show that T_1 implies T'_1: We apply Sublemma 6 with $\Gamma_0 = \emptyset$, $W = T$, $W' = T'$, $\beta_i = \alpha_i$, $\beta'_i = \alpha'_i$ and $a = n$. By that Sublemma, we will be done if we can show the following: In the case that n is odd, $\Gamma_n \vdash x$. In the case that n is even, $\Gamma_n \vdash y$.

If n is odd, then Γ_n contains T_n. Since it also contains $T'_{n+1,0}, \ldots T'_{n+1,g(n+1)}$ which are equal to $T_{n+1,0}, \ldots, T'_{n+1,g(n+1)}$, $\Gamma_n \vdash l(\alpha_n)$, so $\Gamma_n \vdash y$. Applying Sublemma 7, we see that $\Gamma_n \vdash x$ as required.

If n is even, everything is the same except we must now apply Sublemma 9 if j is odd and Sublemma 10 if j is even.

To show that T'_1 implies T_1: Apply Sublemma 6 with $\Gamma_0 = \emptyset$, $W = T'$, $W' = T$, $\beta_i = \alpha'_i$, $\beta'_i = \alpha_i$ and $a = n$. We will be done if we can show the following: In the case that n is even, $\Gamma_n \vdash x$. In the case that n is odd, $\Gamma_n \vdash y$. In the case that n is even, we reason as above and apply Sublemma 8. In the case that n is odd we apply Sublemma 10 if j is odd and Sublemma 4 if j is even.

CHAPTER 4
TREE AUTOMATA

4.1 Association between Formulas and Automata

Definition 24 (Basic Tree Automata Notions)**.** A(n unbounded) deterministic bottom-up tree automaton over alphabet $\mathcal{A}$ is a tuple $\mathcal{T} = (\textbf{states}_{\mathcal{T}}, \textbf{accept}_{\mathcal{T}}, \Delta_{\mathcal{T}})$ where $\textbf{states}_T$ is a set, $\textbf{accept}_{\mathcal{T}} \subseteq \textbf{states}_T$, and $\Delta_{\mathcal{T}}$ is a function with domain $\mathcal{A}$ such that for $a \in \mathcal{A}$, $\Delta_{\mathcal{T}}(a)$ is a function from $\textbf{states}_{\mathcal{T}}^{<\omega}$ to $\textbf{states}_{\mathcal{T}}$.

We define $\mathcal{T}(t)$ for $t \in T(\mathcal{A})$ inductively by letting

$$\mathcal{T}(a(t_1, \ldots, t_m)) = \Delta_{\mathcal{T}}(a)(\mathcal{T}(t_1), \ldots, \mathcal{T}(t_n)).$$

For $t \in T(\mathcal{A})$ we say that $\mathcal{T}$ accepts t if $\mathcal{T}(t) \in \textbf{accept}_{\mathcal{T}}$.

Definition 25 (Prefix-Closed Tree Automata)**.** We say that $\mathcal{T}$ is prefix-closed if whenever $q \in \textbf{states}_{\mathcal{T}} - \textbf{accept}_{\mathcal{T}}$ and s is a sequence in $\textbf{states}_{\mathcal{T}}^{<\omega}$ containing q, $\Delta(a)_{\mathcal{T}}(s) \in \textbf{states}_{\mathcal{T}} - \textbf{accept}_{\mathcal{T}}$ for all $a \in \mathcal{A}$.

Definition 26 (Tree Automata and Partial Orders)**.** Let $\leq$ be a partial order on $\textbf{states}_{\mathcal{T}}$. We say that $\mathcal{T}$ supports $\leq$ whenever $a \in \mathcal{A}$ and s is a sequence in $\textbf{states}_{\mathcal{T}}^{<\omega}$ containing q, $\Delta_{\mathcal{T}}(a)(s) \leq q$.

Definition 27 (Injective Tree Automata, $\Delta_{\mathcal{T}}^{-1}$)**.** We say that $\mathcal{T}$ is *injective* if for all elements $a \neq b$ of $\mathcal{A}$, $\operatorname{ran} \Delta_{\mathcal{T}}(a) \cap \operatorname{ran} \Delta_{\mathcal{T}}(b)$ contains only non-accept states.

If $\mathcal{T}$ is injective, then we define the partial function $\Delta_{\mathcal{T}}^{-1}$ from $\textbf{states}_T$ to $\mathcal{A}$ by letting

$$\Delta_{\mathcal{T}}^{-1}(q) = a \text{ such that } q \in \operatorname{ran} \Delta_{\mathcal{T}}(a)$$

if there is such an a, and letting $\Delta_{\mathcal{T}}^{-1}(q)$ be undefined if there is no such a. If $\Delta_{\mathcal{T}}^{-1}(q)$ is undefined, then we write $\Delta_{\mathcal{T}}^{-1}(q) = \uparrow$.

Note that $\Delta_{\mathcal{T}}^{-1}$ isn't the literal inverse of $\Delta_{\mathcal{T}}$.

It is clear that every tree automaton is equivalent to an injective one. From here on out, we will assume that all tree automata are injective whenever it is convenient.

Definition 28 (Tree Automata which Accept Only Kripke Models)**.** The automaton $\mathcal{T}$ is said to *accept only Kripke models* if the following hold:

1. The alphabet of $\mathcal{T}$ is $\mathcal{A}_n$ for some n.
2. $\mathcal{T}$ is injective.
3. Whenever $\Delta_{\mathcal{T}}^{-1}(q) = a$, $b \not\subseteq a$ and s is a sequence in $\mathbf{states}_{\mathcal{T}}^{<\omega}$ containing q, $\Delta(b)(s) \in \mathbf{states}_{\mathcal{T}} - \mathbf{accept}_{\mathcal{T}}$.

The set of elements of $T(\mathcal{F})$ accepted by $\mathcal{T}$ is called $L(\mathcal{T})$. It is called the set *recognized* by $\mathcal{T}$. If a set is recognized by some $\mathcal{T}$, then it is called *recognizable.*

Definition 29 (Intuitionistic-Equivalence-Respecting Automata)**.** An automaton $\mathcal{T}$ is called *intuitionistic-equivalence-respecting* if satisfies the following properties:

1. It accepts only Kripke models.
2. For every (accept) state $q \in \mathbf{states}_{\mathcal{T}}$, the set $\{\mathcal{M} \mid \mathcal{T}(\mathcal{M}) = s\}$ is closed under intuitionistic-equivalence.

The following lemma will be a useful characterization of intuitionistic-equivalence-respecting automata.

Lemma 15. *An automaton $\mathcal{T}$ is* intuitionistic-equivalence-respecting *iff it satisfies the following properties:*

1. *It accepts only Kripke models.*
2. *It supports a partial order $\leq$.*
3. *The transition functions disregard order and multiplicity. That is, if $\{q_1, \ldots, q_n\} = \{q'_1, \ldots, q'_m\}$, then $\Delta(a)(\langle q_1, \ldots, q_n\rangle) = \Delta(a)(\langle q'_1, \ldots, q'_m\rangle)$.*
4. *If $\Delta_{\mathcal{T}}^{-1}(q) = a$, then $\Delta(a)(q) = q$.*
5. *If $\Delta(a)(S) = q$, and q' is such that there is an $q'' \in S$ with $q' \geq_P q''$, then $\Delta(a)(\{q'\} \cup S) = q$.*

Proof. First suppose that $\mathcal{T}$ is intuitionistic-equivalence-respecting. By definition, it accepts only Kripke models.

We may define a partial order on the states as follows: Let $q \leq q'$ iff there is a tree t with (upward-closed) subtree t' such that $\mathcal{T}(t) = q$ and $\mathcal{T}(t') = q'$. This is clearly reflexive and transitive. By lemma 8, it is antisymmetric.

It is clear that the transition function must disregard order since intuitionistic-equivalence disregards order. They must disregard multiplicity since the two branches may be merged into one with Type II reductions.

The third requirement follows from the fact that Type I reductions preserve intuitionistic equivalence.

The fourth requirement follows from the fact that Type II reductions preserve intuitionistic equivalence.

Conversely, suppose that an automaton satisfies all four of the above conditions. The third requirement says that the sets $\mathcal{T}^{-1}(q)$ for $q \in \mathbf{states}_{\mathcal{T}}$ are closed under Type I expansions and the fourth that the sets $\mathcal{T}^{-1}(q)$ are closed under Type II expansions. Since this holds for all sets, they are all closed under Type I and II expansions and reductions and therefore under intuitionistic-equivalence. $\square$

Theorem 1. *Let $S \subseteq T(\mathcal{A}_n)$. There is a $\phi \in F_n$ such that $S = K(\phi)$ iff there is a prefix-closed intuitionistic-equivalence-respecting tree automaton $\mathcal{T}$ whose alphabet is $\mathcal{A}_n$ such that $S = L(\mathcal{T})$.*

Proof. We will first show right-to-left. Let $\mathcal{T}$ be a prefix-closed intuitionistic-equivalence-respecting tree automaton.

For $q \in \mathbf{states}_{\mathcal{T}}$, let $\mathcal{T}_q$ be the tree automaton defined to be the same as $\mathcal{T}$ except that $\mathbf{accept}_{\mathcal{T}_q} = \{q' \in P \mid q' \geq q\}$.

Let $B_n = \{q \in \mathbf{states}_{\mathcal{T}} \mid |\{q' \mid q' \geq q\}| = n\}$. Let $B_{\leq n} = \bigcup_{i \leq n} B_i$.

We will show the following by induction on n: For each $q \in B_n$, there is a formula ϕ_q such that $\mathcal{M} \Vdash \phi_q$ iff $\mathcal{M}$ is accepted by $\mathcal{T}_q$. There is also a formula ϕ'_q such that $\mathcal{M} \Vdash \phi'_q$ iff for all $\alpha \in \mathcal{M}$, $\mathcal{T}(\alpha) \neq q$.

We will need the following lemma:

Lemma 16. *Let $q_1, \ldots, q_n \in \mathbf{states}_{\mathcal{T}}$ be pairwise incomparable and such that there exist formulas $\phi_{q_1}, \ldots, \phi_{q_n}, \phi'_{q_1}, \ldots, \phi'_{q_n}$ as above.*

Let $a \subseteq V_n$.

Then there is a formula ψ such that whenever $\mathcal{M}$ is finite, $\mathcal{M} \Vdash \psi$ iff there does not exist a node $\alpha \in \mathcal{M}$ such that $\text{Forces}(\alpha) \cap V_n = a$*, each immediate successor α_i of α forces some ϕ_{q_j}, for each j there is some immediate successor α_i of α such that $\mathcal{T}(\alpha_i) = q_j$, and α doesn't force $\bigvee \phi_{q_i}$.*

Proof. Let ψ^0 be

$$\bigvee \phi'_{q_i} \to \bigvee \phi_{q_i}$$

Let ψ^1 be $\bigwedge a$. Let ψ^2 be $(\bigvee_{v \in V_n - a} v) \to \bigvee \phi_{q_i}$. Let ψ be

$$(\psi^0 \wedge \psi^1 \wedge \psi^2) \to \bigvee \phi_{q_i}$$

Suppose $\mathcal{M} \Vdash \psi$. Let $\alpha \in \mathcal{M}$ be such that $\text{Forces}(\alpha) \cap V_n = a$, each immediate successor of α forces some ϕ_{q_i}, and each q_i is the $\mathcal{T}$-state of some immediate successor of α. Then we would like to show that α forces $\psi^0 \wedge \psi^1 \wedge \psi^2$.

First, ψ^0. All of its successors force ψ^0, since they all force $\bigvee \phi_{q_i}$. Therefore, $\alpha \Vdash \psi^0$, since it does not force $\bigvee \phi'_{q_i}$: If it did, it would force some ϕ'_{q_i}, but it has a successor with $\mathcal{T}$-state q_i for each i.

We have that α forces ψ^1 by construction. We also have that α forces ψ^2: all of its successors do, since they all force its consequent, and α does since it does not force its hypothesis.

Conversely, let $\mathcal{M}$ have the property in statement in the lemma. We would like to show that every element of $\mathcal{M}$ forces ψ. For a contradiction, take an $\alpha \in \mathcal{M}$ such that $\alpha \nVdash \psi$ but all of its successors force ψ.

Then α must force $\psi^0 \wedge \psi^1 \wedge \psi^2$ but not $\bigvee \phi_{q_i}$. Since $\alpha \Vdash \psi^1$, $\text{Forces}(\alpha) \cap V_n \supset a$. Since $\alpha \Vdash \psi^2$ and $\alpha \nVdash \bigvee \phi_{q_i}$, $\text{Forces}(\alpha) \cap V_n \subseteq a$, so $\text{Forces}(\alpha) \cap V_n = a$.

Since α forces $\bigvee \phi'_{q_i} \to \bigvee \phi_{q_i}$, and does not force $\bigvee \phi_{q_i}$, it must not force $\bigvee \phi'_{q_i}$. Therefore, it must not force any ϕ'_{q_i}. Since it doesn't force any ϕ'_{q_i}, α must have successors of $\mathcal{T}$-state q_i for each i.

We want to show that all of α's successors force $\bigvee \phi_{q_i}$ and that for each i, α has an immediate successor of $\mathcal{T}$-state q_i. Suppose first that α had a successor which did not force any ϕ_{q_i}. Then, since it forces ψ, it must not force $\bigvee \phi'_{q_i} \to \bigvee \phi_{q_i}$. But α forces this, which is a contradiction.

We now show that for each i, α has an immediate successor of $\mathcal{T}$-state q_i. Suppose there is an i for which α does not have such an immediate successor. It does have some successor of $\mathcal{T}$-state q_i. Let β be a successor with $\mathcal{T}$-state q_i of minimal distance from α. Let its immediate predecessor be β'. Since β' is a successor of α, $\beta' \Vdash \bigvee \phi_{q_i}$. Since the q_i are incomparable and β''s successor has $\mathcal{T}$-state q_i, β' must have $\mathcal{T}$-state q_i. This contradicts the minimality of β.

Thus, by the assumed property of $\mathcal{M}$, α must force $\bigvee \phi_{q_i}$, which is a contradiction. □

If $S = \{q_1, \ldots q_n\}$ and $a \subseteq V_n$, we will call the ψ given by the lemma $\phi'_{S,a}$.

Let $q \in B_1$. Then q has no successors in $\textbf{states}_{\mathcal{T}}$. If $\Delta_{\mathcal{T}}^{-1}(q) = \uparrow$, then $\mathcal{T}_q$ accepts no trees and we can take $\phi_q = \bot$, $\phi'_q = \top$.

Otherwise, let $a = \Delta_{\mathcal{T}}^{-1}(q)$. Since q has no successors, we must have $\Delta(a)() = q$, or else we would again have that $\mathcal{T}_q$ accepts no trees. So $\mathcal{T}_q$ accepts exactly those Kripke models where every node forces exactly the elements of a.

We may thus let $\phi_q = \bigwedge_{v \in a} v \wedge \bigwedge_{v \in V - a} \neg v$ and $\phi'_q = \neg \phi_q$.

We are done with the $n = 1$ case. Suppose we have proven the induction hypothesis for n, and we will show it for $n + 1$.

Let $q \in B_{n+1}$. As above, we may assume that $\Delta_{\mathcal{T}}^{-1}(q)$ exists and equals a. Let $\{S_1, \ldots, S_n\}$ be the set of pairwise incomparable $S \subseteq B_n$ such that $\Delta(a)(S) = q$. As above, we may assume that there is at least one such S.

Let ϕ_q^0 be the formula

$$(\bigwedge_i \phi'_{S_i,a}) \to \bigvee_{q' > q} \phi_{q'}$$

Let ϕ_q^1 be the formula $\bigwedge_{v \in a} v$. Let ϕ_q^2 be the formula $(\bigvee_{v \in V - a} v) \to \bigvee_{q' > q} \phi_{q'}$.

We let $\phi_q = \bigwedge_{i=0,1,2} \phi_q^i$ and $\phi'_q = \phi_q \to \bigvee_{q' > q} \phi_{q'}$.

Suppose $\mathcal{M}$ is a rooted Kripke model which forces ϕ_q. Let α be the root node of $\mathcal{M}$. We want to show that it is accepted by $\mathcal{T}_q$, i.e., that $\mathcal{T}(\alpha) \geq q$. If α forces $\phi_{q'}$ for some $q' > q$, then by induction we are done. So assume that α doesn't force any such $\phi_{q'}$.

We will show this by induction on the number of α's successors which do not have $\mathcal{T}$-state strictly greater than q.

Suppose first that α has no such successors.

Then, since $\alpha \Vdash \phi_q^1 \wedge \phi_q^2$, we know that $\text{Forces}(\alpha) \cap V_n = a$. Since $\alpha \Vdash \phi_q^0$ but $\alpha \nVdash \bigvee_{q' > q} \phi_{q'}$, α must not force $\bigwedge_i \phi'_{S_i,a}$. Therefore, there is some i such that α does not force $\phi'_{S_i,a}$. By the lemma above, that means that α must have some successor β such that $\text{Forces}(\beta) \cap V_n = a$ and for each $q' \in S_i$, there is an immediate successor of β of $\mathcal{T}$-state q'. Thus, the $\mathcal{T}$-state of β is q. Since all of α's successors have $\mathcal{T}$-state strictly greater than q, β must equal α, and the $\mathcal{T}$-state of α is q.

For the induction step, we note that that by induction, α has some successor of $\mathcal{T}$-state q. It follows by the properties of the intuitionistic-equivalence-respecting automaton that α must have $\mathcal{T}$-state q.

Conversely, we want to show that if α is a node such that $T(\alpha) \geq q$ then α forces ϕ_q. If $T(\alpha) > q$ then we are done by induction, so suppose that $T(\alpha) = q$. We will prove this by induction on the number of successors β of α such that $T(\beta) = q$.

Suppose first that α has no such successors. Let α's immediate successors be $A = \{\alpha_1, \ldots, \alpha_r\}$. Let $\mathcal{T}(A) = \{\mathcal{T}(\alpha_1), \ldots, \mathcal{T}(\alpha_r)\}$. It is immediate that α forces ϕ_q^1. It is also the case that α forces ϕ_q^2, since α does not force any atomic formulas other than those in a and all of its successors force $\phi_{q'}$ for $q' > q$.

All of α's successors force ϕ_q^0, since they all force its consequent. So we just have to show that α does not force its hypothesis. This follows from the fact that the set $\mathcal{T}(A)$ must be some S_i.

Now, by induction, we just have to prove that if one of α's immediate successors β is such that $T(\beta) = q$, then α will force ϕ_q. That α forces ϕ_q^1 and ϕ_q^2 is clear, using the inductive hypothesis. By induction, all of α's successors force ϕ_q^0. But then α forces ϕ_q^0 since it doesn't force its hypothesis, since β doesn't force its hypothesis.

The definition for ϕ_q' works just as in [4].

This completes the right-to-left direction, as we may now take the formula ϕ to be the disjunction of ϕ_q for all accept states q.

For the left-to-right direction: Let ϕ be a formula. Let $\mathbf{states}_{\mathcal{T}}$ be the power set of the set of subformulas of ϕ, ordered by inclusion. Let $\mathbf{accept}_{\mathcal{T}}$ be the set of sets of subformulas of ϕ that include ϕ.

Given $a \subseteq V_n$, define $\Delta(a)(S)$ where S is a set of sets of subformulas of ϕ as follows: If there is an $s \in S$ such that there is no rooted Kripke model $\mathcal{M}$ such that for all subformulas ψ of ϕ, $\mathcal{M} \Vdash \psi$ iff $\psi \in s$, then let $\Delta(a)(S) = \emptyset$.

Otherwise, for each $s \in S$, let $\mathcal{M}_s$ be a rooted Kripke model K such that for all subformulas ψ of ϕ, $\mathcal{M} \Vdash \psi$ iff $\psi \in s$. Let $\mathcal{M}$ be the Kripke model gotten by taking the disjoint union of the $\mathcal{M}_s$'s and adding a new root node α below each of them such that $\mathrm{Forces}(\alpha) \cap V_n = a$. (If this is not possible because some s doesn't contain all of the formulas in a, then let $\Delta(a)(S) = \emptyset$.)

Let $\Delta(a)(S)$ be the set of subformulas of ϕ forced by $\mathcal{M}$. This is independent of the particular Kripke models $\mathcal{M}_s$ that are chosen. Everything can be verified easily. $\square$

We also have the following reformulation.

Theorem 2. *Fix n and let S be a set of Kripke models defined over V_n. Then there is a ϕ such that $S = \{\mathcal{M} \mid \mathcal{M} \Vdash \phi\}$ iff S is closed under taking upward-closed submodels, under intuitionistic equivalence, and is recognizable by some tree automaton.*

Proof. The left-to-right direction is clear.

For the right-to-left direction, let S be as in the statement of the theorem, and let it be recognized by $\mathcal{T}$. We must show that there is a prefix-closed intuitionistic-equivalence-respecting automaton whose language is also S.

We may assume that $\mathcal{T}$ is injective. It follows that $\mathcal{T}$ is already prefix-closed. We will show that the reduction of $\mathcal{T}$ in the sense of the Myhill-Nerode theorem for tree automata (see [6]) satisfies the requirements.

By the Myhill-Nerode theorem for tree automata, there is an automaton $\mathcal{T}'$ which is equivalent to $\mathcal{T}$ and which we may also take to be injective with the following property: Let $q, q' \in \mathbf{states}_{\mathcal{T}'}$. If ($\mathcal{T}'(t) \in \mathbf{accept}_{\mathcal{T}'}$ iff $\mathcal{T}'(t') \in \mathbf{accept}_{\mathcal{T}'}$) whenever t' is the same as t except for possibly changing a subtree of $\mathcal{T}'$-state q to one of $\mathcal{T}'$-state q', then $q = q'$.

This is precisely the property we need: If t and t' are intuitionistically equivalent, then the fact that S is closed under intuitionistic-equivalence implies using the above that $\mathcal{T}'(t) = \mathcal{T}'(t')$. $\square$

Definition 30. If $\mathcal{T}$ is a tree automaton which accepts only Kripke models with alphabet $\mathcal{A}_n$, $\phi \in F_n$, and $L(\mathcal{T}) = K(\phi)$, then we say that $\mathcal{T}$ *represents* ϕ or that $\mathcal{T}$ is a representation of ϕ.

Definition 31 (Kripke-Accessible Nodes)**.** If $\mathcal{T}$ is a tree automaton which accepts only Kripke models we define the set of its *Kripke-accessible nodes* to be

$$\{q \in \mathbf{states}_{\mathcal{T}} \mid \exists \text{ a Kripke model } M \text{ such that } \mathcal{T}(M) = q\}.$$

4.2 Some First Applications

4.2.1 A Useful Class of Definable Sets

Definition 32. Let S be a set of pairwise incomparable nodes of $\mathcal{U}(n)$. (In practice, S will always be finite.)

We let

$$f(S) = \{\mathrm{node}(T, V) \mid T \subseteq S \text{ and } V \subseteq V_n \text{ and } \mathrm{node}(T, V) \text{ exists}\}.$$

We let $\bar{S}$ be the closure of S under f. In other words, $\bar{S} = \bigcup_{m \in \omega} f^{(m)}(S)$. We let $S\updownarrow = S\uparrow \cup \bar{S}$.

Proposition 6. *Let $S \subseteq \mathcal{U}(n)$ be finite and pairwise incomparable. Then $S\updownarrow$ is definable.*

Proof. Let $S' = S\uparrow - S$. Construct a prefix-closed intuitionistic-equivalence-respecting automaton $\mathcal{T}$ as follows: Let the set of states, Q be $S' \cup \{\circ_a\}_{a \subseteq V_n} \cup \{\bullet\}$ where $\bullet$ and the $\circ_a$ are new states. The only non-accept state is $\bullet$.

We will define Δ so that $\Delta^{-1}(s) = \mathrm{Forces}(s) \cap V_n$ for $s \in S'$ and $\Delta^{-1}(\circ_a) = a$. For $a \subseteq V_n$ and $T \subseteq \mathbf{states}_{\mathcal{T}}$, we let

$$\Delta(a)(T) = \begin{cases} \mathrm{node}(T, a) & T \subseteq S' \text{ and } \mathrm{node}(T, a) \text{ exists and is } \in S' \\ \circ_a & T \subseteq S' \text{ and } \mathrm{node}(T, a) \text{ exists and is } \in S \\ \circ_a & \exists b\, \circ_b \in S \text{ and } \bullet \notin S \text{ and } (\forall q \in S)\, a \subseteq \Delta^{-1}(q) \\ \bullet & \text{otherwise} \end{cases}$$

By construction, $\mathcal{T}$ accepts a node β iff $\beta \in S\updownarrow$, so we may take ϕ_S to be the formula associated to $\mathcal{T}$. □

4.2.2 Proof of Proposition 2

Recall the statement of the proposition:

Proposition. *For all $\alpha \in \mathcal{U}(n)$, $\{\alpha\}\uparrow$ and $\mathcal{U}(n) - \{\alpha\}\downarrow$ are definable.*

Proof. Let α be a node in $\mathcal{U}(n)$.

First we show that ϕ_α exists. Construct a prefix-closed intuitionistic-equivalence-respecting automaton $\mathcal{T}$ as follows: Let $\mathbf{states}_{\mathcal{T}} = \{\alpha\}\uparrow \cup \{\bullet\}$, where $\bullet$ is a new state. The only non-accept state is $\bullet$.

For $a \subseteq V_n$ and $S \subseteq \mathbf{states}_{\mathcal{T}}$, we let

$$\Delta(a)(S) = \begin{cases} \mathrm{node}(S, a) & \mathrm{node}(S, a) \text{ exists and is } \geq \alpha \\ \bullet & \text{otherwise} \end{cases}$$

By construction, $\mathcal{T}$ accepts a node β iff $\beta \geq \alpha$, so we may take ϕ_α to be the formula associated to $\mathcal{T}$.

That ϕ'_α exists is a consequence of Proposition 6, as $\mathcal{U}(n) - \{\alpha\}\downarrow = S\updownarrow$, where $S = L_n^{\mathrm{Lev}(\alpha)} - \{\alpha\}$.

4.2.3 H_n for $n \geq 2$ is not complete

A lattice is called complete if every set has a least upper bound. In [4] it is proven that H_n is not complete for $n \geq 2$. The proof is as follows: A countably infinite set of pairwise incomparable elements $A = \{\alpha_0, \alpha_1, \ldots\} \subseteq \mathcal{U}(n)$ is constructed. It is proven that if $S_1, S_2 \subseteq A$ are distinct, then the least upper bounds of S_1 and S_2 must be distinct (if they exist). The fact that H_n is not complete then follows by a cardinality argument.

Using Proposition 4, we can give an easy proof that $A_\phi = \{\phi_{\alpha_0}, \phi_{\alpha_1}, \ldots\}$ itself has no least upper bound.

Let ϕ be an upper bound for A_ϕ, and we will find a strictly lower upper bound. Let $\mathcal{T}$ be a prefix-closed intuitionistic-equivalence-respecting automaton associated to ϕ. By the pigeonhole principle, there must be $i \neq j$ such that $\mathcal{T}(\alpha_i) = \mathcal{T}(\alpha_j)$. Let $\beta = \mathrm{node}(\{\alpha_i, \alpha_j\}, \mathrm{Forces}(\alpha_i) \cap V_n)$. Since $\mathcal{T}$ is intuitionistic-equivalence-respecting, we must have $\mathcal{T}(\beta) = \mathcal{T}(\alpha_i)$. Thus β is accepted by $\mathcal{T}$, and it therefore forces ϕ.

Since A is incomparable, $\beta \notin A\uparrow$. Therefore $\phi \wedge \phi'_\beta < \phi$ is also an upper bound for A. $\square$

CHAPTER 5
THE ORDER-THEORETIC STRUCTURE OF H_N

5.1 Statement of the Theorems

Definition 33 (Quasisemilattices)**.** A quasisemilattice (qsl) is a poset $(P, \leq)$ such that for any two elements p and q, the set $\{r \in P \mid r \leq p, r \leq q\}$ of lower bounds of p and q has only finitely many maximal elements and such that every lower bound of p and q is below at least one such maximal element.

If p and q are elements in a qsl, we define

$$p \wedge q = \{r \mid r \text{ a maximal lower bound of } p \text{ and } q\}$$

A qsl is called bounded (a bqsl) if it has a minimum element.

A qsl is called locally finite if it is locally finite under the relation $R(p, q, r)$ which holds iff r is a maximal lower bound of p and q.

A bqsl embedding between two bqsls Q_1 and Q_2 is an order-embedding f that respects the minimal element $\perp$ and is such that for all $p, q \in Q_1$, $f(p \wedge q) = f(p) \wedge f(q)$. (Note that $p \wedge q$, $f(p \wedge q)$, and $f(p) \wedge f(q)$ are sets, while $f(p)$ and $f(q)$ are elements.)

A qsl is universal countable bounded locally finite if it is countable, bounded, and locally finite, and embeds all countable, bounded, locally finite qsls.

By a standard Fraïssé argument, there is a unique universal countable homogeneous locally finite bounded quasisemilattice. Let it be Q.

Definition 34 (Join-Irreducibles, $J_{1,n}$, $J_{2,n}$, $J_{3,n}$, $J_{4,n}$)**.** A *join-irreducible formula* is a $\perp \neq \phi \in H_n$ such that if ϕ is equivalent to $\psi \vee \chi$, then ϕ is equivalent to ψ or ϕ is equivalent to χ.

Let $J_n = \{\phi \in H_n \mid \phi \text{ is join-irreducible}\}$.

For each n, let $J_{1,n} = \{\phi \in J_n \mid (\exists^{<\infty}\psi)\ \psi < \phi\}$.

Let $J_{2,n}$ be the set of minimal elements of $J_n - J_{1,n}$.

Let $J_{3,n} = J_n - (J_{1,n} \cup J_{2,n} \cup \{\top\})$.

Let $J_{4,n} = \{\top\}$.

As discussed below in Section 5.2, $J_{1,n}$ is characterized completely by the properties of $\mathcal{U}(n)$ (and is, in fact, equal to the dual of the underlying partial order of $\mathcal{U}(n)$). We clearly understand $J_{4,n}$.

Theorem 3. *For all $n \geq 2$, $J_{2,n}$ is a countably infinite antichain and $J_{3,n} \cup \{\bot\}$ is isomorphic to Q.*

Every element of $J_{2,n}$ has an element of $J_{1,n}$ below it. Every element of $J_{3,n}$ has an element of $J_{2,n}$ below it.

If $x \in J_{i,n}$ and $y \in J_{j,n}$ and $x \leq y$, then $i \leq j$.

Proof. Proposition 7 states that every element of J_n has an element of $J_{1,n}$ below it.

Proposition 9 states that $J_{2,n}$ is a countably infinite antichain and that every element of $J_{3,n}$ has an element of $J_{2,n}$ below it.

Proposition 11 states that $J_{3,n} \cup \{\bot\}$ is isomorphic to Q.

The final statement follows immediately from the definitions. $\square$

5.2 $J_{1,n}$

Let $P_{1,n}$ be the underlying partial order of $\mathcal{U}(n)$ with the ordering reversed.

Proposition 7 (Implicit in [4])**.** *$P_{1,n}$ and $J_{1,n}$ are order-isomorphic. Every element of H_n besides the minimal element has an element of $J_{1,n}$ below it. A join-irreducible formula ϕ is in $J_{1,n}$ iff $\mathrm{Forces}^{-1}_{\mathcal{U}(N)}(\phi)$ is finite.*

The isomorphism sends α to ϕ_α.

5.3 Join-Irreducibles

Here we will collect some useful lemmas and propositions.

Proposition 8. *Let ϕ be a formula, and $\mathcal{T}$ be an automaton representing it. Then ϕ is join-irreducible iff $\mathcal{T}$ has a minimum Kripke-accessible accepting state.*

Proof. Suppose ϕ is join-irreducible, represented by $\mathcal{T}$, and let M be the set of $\mathcal{T}$'s minimal Kripke-accessible accepting states. We know from above that ϕ is logically equivalent to

$$\bigvee_{q \in M} \phi_q$$

(since they are forced by the same finite tree Kripke models). Furthermore, if $q_1, q_2 \in M$ are distinct, then ϕ_{q_1} is not logically equivalent to ϕ_{q_2}: Since q_i is Kripke-accessible there is some finite Kripke tree N_i such that $\mathcal{T}(n_i) = q_i$. Then, ϕ_{q_1} is not forced by N_2 and vice versa.

Thus, if $|M| \geq 2$, ϕ is not join-irreducible.

Conversely, let $|M| = 1$ (say $M = \{m\}$), and suppose that $\phi = \phi_1 \vee \phi_2$. For a contradiction, suppose that $\phi_1 \nVdash \phi_2$ and $\phi_2 \nVdash \phi_1$. Then there is a finite Kripke tree N_1 which forces ϕ_1 but not ϕ_2 and a finite Kripke tree N_2 which forces ϕ_2 but not ϕ_1, as well as a finite Kripke tree N_3 such that $\mathcal{T}(N_3) = m$. Consider the Kripke tree N formed by taking the disjoint union of N_1, N_2, and N_3, and adding a new node below each N_i and which forces exactly the propositional variables forced by N_3.

By the properties of intuitionistic-equivalence-respecting tree automata, $\mathcal{T}(N) = m$, so N forces ϕ. So either ϕ_1 or ϕ_2 must be forced by N, and in either case we have a contradiction to the assumed properties of either N_2 or N_1. □

There is a corollary to this which is interesting in its own right.

Corollary 5. *If ϕ is not join-irreducible, then it is equivalent to $\phi_0 \vee \cdots \vee \phi_r$, where each ϕ_i is a conjunction of subformulas of ϕ, each ϕ_i is join-irreducible, and no ϕ_i is equivalent to ϕ.*

Proof. This follows by taking the particular automaton $\mathcal{T}$ representing ϕ constructed in the last section where the states are sets of subformulas of ϕ. □

Lemma 17. *Suppose $\phi_1, \ldots, \phi_m$ are any formulas, ψ is join-irreducible, and $\psi \leq \phi_1 \vee \cdots \vee \phi_m$. Then there is an i such that $\psi \leq \phi_i$*

Proof. Since $\psi \leq \phi_1 \vee \cdots \vee \phi_m$, ψ is equivalent to $\psi \wedge (\phi_1 \vee \cdots \vee \phi_m)$ and thus to $(\psi \wedge \phi_1) \vee \cdots \vee (\psi \wedge \phi_m)$. Since ψ is join-irreducible, it must be equivalent to some $\psi \wedge \phi_i$, and therefore must be less than ϕ_i. □

5.4 $J_{2,n}$

Proposition 9. *$J_{2,n}$ is a countably infinite antichain. Every element of $J_n - (J_{2,n} \cup J_{1,n})$ has an element of $J_{2,n}$ below it.*

An element ϕ of J_n is in $J_{2,n}$ iff for all but finitely many m, $|L_n^m \cap \mathrm{Forces}_{\mathcal{U}(n)}^{-1}(\phi)| =$ 2.

Proof. We begin with a few definitions.

Definition 35 (Well-Positioned Triplets of Nodes)**.** Let α, β, and γ be three distinct nodes in $\mathcal{U}(n)$. The ordered triplet (α, β, γ) is called well-positioned if the following properties hold:

1. $\mathrm{Lev}(\alpha) + 1 = \mathrm{Lev}(\beta) + 1 = \mathrm{Lev}(\gamma)$.
2. $\gamma < \beta$.
3. $\gamma \not< \alpha$.
4. $\mathrm{Forces}(\alpha) \cap V_n = \mathrm{Forces}(\beta) \cap V_n = \mathrm{Forces}(\gamma) \cap V_n$.

Definition 36 ($A_{\alpha,\beta,\gamma}, \chi_i^m$)**.** Let (α, β, γ) be a well-positioned triplet of nodes with $\alpha \in L_n^i$ and $\mathrm{Forces}(\alpha) \cap V_n = U$. For each $m \in \mathbb{N}$ with $m \geq \mathrm{Lev}(\alpha)$, define two nodes χ_0^m and χ_1^m as follows:

$\chi_0^{\mathrm{Lev}(\alpha)} = \alpha, \chi_1^{\mathrm{Lev}(\alpha)} = \beta, \chi_1^{\mathrm{Lev}(\alpha)+1} = \gamma.$

$$\chi_0^{m+1} = \mathrm{node}(\{\chi_0^m, \chi_1^m\})$$

$$\chi_1^{m+2} = \mathrm{node}(\{\chi_0^m, \chi_1^{m+1}\})$$

Note that $\chi_j^m \in \mathrm{Lev}_m$.

Let $A_{\alpha,\beta,\gamma} = \{\rho \mid \rho \geq \alpha \text{ or } \gamma\} \cup \{\chi_i^m \mid m \in \mathbb{N}, i \in \{0,1\}\}$. Note that $A_{\alpha,\beta,\gamma} = \{\chi_0^{\mathrm{Lev}(\alpha)+1}, \gamma\}\updownarrow$.

The sets of the form $A_{\alpha,\beta,\gamma}$ will turn out to be exactly the sets in $\{\mathrm{Forces}_{\mathcal{U}(n)}^{-1}(\phi) \mid \phi \in J_{2,n}\}$.

By Proposition 6, we know that for each well-positioned triplet (α, β, γ) of nodes, $A_{\alpha,\beta,\gamma} = \{\chi_0^{\mathrm{Lev}(\alpha)+1}, \gamma\}\updownarrow$ is definable by a formula $\phi_{\{\chi_0^{\mathrm{Lev}(\alpha)+1}, \gamma\}\updownarrow}$. We will write $\phi_{\alpha,\beta,\gamma}$ for $\phi_{\{\chi_0^{\mathrm{Lev}(\alpha)+1}, \gamma\}\updownarrow}$ when (α, β, γ) is a well-positioned triple.

The fact that these $\phi_{\alpha,\beta,\gamma}$ are in $J_{2,n}$ is seen by observing that for sufficiently large m, $|A_{\alpha,\beta,\gamma} \cap L_n^m| = 2$, that every $\rho \in A_{\alpha,\beta,\gamma}$ has some $\rho' > \rho$ such that $\rho' \in A_{\alpha,\beta,\gamma}$, and the following simple lemma.

Lemma 18. *Let ϕ be such that there is an m such that $|\operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \cap L_n^m| = 1$. Then for all $m' \geq m + n$, $|\operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \cap L_n^m| = 0$. In particular, $|\operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi)| < \infty$.*

Proof. Straightforward. □

We now show that there are no other minimal formulas and that every element of $J_n - J_{1,n}$ has a minimal element of $J_n - J_{1,n}$ below it with the following proposition.

Proposition 10. *Let ϕ be a join-irreducible such that $\operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ is infinite. Then there is a well-posititioned triplet $(\alpha, \beta, \gamma) \in \operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ such that $A_{\alpha,\beta,\gamma} \subseteq \operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$.*

Proof. Let $\mathcal{T}$ represent ϕ, and let q be the minimum Kripke-accessible accepting state of $\mathcal{T}$. By the properties of intuitionistic-equivalence-respecting automata, if we find a well-positioned tuple (α, β, γ) such that $\mathcal{T}(\alpha) = \mathcal{T}(\beta) = \mathcal{T}(\gamma)$ we will be done.

Let α_0 be such that $\mathcal{T}(\alpha_0) = q$. Since $\operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ is infinite, there must be some $\beta_0 \neq \alpha_0$ such that $\mathrm{Lev}(\beta_0) = \mathrm{Lev}(\alpha_0)$.

Let $\alpha_1 = \mathrm{node}(\{\beta_0, \alpha_0\})$. We have that $\mathcal{T}(\alpha_1) = q$. Since $\operatorname{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ is infinite, there must be a $\beta_1 \neq \alpha_1$ such that $\mathrm{Lev}(\beta_1) = \mathrm{Lev}(\alpha_1)$

First assume that $\beta_1 < \alpha_0$. If there is any γ' such that $\mathrm{Lev}(\gamma') < \mathrm{Lev}(\alpha_2)$ and $\gamma' \not\geq \alpha_2$ then we may take $\alpha = \beta_2$, $\beta = \alpha_2$, $\gamma = \mathrm{node}(\{\alpha_2, \gamma'\})$. Otherwise, there must be some γ' such that $\mathrm{Lev}(\gamma') < \mathrm{Lev}(\alpha_2)$ and $\gamma' \not\geq \beta_2$ (otherwise α_2 would equal β_2). Then we may take $\alpha = \alpha_2$, $\beta = \beta_2$ and $\gamma = \mathrm{node}(\{\beta_2, \gamma'\})$.

If $\beta_1 \not< \alpha_0$ then repeat the argument of the above paragraph with $\beta_2 = \mathrm{node}(\{\alpha_0, \beta_1\})$ and $\alpha_2 = \mathrm{node}(\{\beta_1, \alpha_1\})$. □

□

Corollary 6. *For any join-irreducible ϕ with representing automaton $\mathcal{T}$ and minimum Kripke-accessible accepting state q, the following are equivalent:*

1. $\phi \in J_{3,n} \cup J_{4,n}$.

2. *There are three incomparable nodes $\alpha_1, \alpha_2, \alpha_3$ such that for $1 \leq i \leq 3$, $\mathcal{T}(\alpha_i) = q$.*

3. *For any m, there is an r such that $|\mathcal{T}^{-1}(q) \cap L_n^r| \geq m$.*

Proof. 1 $\implies$ 2: Let $\phi \in J_{3,n} \cup J_{4,n}$ and let (α, β, γ) be a well-positioned tuple of nodes in $\text{Forces}_{\mathcal{U}(n)}^{-1}(\phi)$.

By assumption, there is a $\delta \in \text{Forces}_{\mathcal{U}(n)}^{-1}(\phi) - A_{\alpha,\beta,\gamma}$. Let $\mathcal{T}(\mu) = q$. First assume that $\mu \in A_{\alpha,\beta,\gamma}$.

Then all $\nu \in A_{\alpha,\beta,\gamma}$ of sufficiently high level must satisfy $\mathcal{T}(\nu) = q$. Since $\text{Forces}_{\mathcal{U}(n)}^{-1}(\phi)$ is infinite, every element of $\text{Forces}_{\mathcal{U}(n)}^{-1}(\phi)$ must have a predecessor in $\text{Forces}_{\mathcal{U}(n)}^{-1}(\phi)$: if some μ' didn't, then ϕ would not be join-irreducible as it would be equivalent to $(\phi \wedge \phi'_{\mu'}) \vee \phi_{\mu'}$.

Thus, we can find some predecessor δ' of δ on the same level as two elements χ_0, χ_1 of $A_{\alpha,\beta,\gamma}$ satisfying $\mathcal{T}(\chi_0) = \mathcal{T}(\chi_1) = q$. Then we can take our three incomparable nodes to be $\text{node}(\{\delta', \chi_0\})$, $\text{node}(\{\delta', \chi_1\})$, and $\text{node}(\{\delta', \chi_0, \chi_1\})$.

Now assume that $\mu \notin A_{\alpha,\beta,\gamma}$. Take a predecessor μ' of μ that is on the same level as two elements χ_0, χ_1 of $A_{\alpha,\beta,\gamma}$. Then we may take our three incomparable nodes to be $\text{node}(\{\mu', \chi_0\})$, $\text{node}(\{\mu', \chi_1\})$ and $\text{node}(\{\mu', \chi_0, \chi_1\})$.

2 $\implies$ 3: Let α_1, α_2, and α_3 be three incomparable nodes with $\mathcal{T}(\alpha_i) = q$ for all $1 \leq i \leq 3$. We may suppose that $\text{Lev}(\alpha_1) = \text{Lev}(\alpha_2) = \text{Lev}(\alpha_3)$: If not, say without loss of generality that $\text{Lev}(\alpha_1) \geq \text{Lev}(\alpha_2), \text{Lev}(\alpha_3)$. Then we may replace α_1, α_2, α_3 with $\alpha_1' = \text{node}(\{\alpha_1, \alpha_2\})$, $\alpha_2' = \text{node}(\{\alpha_1, \alpha_3\})$, $\alpha_3' = \text{node}(\{\alpha_1, \alpha_2, \alpha_3\})$.

So suppose the common level of the α_i's is l.

Then at level $l+1$ we have four nodes of $\mathcal{T}$-state q: $\text{node}(\{\alpha_1, \alpha_2, \alpha_3\})$, $\text{node}(\{\alpha_1, \alpha_2\})$, $\text{node}(\{\alpha_2, \alpha_3\})$, $\text{node}(\{\alpha_1, \alpha_3\})$. Clearly, if at any level s', we have r nodes of $\mathcal{T}$-state q, then at level $s'+1$, we have at least $\binom{r}{2}$ nodes of type q. Since the function $r \mapsto \binom{r}{2}$ is strictly increasing for $r \geq 4$, we are done.

3 $\implies$ 1: Clearly $\text{Forces}_{\mathcal{U}(n)}^{-1}(\phi)$ is infinite, so $\phi \notin J_{1,n}$. If $\phi \in J_{2,n}$, then $|k(\phi) \cap L_n^m|$ would be 2 for all sufficiently large m. $\square$

5.5 $J_{3,n}$

We begin with some facts about qsls.

Lemma 19. *Any finite set of elements in a bqsl has only finitely many maximal lower bounds, and any element less than all elements of a finite set is less than some maximal lower bound of that set.*

Proof. We will prove it for a set of three elements. The general case is by induction.

Let p, q, and r be elements of a bqsl. Let $p \wedge q = \{s_1, \ldots, s_m\}$. Then $\bigcup_i (r \wedge s_i)$ is finite and every element which is less than all three of p, q and r is less than some element of $\bigcup_i (r \wedge s_i)$: since it's less than p and q, it's less than some s_i and therefore is less than some element of $r \wedge s_i$. The conclusion follows. $\square$

Lemma 20. *If $Q^* \subseteq Q^{**}$ are finite bqsl's, then there is a sequence $Q^* = Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_m = Q^{**}$ such that each Q_i is a bqsl, $|Q_{i+1}| = |Q_i| + 1$, and each inclusion of Q_i in Q_{i+1} is a bqsl embedding.*

Proof. Let q be a minimal element of $Q^{**} - Q^*$. Let $Q_1 = Q^* \cup \{q\}$. This is a bqsl: for any $p \in Q^*$, the maximal lower bounds of p and q must be in Q^{**}, therefore they must be in $Q^* \cup \{q\}$ since q was minimal in $Q^{**} - Q^*$. Repeat the process. $\square$

Lemma 21. *Let $\phi \in J_{3,n}$. Let S be a finite collection of join-irreducible formulas such that for all $\psi \in S$, $\phi \not\leq \psi$. There is a $\rho \in J_{3,n}$ such that $\rho < \phi$ and for all $\psi \in S$, $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\rho) \cap \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$ is finite.*

Proof. Let $\mathcal{T}$ be a prefix-closed intuitionistic-equivalence-respecting automaton associated to ϕ, and let q be the minimum Kripke-accessible accepting state of $\mathcal{T}$. Let α be such that $\mathcal{T}(\alpha) = q$. Let $S = \{\psi_1, \ldots, \psi_m\}$. For $1 \leq i \leq m$, let $\alpha_i \in \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) - \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi_i)$ By Corollary 6, find an r greater than the level of each α_i such that there are four nodes $\gamma_1, \ldots, \gamma_4$ in L^r_n such that for $1 \leq i \leq 4$, $\mathcal{T}(\gamma_i) = q$.

Let $\delta_j = \mathrm{node}(\{\alpha_i \mid 1 \leq i \leq m\} \cup \{\gamma_j\})$ for $1 \leq j \leq 3$. Then a ρ satisfying the required properties is one which defines $\{\delta_1, \delta_2, \delta_3\}\updownarrow$.

$\square$

Proposition 11. *The set $J_{3,n} \cup \{\bot\}$ is isomorphic to Q.*

Proof. Let J be $J_{3,n}$ with a minimum element $\perp$ added.

First we observe that J is a bounded locally finite quasisemilattice. It has a minimum element. Given any two formulas ϕ and ψ, $\phi \wedge \psi$ is their greatest lower bound in H_n. Although it may not be join-irreducible, it can be written as $\rho_1 \vee \cdots \vee \rho_m$ with each ρ_i join-irreducible. The maximal elements among those ρ_i in $J_{3,n}$ are then the maximal lower bounds of ϕ and ψ in $J_{3,n}$ and there are only finitely many of them. If no ρ_i is in $J_{3,n}$, then the maximum lower bound of ϕ and ψ in J is $\perp$.

We now show that J is locally finite. Given join-irreducible ϕ and ψ, their maximal lower bounds are the maximal join-irreducibles less than $\phi \wedge \psi$. By Corollary 5 and Lemma 17 these are formed out of subformulas of ϕ and ψ by $\wedge$ and $\vee$. Iterating the process still yields formulas formed out of subformulas of ϕ and ψ by $\wedge$ and $\vee$. Thus, there can only be finitely many such formulas.

We now prove that it is the universal countable homogeneous locally finite bounded quasisemilattice by showing that for any finite bounded quasisemilattice Q_1, any bqsl embedding f from Q_1 into J_n, and any bqsl embedding g from Q_1 into Q_2 where $|Q_2| = |Q_1| + 1$, there is an extension h of f along g from Q_2 into J_n which is also a bqsl embedding.

Let q be the unique element of $Q_2 - Q_1$. Let $U = \{f(p) \mid p \in Q_1, p > q\}$, $K = \{f(p) \mid p \in Q_1, p \not< q, p \not> q\}$ and $L = \{f(p) \mid p \in Q_1, p < q\}$.

Assume U is nonempty. Then by Lemma 19, there must be a minimum element of U. Let it be u^*. If U is empty, let u^* be $\top$.

Let $\mathcal{T}$ be a prefix-closed intuitionistic-equivalence-respecting automaton representing u^* and let q be the minimum accepting Kripke-accessible state of $\mathcal{T}$.

First assume that L has more than one maximal element.

Since u^* is not less than any element of $K \cup L$, we can find, for each $\psi \in K \cup L$, an element $\alpha_\psi \in \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(u^*)$ that is not in $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$. By Corollary 6, we can find two incomparable nodes β_1 and β_2 at a level greater than any $\alpha_\psi \in K \cup L$ and such that $\mathcal{T}(\beta_1) = \mathcal{T}(\beta_2) = q$.

Let $\beta = \mathrm{node}(\{\alpha_\psi \mid \psi \in K \cup L\} \cup \{\beta_1\})$ and let $\beta' = \mathrm{node}(\{\alpha_\psi \mid \psi \in K \cup L\} \cup \{\beta_2\})$. Note that β and β' are incomparable, $\mathcal{T}(\beta) = \mathcal{T}(\beta') = q$ and for all $\psi \in K \cup L$, $\beta, \beta' \notin \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$.

Our new element ϕ (representing q) will be $\phi'_\beta \to (\bigvee_{\rho \in L} \rho \vee \phi_\beta)$.

Let $R_1 = \{\beta\}$. For $n > 1$, let R_n be

$$\{\text{node}(R) \mid R \subseteq \bigcup_{i<n} R_i \cup k(\bigvee L) \text{ and } R \cap \bigcup_{i<n} R_i \neq \emptyset\}.$$

Lemma 22. $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi) = \bigcup_i R_i \cup \{\beta\}\uparrow \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$.

Proof. We will first show that $\bigcup_i R_i \cup \{\beta\}\uparrow \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L) \subseteq \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$.

Clearly $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi_\beta)$, $\text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$, and R_1 are subsets of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$.

We will also show that no element of any R_i forces ϕ'_β. Clearly the sole element of R_1 does not force it.

Assume $R_n \subseteq \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ and that no element of R_i for $i \leq n$ forces ϕ'_β. Let $R \subseteq \bigcup_{i<n} R_i \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$ and $R \cap \bigcup_{i<n} R_i \neq \emptyset$.

Since there is an element of R that does not force ϕ'_β, $\text{node}(R)$ doesn't force it either, and thus forces $\phi'_\beta \to (\bigvee_{\rho \in L} \rho \vee \phi_\beta)$ since all of its successors force ϕ. Thus, $\text{node}(R)$ forces ϕ.

We will now show that $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \subseteq \bigcup_i R_i \cup \{\beta\}\uparrow \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$. Suppose $\alpha \in \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$, $\alpha \notin \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi_\beta)$, and $\alpha \notin \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$.

Since $\alpha \nVdash \phi_\beta \vee \bigvee L$, we must have $\alpha \nVdash \phi'_\beta$. Thus $\alpha \leq \beta$. It follows that every node in $\bigcup_i R_i$ is less than β

Given any $\alpha' \leq \beta$, let $\alpha' = \alpha'_0 < \alpha'_1 < \cdots < \alpha'_{n(\alpha')} = \beta$ where α'_{i+1} is an immediate successor of α'_i and $n(\alpha')$ is as large as possible.

We will show by induction on m that for all $\alpha' \leq \beta$ such that $\alpha' \Vdash \phi$, $\alpha' \in R_{m+1}$ iff $n(\alpha') \leq m$.

The case where $m = 0$ is clear.

Suppose that it's true for m and we'll show it true for $m+1$. First, let $\alpha' \in R_{m+1}$. Then $\alpha' = \text{node}(R)$, where $R \subseteq \bigcup_{i \leq m} R_i \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi_\beta) \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$. The maximum distance to β is given by the maximum distance from one if its immediate successors plus one.

Conversely, if $n(\alpha') \leq m$, then since all of its successors must force ϕ, the ones less than β are in R_m, and the ones not less than β must be in $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi_\beta) \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L)$, α' must be in R_{m+1}.

Thus we are done, as each $\alpha' \leq \beta$ such that $\alpha \Vdash \phi$ is in some R_m (namely, $R_{n(\alpha')}$). $\square$

We must show that ϕ is different from every element of L, K, and U, that it is less than every element of u^*, greater than every element of L, incomparable with every element of K, and for each $\psi \in K$, the maximal lower bounds of ϕ and ψ are in L.

Clearly, ϕ is above every element of L. We have that ϕ is different from each $\psi \in L$, since $\beta \Vdash \phi$ but $\beta \nVdash \psi$.

By construction, ϕ is below u^*, and it is different from u^* as $\beta' \Vdash \phi'_\beta$ but $\beta' \nVdash \bigvee L$, so $\beta' \in \text{Forces}^{-1}_{\mathcal{U}(n)}(u^*) - \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$.

We will show that ϕ is incomparable with each element of K. Fix a $\psi \in K$. Since $\beta \in \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ but not in $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$, we have $\psi \not\leq \phi$. On the other hand, the intersection of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$ with $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ must be contained in $\text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L) \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi_\beta)$, as every element of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ is either below β, or contained in $\text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L) \cup \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi_\beta)$ by Lemma 22. Since ψ cannot be below $\bigvee L \vee \phi_\beta$ by Lemma 17, there must be an element $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$ not in $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$. Of course, it follows that ϕ is not equal to any element of K.

Let $\psi \in K$. In order to show that ψ and ϕ have the same maximal lower bounds in $K \cup L \cup U \cup \{\phi\}$ as they do in $J_{3,n} \cup \{\perp\}$, we must show that every $\chi \in J_{3,n}$ such that $\chi \leq \psi \wedge \phi$ is less than some element of L. But, as observed above $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ is contained in $\text{Forces}^{-1}_{\mathcal{U}(n)}(\bigvee L \vee \phi_\beta)$. Since χ is join-irreducible and $\text{Forces}^{-1}_{\mathcal{U}(n)}(\chi)$ is infinite, χ is less than some $\rho \in L$. $\square$

CHAPTER 6
EMBEDDINGS

6.1 A Lattice-Embedding from H_m to H_n for $m \geq 1, n \geq 2$

Theorem 4. *Let $n \geq 2$, $m \geq 1$. Then there are $\phi, \psi \in H_n$ such that H_m is isomorphic to $[\phi, \psi]$. In addition, the isomorphism from $[\phi, \psi]$ to H_m can be extended to a surjective lattice-homomorphism from H_n to H_m.*

Proof. The main work is contained in the following proposition.

Proposition 12. *Let $m \geq 2$ and n be such that there is a level L_n^i of $\mathcal{U}(n)$ and a set $A \subseteq L_n^i$ such that $|A| = m$ and each $\alpha \in A$ has some immediate successor not above any other $\alpha' \in A$. Then there is are $\phi, \psi \in H_n$ such that H_m is lattice-isomorphic to $[\phi, \psi]$ and in addition, the isomorphism from $[\phi, \psi]$ to H_m can be extended to a surjective lattice-homomorphism from H_n to H_m.*

Proof. Fix A and i from the hypothesis.

Let $A = \{\alpha_1, \ldots, \alpha_m\}$. For each i, let α_i be an immediate successor of α_i not above any other α_j. Let ϕ be

$$\bigvee_i \phi_{\alpha_i}.$$

For each $A' \subseteq A$, let $\gamma_{A'} = \text{node}(A' \cup \bigcup_{\alpha \notin A'} \alpha, \emptyset)$. This is valid as the elements of T are pairwise incomparable and $|T| \geq 2$ since $m \geq 2$.

Note that $\gamma_{A'}$ is at level $i+1$ if A' is nonempty and at level i if A' is empty. Since each α_i has a successor not above any other α_j, if $A' \neq A''$, $\gamma_{A'} \neq \gamma_{A''}$.

Let ψ define $\{\gamma_{A'} \mid A' \subseteq A\}\updownarrow$.

Define $g \colon \mathcal{U}(m) \to \mathcal{U}(n)$ by:

1. $g(\text{node}_{\mathcal{U}(m)}(\emptyset, U)) = \gamma_{A'}$ where $A' = \{\alpha_k \mid x_k \in V_m - U\}$.

2. $g(\text{node}_{\mathcal{U}(m)}(T, U)) = \text{node}_{\mathcal{U}(n)}(T', \emptyset)$, where $T' = \{g(\delta) \mid \delta \in T\} \cup \{\alpha_k \mid x_k \in V_m - U\}$.

We must show that g is well-defined, i.e., that $\text{node}(T', \emptyset)$ exists, which it might not, if, for example, T' has a single minimal element.

Lemma 23. *The function g is well-defined and preserves order and nonorder. For all $\beta \in \mathcal{U}(m)$ and $x_k \in V_m$, $\beta \Vdash x_k$ iff $g(\beta) \not\leq \alpha_k$.*

Proof. We will prove by induction on i that g restricted to $\bigcup_{j\leq i} L^j_m$ satisfies the conditions in the statement of the lemma.

For $i = 0$, observe that $\{g(\mathrm{node}(\emptyset, U)) \mid U \subseteq V_m\}$ is pairwise incomparable and that if $U \neq U'$, $g(\mathrm{node}(\emptyset, U)) \neq g(\mathrm{node}(\emptyset, U'))$ as they have different immediate successors. It is also the case that for all $\mathrm{node}(\emptyset, U) \in L^0_m$ and $x_k \in V_m$, $\mathrm{node}(\emptyset, U) \Vdash x_k$ iff $x_k \in U$ iff $\gamma_{A'} \not\leq \alpha_k$, where $A' = \{\alpha_k \mid x_k \in V_m - U\}$.

Now suppose g restricted to $\bigcup_{j\leq i} L^i_m$ satisfies the hypotheses in the statement of the lemma.

Let $\mathrm{node}(T, U) \in L^{i+1}_m$. If T has at least two minimal elements, we are done. If $T = \{\beta\}\uparrow$, then $U \subsetneq \mathrm{Forces}(\beta) \cap V_m$ and T' must contain both $g(\beta)$ and α_k, where $x_k \in \mathrm{Forces}(\beta) \cap V_m - U$. Since $\beta \Vdash x_k$, $g(\beta) \not\leq \alpha_k$. Since it is fairly easy to see that each α_k is not less than any element of the range of g, we must have that T' has at least 2 minimal elements.

It is immediate then that g restricted to L^{i+1}_m is preserves order and the immediate successor relation. Each element of L^{i+1}_m is of the form $\mathrm{node}(T, U)$. Observe that if $U \neq U'$ and $\mathrm{node}(T, U), \mathrm{node}(T, U') \in \mathcal{U}(m)$, then $g(\mathrm{node}(T, U)) \neq g(\mathrm{node}(T, U'))$ as they have different immediate successors. Similarly, if $T\uparrow \neq T'\uparrow$ then $g(\mathrm{node}(T, U)) \neq g(\mathrm{node}(\langle T', U'\rangle))$ as they have different immediate successors. We can now conclude that g preserves nonorder by using the inductive hypothesis and the fact that g preserves the immediate successor relation.

□

Lemma 24. *The sets $\mathrm{ran}(g)$ and $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ are disjoint and $\mathrm{ran}(g) \cup \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) = \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$ (= $\{\gamma_{A'} \mid A' \subseteq A\}\updownarrow$).*

Proof. It is immediate that $\mathrm{ran}(g)$ and $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ are disjoint.

We will first show that $\mathrm{ran}(g) \cup \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \subseteq \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$. It is clear that $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \subseteq \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$. Since every node in $\mathrm{ran}(g)$ is at level $\geq i+1$ and every successor of a node in $\mathrm{ran}(g)$ is in $\mathrm{ran}(g)$ or $k(\phi)$, by induction every element of $\mathrm{ran}(g)$ is in $\{\gamma_{A'} \mid A' \subseteq A\}\updownarrow$.

We will now show that $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \subseteq \mathrm{ran}(g) \cup \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$. By construction, $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap L^i_n = \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \cup \{\gamma_A\}$ and $\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap L^{i+1}_n =$

$\operatorname{ran}(g) \cap L_n^{i+1} = \operatorname{ran}(g|L_m^0) - \{\gamma_A\}$. We will show that $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap L_n^j \subseteq \operatorname{ran}(g|L_m^{j-(i+1)})$ for all $j \geq i+1$ by induction on j. We just observed that this holds for $j = i+1$.

Suppose it holds for j. A node of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap L_n^{j+1}$ must be of the form $\text{node}(T, \emptyset)$ for $T \subseteq \text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap (\bigcup_{k \leq j} L_n^k)$. Since T must contain an element of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap L_n^j$ and every such node is below every element of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap L_n^{i-1}$, T must be a subset of $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) \cap (\bigcup_{i \leq k \leq j} L_n^k)$. Let $S = g^{-1}(T \cap (K_n^j - K_n^i))$ and $U = \{x_k \mid \alpha_k \in T \cap L_n^i\}$. Then $g^{-1}(\text{node}_{\mathcal{U}(n)}(T, \emptyset))$ is $\text{node}_{U(m)}(\langle S, \bigcap_{\mu \in S} \text{Forces}(\mu) \cap V_m - U \rangle)$

□

It follows from Lemmas 23 and 24 that g is an order-isomorphism from $\mathcal{U}(m)$ to $\text{Forces}^{-1}_{\mathcal{U}(n)}(\phi) - \text{Forces}^{-1}_{\mathcal{U}(n)}(\psi)$.

Define $f \colon F_m \to F_n$ by:

1. $f(\bot) = \phi$
2. $f(x_i) = (\phi'_{\alpha_i} \vee \phi) \wedge \psi$
3. $f(\rho_0 \wedge \rho_1) = f(\rho_0) \wedge f(\rho_1)$.
4. $f(\rho_0 \vee \rho_1) = f(\rho_0) \vee f(\rho_1)$.
5. $f(\rho_0 \to \rho_1) = (f(\rho_0) \to f(\rho_1)) \wedge \psi$.

Lemma 25. *For any $\rho \in F_m$, $\phi \vdash f(\rho) \vdash \psi$. If $\delta = g(\gamma)$ then $\gamma \Vdash \rho$ iff $\delta \Vdash f(\rho)$.*

Proof. The proof that $\phi \vdash f(\rho) \vdash \psi$ is an easy proof by induction on ρ.

We now prove the second part of the lemma by induction on ρ.

For $\rho = \bot$, the result is immediate. The observation that $\gamma \Vdash x_i$ iff $\delta \not\leq \alpha_i$ furnishes the case where ρ is x_i. The inductive steps follow from the existence of the order-isomorphism g from $\mathcal{U}(m)$ to $\text{Forces}^{-1}_{\mathcal{U}(n)}(\psi) - \text{Forces}^{-1}_{\mathcal{U}(n)}(\phi)$ and the fact that $\phi \vdash f(\rho)$ for all ρ. □

Note that it follows from Lemma 25 that f is injective and hence an embedding.

We now define a function from F_n to F_m that is an inverse to f when restricted to $[\phi, \psi]$. Define h from F_n to F_m as follows:

1. $h(\bot) = h(x_j) = \bot$.
2. $h(\rho_0 \wedge \rho_1) = h(\rho_0) \wedge h(\rho_1)$.
3. $h(\rho_0 \vee \rho_1) = h(\rho_0) \vee h(\rho_1)$.
4. If there is some $\delta \in \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \cap \bigcup_{j<i} L^j_n$ such that $\delta \nVdash \rho_0 \to \rho_1$, then $h(\rho_0 \to \rho_1) = \bot$. Otherwise,

$$h(\rho_0 \to \rho_1) = (h(\rho_0) \to h(\rho_1)) \wedge \bigwedge\{x_j \mid \alpha_j \nVdash \rho_0 \to \rho_1\}$$

Lemma 26. *Let $\delta = g(\gamma)$. For all $\rho \in F_n$, $\delta \Vdash \rho$ iff $\gamma \Vdash h(\rho)$.*

Proof. We will prove this by induction on the level of γ and the structure of ρ.

If ρ is $\bot$ or x_i, then $\delta \nVdash \rho$ and $\gamma \nVdash h(\rho)$.

The inductive step for $\rho = \rho_0 \vee \rho_1$ and $\rho = \rho_0 \wedge \rho_1$ is straightforward.

Let ρ be $\rho_0 \to \rho_1$. Suppose $\delta \Vdash \rho$. Then, since for every $\mu \in \mathrm{Forces}^{-1}_{\mathcal{U}(n)}(\phi) \cap \bigcup_{j<i} L^j_n$, $\delta < \mu$, $h(\rho) = (h(\rho_0) \to h(\rho_1)) \wedge \bigwedge\{x_j \mid \alpha_j \nVdash \rho\}$. Since $\delta \Vdash \rho$, if $\alpha_j \nVdash \rho$, $\delta \not< \alpha_j$. It follows that $\gamma \Vdash x_j$. Thus γ forces the right conjunct of $h(\rho)$.

Suppose $\delta \Vdash \rho_0$ and $\delta \Vdash \rho_1$. Then we are done by the inductive hypothesis on the structure of ρ. Otherwise, suppose $\delta \nVdash \rho_0$. Then we are done by the inductive hypothesis on the structure of ρ and the level of γ.

Now suppose $\delta \nVdash \rho$. Then there is some $\mu \geq \delta$ such that $\mu \Vdash \rho_0$ and $\mu \nVdash \rho_1$. If μ is in the range of g then we are done by induction. If $\mu \in \bigcup_{j<i} L^j_n$, then $h(\rho) = \bot$ and we are done. Otherwise $\mu \in L^i_n$ and is some α_j. Since $\delta < \alpha_j$, $\gamma \nVdash x_j$ and $\gamma \nVdash h(\rho)$. □

It follows from Lemma 26 and Lemma 25 that if $\phi \vdash \rho \vdash \psi$, then $f(h(\rho)) = \rho$.

□

If $n \geq 2$, $m \geq 2$ then we can find a level in $\mathcal{U}(n)$ satisfying the hypotheses of the Proposition. For example, we may pick a level in $\mathcal{U}(n)$ of cardinality greater than $2m$, call $2m$ of its elements $\beta_1, \ldots, \beta_{2m}$, and let

$$A = \{\mathrm{node}(\{\beta_1, \beta_2\}, \emptyset), \ldots, \mathrm{node}(\{\beta_{2m-1}, \beta_{2m}\}, \emptyset)\}.$$

If $m = 1$, then we may let ϕ be $\bot$ and ψ be $x_2 \wedge \ldots \wedge x_n$. The embedding f from H_1 to $[\phi, \psi] \subseteq H_n$ sends ρ to $\rho \wedge x_2 \wedge \ldots \wedge x_n$. We may define a surjective lattice homomorphism h from H_n to H_1 that is an inverse to f as follows:

$$h(x_1) = x_1$$

$$h(x_i) = \top \text{ for } 1 < i \leq n$$

$$h(\phi \wedge \psi) = h(\phi) \wedge h(\psi)$$

$$h(\phi \vee \psi) = h(\phi) \vee h(\psi)$$

$$h(\phi \rightarrow \psi) = h(\phi) \rightarrow h(\psi)$$

□

By [4], H_n for $n \geq 2$ has an infinite descending chain, while H_1 does not, so there is no embedding of H_n into H_1 for $n \geq 2$.

6.2 A Lattice-Embedding from H_ω to H_n for $n \geq 2$

Theorem 5. *There is a lattice-embedding from H_ω into H_2*

Proof. Pick $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in \mathcal{U}(2)$, all at the same level, say i.

Let ϕ be $\phi_{\{\alpha_j | 1 \leq j \leq 4\}\updownarrow}$ and let ψ be $\phi_{\{\alpha_j | 1 \leq j \leq 5\}\updownarrow}$.

Define a sequence $\{\beta_i^j \mid i \in \omega, j \in \{1,2,3,4\}\}$ as follows: let $\beta_0^j = \alpha_j$. For $i \geq 0$, let $\{\beta_{i+1}^j \mid j = 1,2,3,4\}$ be a collection of four distinct nodes of the same level, with $\mathrm{Lev}(\beta_{i+1}^1) > \mathrm{Lev}(\beta_i^1)$ and such that they all force $\neg\neg(\phi_{\beta_i^2} \vee \phi_{\beta_i^3} \vee \phi_{\beta_i^4})$. For example, we may take $\beta_{i+1}^1 = \mathrm{node}(\{\beta_i^2, \beta_i^3\}, \emptyset)$, $\beta_{i+1}^2 = \mathrm{node}(\{\beta_i^2, \beta_i^4\}, \emptyset)$, $\beta_{i+1}^3 = \mathrm{node}(\{\beta_i^3, \beta_i^4\}, \emptyset)$, and $\beta_{i+1}^4 = \mathrm{node}(\{\beta_i^2, \beta_i^3, \beta_i^4\}, \emptyset)$.

As in [4] (where a very similar construction is done), the nodes of $\{\beta_i^1 \mid i \in \omega\}$ are pairwise incomparable, and they all force ϕ.

Define a Kripke model K over V_ω as follows: The set of nodes of K is the set $\mathrm{Forces}_{\mathcal{U}(2)}^{-1}(\psi) - \mathrm{Forces}_{\mathcal{U}(2)}^{-1}(\phi)$ and a node α forces x_i iff $\alpha \not\leq \beta_i^1$.

Lemma 27. *For all $\phi, \psi \in F_\omega$, $\mathrm{Forces}_K^{-1}(\phi) \subseteq \mathrm{Forces}_k^{-1}(\psi)$ iff $\phi \vdash \psi$.*

Proof. Since K is a Kripke model, if $\phi \vdash \psi$, $\mathrm{Forces}_K^{-1}(\phi) \subseteq \mathrm{Forces}_K^{-1}(\psi)$.

Suppose $\phi \nvdash \psi$. Then there is a rooted finite Kripke model K' over V_ω such that $K' \Vdash \phi$ and $K' \nVdash \psi$. Since variables not occurring in ϕ or ψ are irrelevant, we may assume that each node of K' forces cofinitely many propositional variables.

Define a map $a\colon K' \to K$ inductively on K' as follows: If $\gamma \in K'$ is a node such that $a(\gamma')$ has defined for all immediate successors of γ, then let $a(\gamma)$ be a node whose set of successors in $\mathcal{U}(2)$ is the upward-closure of the set $\{\beta_i^1 \mid \gamma \nVdash x_i\} \cup \{a(\gamma') \mid \gamma' \geq \gamma\} \cup \{\alpha_5\}$.

For each i, $\gamma \Vdash x_i$ iff $a(\gamma) \Vdash x_i$. Since a is also order-preserving and its range is upward-closed in K, we have that if γ is the root of K', $a(\gamma) \Vdash \phi$ and $a(\gamma) \nVdash \psi$. $\square$

Now, as before, define $f\colon F_\omega \to F_2$ by:

1. $f(\bot) = \phi$
2. $f(x_i) = (\phi'_{\beta_i^1} \vee \phi) \wedge \psi$
3. $f(\rho_0 \wedge \rho_1) = f(\rho_0) \wedge f(\rho_1)$.
4. $f(\rho_0 \vee \rho_1) = f(\rho_0) \vee f(\rho_1)$.
5. $f(\rho_0 \to \rho_1) = (f(\rho_0) \to f(\rho_1)) \wedge \psi$.

By precisely the same argument as before, this is an embedding. $\square$

Note that, by [4], in any interval $[\phi, \psi] \subseteq H_n$, there are atomic elements. As there are no atomic elements in H_ω, H_ω cannot be embedded in H_n as an interval.

6.3 Impossibility of Lattice-Embedding B_ω into H_ω

Let B_ω be the countable atomless Boolean algebra. We will think of it as the Lindenbaum algebra of classical propositional logic on V_ω.

Proposition 13. *There is no lattice embedding from B_ω into H_n for any n or into H_ω.*

Proof. By the previous theorem, it suffices to prove the proposition for H_2. Suppose there is a lattice embedding of B_ω into H_2. Call it f.

Let $\mathcal{T}$ be a prefix-closed intuitionistic-equivalence-respecting automaton representing $f(\top)$ and suppose it has n states. Consider the 2^n formulas $\phi_1 = x_1 \wedge \cdots \wedge x_n$, $\phi_2 = x_1 \wedge \cdots \wedge \neg x_n, \ldots, \phi_{2^n} = \neg x_1 \wedge \cdots \wedge \neg x_n$. Since f preserves $\wedge$ and $\vee$ we must have that $\{\mathrm{Forces}^{-1}_{\mathcal{U}(n)}(f(\phi_i)) \mid 1 \leq i \leq 2^n\}$ is a partition of $\text{Forces}^{-1}_{\mathcal{U}(n)}(f(\top)) - \text{Forces}^{-1}_{\mathcal{U}(n)}(f(\bot))$ and that $\text{Forces}^{-1}_{\mathcal{U}(n)}(f(\phi_i)) \cap (\text{Forces}^{-1}_{\mathcal{U}(n)}(f(\top)) - \text{Forces}^{-1}_{\mathcal{U}(n)}(f(\bot)))$ is non-empty for each i.

For each i, let $\beta_i \in \text{Forces}^{-1}_{\mathcal{U}(n)}(f(\phi_i)) \cap (\text{Forces}^{-1}_{\mathcal{U}(n)}(f(\top)) - \text{Forces}^{-1}_{\mathcal{U}(n)}(f(\bot)))$. By the pigeonhole principle, there must be some i and j, $i \neq j$, such that $\mathcal{T}(\beta_i) = \mathcal{T}(\beta_j)$. Let β be $\text{node}(\{\beta_i, \beta_j\})$. By the properties of intuitionistic-equivalence-respecting automata, $\mathcal{T}(\beta) = \mathcal{T}(\beta_i)$, so $\beta \Vdash f(\top)$. Thus, β is in $\text{Forces}^{-1}_{\mathcal{U}(n)}(f(\phi_m)) \cap (\text{Forces}^{-1}_{\mathcal{U}(n)}(f(\top)) - \text{Forces}^{-1}_{\mathcal{U}(n)}(f(\bot)))$ for some m. Without loss of generality, say $m \neq i$. Then $\beta_i \Vdash f(\phi_m)$ and $\beta_i \Vdash f(\phi_i)$ but $\beta_i \nVdash f(\bot)$, a contradiction.

$\square$

6.4 Order-Embeddings

As an immediate corollary of the characterization in Theorem 3, we have the following.

Proposition 14. *All countable semilattices can be semilattice-embedded into H_n for $n \geq 2$.*

Since all countable partial orders P can be order-embedded into some countable lattice P' (for example, let P' be the lattice generated by the elements of $\mathcal{P}(P)$ of the form $p\downarrow$ for $p \in P$), this gives the following:

Proposition 15. *All countable partial orders can be order-embedded into H_n for $n \geq 2$.*

A natural further question is to ask when a countable lattice can be lattice-embedded into H_n for $n \geq 2$. Since each H_n is distributive, distributivity is a necessary condition. It is easy to see that if a distributive lattice L is such that every element is a finite join of join-irreducibles, then its join-irreducibles form a quasisemilattice. From Theorem 3, we get the following:

Proposition 16. *If L is a countable distributive lattice with the property that every element is a finite join of join-irreducibles and such that its quasisemilattice of join-irreducibles is locally finite, then L lattice-embeds into H_n for $n \geq 2$.*

Definition 37. A division of an element a of a distributive lattice is a finite set $\{b_i\}$ such that $a = \bigvee\{b_i\}$ and for each i, $b_i \not\leq \bigvee_{j \neq i}\{b_j\}$.

An element a of a distributive lattice is called arbitrarily divisible if for each n it has a division of cardinality n.

For example, every element of the countable atomless boolean algebra is arbitrarily divisible. It is the case that no arbitrarily divisible element of a lattice can be a finite join of join-irreducibles. Therefore, the following proposition is a partial converse to Proposition 16.

Proposition 17. *Suppose L is a distributive lattice with an arbitrarily divisible element. Then it cannot be lattice-embeded into any H_n.*

Proof. It suffices to show that no element of H_n is arbitrarily divisible. Suppose that a is, and consider its representation as a tree automaton $\mathcal{T}$. Let $m = |\mathbf{states}_{\mathcal{T}}|$ and let $\{b_i\}$ be a division of cardinality $m+1$.

For $1 \leq i \leq m+1$, let α_i be a node of $\mathcal{U}(n)$ such that $\alpha_i \Vdash b_i$ and $\alpha_i \nVdash \bigvee_{j \neq i} b_j$. By the pigeonhole principle, there must be i_0 and i_1 such that $\mathcal{T}(\alpha_{i_0}) = \mathcal{T}(\alpha_{i_1})$. Therefore, node$(\{\alpha_{i_0}, \alpha_{i_1}\})$ is also accepted by $\mathcal{T}$ and therefore forces a. But, since it is below both α_{i_0} and α_{i_1}, it cannot force any b_i, which is a contradiction. $\square$

CHAPTER 7
EXISTENCE PROBLEMS

7.1 Automata Substitution

Definition 38. Let $\mathcal{T}$ be an automaton whose alphabet is $\mathcal{A}_{n+1}$. Let $\mathcal{S}$ be an automaton with alphabet $\mathcal{A}_n$. The automaton $\mathcal{T}[\mathcal{S}/x_{n+1}]$ is defined as follows: Its alphabet is the same as the alphabet of $\mathcal{S}$. The set $\mathbf{states}_{\mathcal{T}[\mathcal{S}/x_{n+1}]}$ is equal to

$$\{(t,s) \mid t \in \mathbf{states}_{\mathcal{T}}, s \in \mathbf{states}_{\mathcal{S}}, x \in \Delta_{\mathcal{T}}^{-1}(t) \text{ iff } s \in \mathbf{accept}_{\mathcal{S}}\}.$$

The transition is given by

$$\Delta_{\mathcal{T}[\mathcal{S}/x_{n+1}]}(a)(S) = \begin{cases} \Delta_{\mathcal{T}}(a \cup \{x_{n+1}\})(\pi_1(S)) & \Delta_{\mathcal{S}}(a)(\pi_2(S)) \in \mathbf{accept}_{\mathcal{S}} \\ \Delta_{\mathcal{T}}(a)(\pi_1(S)) & \Delta_{\mathcal{S}}(a)(\pi_2(S)) \notin \mathbf{accept}_{\mathcal{S}} \end{cases}$$

where $\pi_2(S) = \{s \mid (\exists t)\,(t,s) \in S\}$.

A state (t,s) is accepting iff t is accepting in $\mathcal{T}$.

Lemma 28. *Let $\phi \in F_{n+1}$ and $\psi \in F_n$. Suppose that $\mathcal{T}$ represents ϕ and $\mathcal{S}$ represents ψ. Then $\mathcal{T}[\mathcal{S}/x_{n+1}]$ represents $\phi[\psi/x_{n+1}]$.*

Proof. Straightforward. □

Definition 39. Given an automaton $\mathcal{T}$, let $\mathbf{kripke}_{\mathcal{T}}$ be the set of Kripke-accessible states of $\mathcal{T}$.

Definition 40. Suppose that $\mathcal{T}$ is an automaton whose alphabet is $\mathcal{A}_{n+1}$. Let $\mathcal{S}$ be an automaton with alphabet $\mathcal{A}_n$. We say that $\mathcal{S}$ is *finely grained with respect to $\mathcal{T}$* if for all $s \in \mathbf{kripke}_{\mathcal{S}}$ there is exactly one $t \in \mathbf{kripke}_{\mathcal{T}}$ such that (t,s) is a Kripke-accessible node of $\mathcal{T}[\mathcal{S}/x_{n+1}]$. In this case, we let $p_{\mathcal{S}}$ be a function from $\mathbf{kripke}_{\mathcal{S}}$ to $\mathbf{kripke}_{\mathcal{T}}$ such that for all $s \in \mathbf{kripke}_{\mathcal{S}}$, $(p(s),s) \in \mathbf{kripke}_{\mathcal{T}[\mathcal{S}/x_{n+1}]}$.

Lemma 29. *Suppose that $\mathcal{T}$ is an automaton whose alphabet $\mathcal{A}_{n+1}$. Let $\mathcal{S}$ and $\mathcal{S}'$ be automata whose alphabet is $\mathcal{A}_n$. Suppose that $\mathcal{S}$ and $\mathcal{S}'$ are finely grained with respect to $\mathcal{T}$ and that $\mathcal{T}[\mathcal{S}/x_{n+1}]$ accepts all Kripke models.*

Let $f\colon \mathbf{kripke}_{\mathcal{S}'} \to \mathbf{kripke}_{\mathcal{S}}$ be such that for all $q \in \mathbf{kripke}_{\mathcal{S}'}$, $\Delta_{\mathcal{S}'}^{-1}(q) = \Delta^{-1}(f(q))$ and $p(q) = p(f(q))$. Suppose also that for all $S \subseteq \mathbf{kripke}_{\mathcal{S}'}$, $\Delta_{\mathcal{S}'}(V)(S)$ is accepting iff $\Delta_{\mathcal{S}}(V)(f(S))$ is accepting.

Then $\mathcal{T}[\mathcal{S}'/x]$ accepts all Kripke models.

Proof. Let t be a Kripke (tree) model. Then t is of the form $a(t_1, \ldots, t_m)$ where if each t_i is of the form $a_i(\bullet)$, then for all i, $a \subseteq a_i$.

The state $\pi_1(\mathcal{T}[\mathcal{S}'/x](t))$ depends on only two things: the set $\{\pi_1(\mathcal{T}[\mathcal{S}'/x](t_i)) \mid 1 \leq i \leq m\}$ and whether or not $\Delta_{\mathcal{S}'}(a)(\{\pi_2(\mathcal{T}[\mathcal{S}'/x](t_i)) \mid 1 \leq i \leq m\})$ is accepting.

Let t' be $a(t'_1, \ldots, t'_m)$ where for each i, t'_i is a Kripke model such that $\mathcal{S}(t'_i) = f(\mathcal{S}'(t_i))$. Then, by hypothesis, $\{\pi_1(\mathcal{T}[\mathcal{S}'/x](t_i)) \mid 1 \leq i \leq m\} = \{\pi_1(\mathcal{T}[\mathcal{S}/x](t'_i)) \mid 1 \leq i \leq m\}$ and $\Delta_{\mathcal{S}'}(a)(\{\pi_2(\mathcal{T}[\mathcal{S}'/x](t_i)) \mid 1 \leq i \leq m\})$ is accepting iff $\Delta_{\mathcal{S}'}(a)(\{\pi_2(\mathcal{T}[\mathcal{S}/x](t'_i)) \mid 1 \leq i \leq m\})$ is accepting. Therefore, $\pi_1(\mathcal{T}[\mathcal{S}'/x](t)) = \pi_1(\mathcal{T}[\mathcal{S}/x](t'))$.

Since $\mathcal{T}[\mathcal{S}/x]$ accepts all Kripke models, $\pi_1(\mathcal{T}[\mathcal{S}/x](t'))$ must be accepting. Therefore, $\mathcal{T}[\mathcal{S}'/x]$ accepts t. Since t was arbitrary, we are done. $\square$

7.2 Solutions to Equations

Definition 41. Let ϕ be a propositional formula with a variable x. A *solution* to $\phi(x) = \top$ is a propositional formula ψ such that $\phi[\psi/x]$ is a tautology.

Proposition 18. *The set of $\phi(x)$ which have a solution is a decidable set.*

Proof. We prove this by showing the following lemma:

Lemma 30. *Suppose that there is some $\mathcal{S}$ such that $\mathcal{T}[\mathcal{S}/x]$ accepts all Kripke models. Then there is an $\mathcal{S}'$ such that $\mathcal{T}[\mathcal{S}'/x]$ accepts all Kripke models and $\mathbf{states}_{\mathcal{S}'} = \mathbf{states}_{\mathcal{T}}$.*

Proof. We may assume that $\mathcal{S}$ is finely grained with respect to $\mathcal{T}$ and that $x = x_{n+1}$ where the alphabet of $\mathcal{T}$ is $\mathcal{A}_{n+1}$.

Let $\mathbf{states}_{\mathcal{S}'} = \mathbf{states}_{\mathcal{T}}$. The transition function $\Delta_{\mathcal{S}'}$ will be given by the following lemma:

Lemma 31. *There is a transition function $\Delta_{\mathcal{S}'}$ (and a corresponding notion of a Kripke-accessible state of $\mathcal{S}'$) and a function f from the Kripke-accessible states of $\mathcal{S}'$ to the Kripke accessible states of $\mathcal{S}$ such that for all $S \subseteq \mathbf{states}_{\mathcal{S}'}$ and $a \subseteq V_n$,*

$$\Delta_{\mathcal{S}'}(a)(S) = \begin{cases} \Delta_{\mathcal{T}}(a \cup \{x_{n+1}\})(S) & \Delta_{\mathcal{S}}(a)(f(S)) \text{ accepting} \\ \Delta_{\mathcal{T}}(a)(S) & \Delta_{\mathcal{S}}(a)(f(S)) \text{ not accepting} \end{cases}$$

and for all Kripke-accessible states q of $\mathcal{S}'$, $\Delta_{\mathcal{S}'}^{-1}(q) = \Delta_{\mathcal{S}}^{-1}(f(q))$.

Proof. We will define a finite sequence of partial functions $f_0 \subseteq f_1 \subseteq \cdots \subseteq f_m$ from $\textbf{states}_{\mathcal{S}'}$ to the set of Kripke-accessible states of $\mathcal{S}$ and let $f = f_m$.

For each f_i, let

$$(\Delta_{\mathcal{S}'})_i(a)(S) = \begin{cases} \Delta_{\mathcal{T}}(a \cup \{x_{n+1}\})(S) & \Delta_{\mathcal{S}}(a)(f_i(S)) \text{ accepting} \\ \Delta_{\mathcal{T}}(a)(S) & \Delta_{\mathcal{S}}(a)(f_i(S)) \text{ not accepting} \end{cases}$$

for all a and all $S \subseteq \operatorname{dom} f_i$.

Let $f_0 = \emptyset$. Suppose f_i has been defined. If for all a and all $S \subseteq \operatorname{dom} f_i$, $(\Delta_{S'})_i(a)(S) \in \operatorname{dom} f_i$, then stop and let $f = f_i$ and $\Delta_{S'} = (\Delta_{S'})_i$.

Otherwise, pick a a and an S for which $(\Delta_{S'})_i(a)(S) \notin \operatorname{dom} f_i$ and let $f((\Delta_{S'})_i(a)(S)) = \Delta_{\mathcal{S}}(a)(f(S))$.

By construction, we are done. $\square$

Let $\textbf{accept}'_{\mathcal{S}} = \{q \mid q$ Kripke-accessible as a state of $\mathcal{T}$ and $x_{n+1} \in \Delta_{\mathcal{T}}^{-1}(q)\}$.

By Lemma 29, we are done. $\square$

Let $\mathcal{T}$ represent ϕ. By Proposition 30, if there is any automaton $\mathcal{S}$ such that $\mathcal{T}[\mathcal{S}/x]$ accepts all Kripke models, then there is one with exactly as many states as $\mathcal{T}$ has, and there are only finitely many such automata. $\square$

Note that this also implies that there is an algorithm deciding whether an arbitrary equation $\psi(x) = \chi(x)$ has a solution by letting $\phi(x)$ be $\psi(x) \leftrightarrow \chi(x)$.

7.3 An Existentially Closed Embedding

Let $\mathcal{U}^*(n)$ be the set of upward-closed subsets of $\mathcal{U}(n)$, given its natural Heyting algebra structure. There is a canonical embedding of H_n into $\mathcal{U}^*(n)$.

Proposition 19. *The embedding of H_n into $\mathcal{U}^*(n)$ is existentially closed.*

Proof. Let $\mu(x)$ be a formula with one free variable in the language of Heyting algebras.

Let $u \in \mathcal{U}^*(n)$ be such that $\mathcal{U}^*(n) \models \mu(u)$. Let $\nu(x)$ be a formula of the form $\phi(x) = \top \wedge \psi(x) \neq \top$ such that $\mathcal{U}^*(n) \models \nu(u)$ and $\forall x\, (\nu(x) \to \mu(x))$.

We will find a $w \in H_n$ such that $H_n \models \nu(w)$.

Let $\mathcal{T}_0$ be a finite prefix-closed intuitionistic-equivalence-respecting automaton representing ϕ and $\mathcal{T}_1$ a finite prefix-closed intuitionistic-equivalence-respecting automaton representing ψ.

We extend the concept of automaton to include automata with possibly infinitely many states. We view upward-closed subsets $U \subseteq \mathcal{U}(n)$ as automata $\mathcal{T}_U$ where $\mathbf{states}_{\mathcal{T}_U} = \mathcal{U}(n) \cup \{\bullet\}$, $\mathbf{accept}_{\mathcal{T}_U} = U$, and

$$\Delta_{\mathcal{T}_U}(V)(S) = \begin{cases} \mathrm{node}(S,V) & \mathrm{node}(S,V) \text{ exists} \\ \bullet & \text{otherwise} \end{cases}$$

Thus, by assumption, $\mathcal{T}_0[\mathcal{T}_u/x]$ accepts all Kripke models, and $\mathcal{T}_1[\mathcal{T}_u/x]$ does not accept all Kripke models. We must show that there is a finite prefix-closed intuitionistic-equivalence-respecting automaton $\mathcal{S}$ with the same property.

Let $f = \pi_1 \circ \mathcal{T}_0[\mathcal{T}_u/x]$.

Let K be a Kripke (tree) model not accepted by $\mathcal{T}_1[\mathcal{T}_u/x]$. Let $(\mathbf{states}_{\mathcal{S}})_0$ be the set of nodes in K together with $\bullet$. Let

$$(\Delta_{\mathcal{S}})_0(V)(S) = \begin{cases} \mathrm{node}(S,V) & \mathrm{node}(S,V) \in (\mathbf{states}_{\mathcal{S}})_0 \\ \bullet & \mathrm{node}(S,V) \text{ does not exist} \\ \uparrow & \text{otherwise} \end{cases}$$

Suppose $(\mathbf{states}_{\mathcal{S}})_i$ and $(\Delta_{\mathcal{S}})_i$ are defined. If $(\Delta_{\mathcal{S}})_i(V)(S)$ is defined for all V and all $S \subseteq (\mathbf{states}_{\mathcal{S}})_i$ then let $\mathbf{states}_{\mathcal{S}} = (\mathbf{states}_{\mathcal{S}})_i$ and $\Delta_{\mathcal{S}} = (\Delta_{\mathcal{S}})_i$.

Otherwise, pick a V and an $S \subseteq (\mathbf{states}_{\mathcal{S}})_i$ such that $(\Delta_{\mathcal{S}})_i(V)(S)$ is not defined. If there is some $q \in (\mathbf{states}_{\mathcal{S}})_i$ such that $f(q) = f(\mathrm{node}(S,V))$ then let $(\mathbf{states}_{\mathcal{S}})_{i+1} = (\mathbf{states}_{\mathcal{S}})_i$ and let $(\Delta_{\mathcal{S}})_{i+1}(V)(S) = q$ and otherwise be the same as $(\Delta_{\mathcal{S}})_i$.

If there is no such q then let $(\mathbf{states}_{\mathcal{S}})_{i+1} = (\mathbf{states}_{\mathcal{S}})_i \cup \{\mathrm{node}(S,V)\}$ and let $(\Delta_{\mathcal{S}})_{i+1}(V)(S) = \mathrm{node}(S,V)$ and otherwise be the same as $(\Delta_{\mathcal{S}})_i$.

Let $\mathbf{accept}_{\mathcal{S}} = u \cap \mathbf{states}_{\mathcal{S}}$.

That $\mathcal{T}_1[\mathcal{S}/x](K)$ is non-accept is clear, since by construction, $\mathcal{T}_1[\mathcal{S}/x](K) = \mathcal{T}_1[\mathcal{T}_u/x](K)$.

That $\mathcal{T}_0[\mathcal{S}/x]$ accepts all Kripke models follows from Lemma 29.

□

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