REAL SYMPLECTIC GEOMETRY

REYER SJAMAAR

Abstract. These are the notes of an introductory lecture series on convexity properties of the moment map, equivariant cohomology and conjugation spaces.

Introduction

These notes are based on a minicourse of three lectures I taught at the Séminaire Itinérant de Géométrie et Physique Mathématique V, which took place in May 2007 at the Université Cheikh Anta Diop in Dakar, Sénégal. I thank my hosts and the organizers for their hospitality and for offering me the opportunity to speak. The lectures are recorded here almost verbatim. I invite the reader who wishes to learn more details than I have given to consult the bibliography. Where available I have given references that are available at no charge on the web through the arXiv and other eprint repositories.

1. Real Hamiltonian G-manifolds

Real structures. Let X be a symplectic manifold with symplectic form ω. A *real structure* on X is a smooth mapping $\sigma\colon X \to X$ which is an *involution* (i.e. $\sigma^2 = \mathrm{id}_X$) and which is *anti-symplectic* (i.e. $\sigma^*\omega = -\omega$). We call X equipped with the symplectic form ω and the real structure σ a *real symplectic manifold*. We put

$$X^\sigma = \{\, x \in X \mid \sigma(x) = x \,\}$$

and we call X^σ the *real locus* of X. The following example explains why we call X^σ the real locus.

1.1. **Example.** Let $V = \mathbf{C}^n$, the vector space of column vectors with n complex components. We view V as a $2n$-dimensional real vector space equipped with the symplectic form

$$\omega_0 = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j = \sum_{j=1}^n dx_j \wedge dy_j,$$

where $x_j = \mathrm{Re}(z_j)$ and $y_j = \mathrm{Im}(z_j)$. Another useful formula for the symplectic form is

$$\omega_0(z, w) = \mathrm{Im}(z^* w),$$

the imaginary part of the Hermitian inner product $z^*w = \sum_{j=1}^n \bar{z}_j w_j$ of z and w. We define an involution σ_0 on V by component-wise complex conjugation,

$$\sigma_0(z) = \bar{z}.$$

Then $\sigma_0^*\omega_0 = -\omega_0$. The real locus of V is $V^{\sigma_0} = \mathbf{R}^n$. We call ω_0 the *standard* symplectic form and σ_0 the *standard* real structure on V.

Date: 22 October 2007.
The author was partially supported by National Science Foundation Grant DMS-0504641.

1.2. **Lemma.** *Let X be a symplectic manifold with symplectic form ω and real structure σ. If the real locus X^σ is nonempty, then it is a Lagrangian submanifold of X.*

Proof. Suppose that X^σ is nonempty and take $x \in X^\sigma$. Let $n = \frac{1}{2}\dim(X)$ and let V be the symplectic vector space $\mathbf{C}^n$ equipped with the standard symplectic form and real structure as in Example 1.1. A real version of the Darboux theorem (see for instance [9, Lemma 2.3] or [29, Appendix A]) says that there exist an open neighbourhood U of x preserved by σ and a chart $\phi\colon U \to V$ centred at x such that

$$\phi^*\omega = \omega_0, \qquad \phi \circ \sigma = \sigma_0 \circ \phi.$$

We conclude that $X^\sigma \cap U = \phi^{-1}(V^{\sigma_0}) = \phi^{-1}(\mathbf{R}^n)$ is a Lagrangian submanifold of U. QED

Let G be a Lie group. A *Hamiltonian G-manifold* is a manifold X equipped with a smooth G-action, a G-invariant symplectic form ω and a *moment map*, which means a smooth map $\Phi\colon X \to \mathfrak{g}^*$ that satisfies the following conditions:

(i) $d\langle \Phi, \xi \rangle = \iota(\xi_X)\omega$,
(ii) Φ is equivariant: $\langle \Phi(g\cdot x), \xi\rangle = \langle \Phi(x), \mathrm{Ad}(g)^{-1}\xi\rangle$

for all $\xi \in \mathfrak{g}$ and $x \in X$. Here $\langle\cdot,\cdot\rangle$ denotes the dual pairing $\mathfrak{g}^* \times \mathfrak{g} \to \mathbf{R}$,

$$\xi_X(x) = \frac{d}{dt}(\exp(t\xi)\cdot x)\Big|_{t=0}$$

is the vector field on X generated by $\xi \in \mathfrak{g}$, and $\iota(\xi_X)\omega$ denotes the 1-form on X obtained by taking the inner product of the vector field ξ_X with the 2-form ω.

1.3. **Example.** Let $V = \mathbf{C}^n$ with the standard symplectic structure and let $\mathbf{U}(n)$ be the unitary group acting on V by matrix multiplication on the left. This action preserves the symplectic form. The Lie algebra $\mathfrak{u}(n)$ of $\mathbf{U}(n)$ consists of all anti-selfadjoint $n\times n$-matrices. The rule $(\xi, \eta) = -\operatorname{trace}(\xi\eta)$ defines a $\mathbf{U}(n)$-invariant positive definite inner product on $\mathfrak{u}(n)$. It will be convenient to identify $\mathfrak{u}(n)^*$ with $\mathfrak{u}(n)$ by means of this inner product. Since elements of V are column vectors, the product zz^* is a selfadjoint $n\times n$-matrix for all $z \in V$. Then a moment map $\Phi_0\colon V \to \mathfrak{u}(n)^*$ for the $\mathbf{U}(n)$-action is given by

$$\Phi_0(z) = \frac{1}{2i}zz^*.$$

We will call this the *standard* moment map for the $\mathbf{U}(n)$-action on V. Let us check that the condition $d\langle\Phi_0, \xi\rangle = \iota(\xi_V)\omega$ is satisfied. On one hand,

$$\langle \Phi_0(z), \xi\rangle = -\operatorname{trace}(\Phi_0(z)\xi) = \frac{i}{2}\operatorname{trace}(zz^*\xi) = \frac{i}{2}z^*\xi z,$$

so for all $w \in V$

$$\begin{aligned} d\langle\Phi_0,\xi\rangle_z(w) = \frac{d}{dt}\langle\Phi_0(z+tw),\xi\rangle\Big|_{t=0} &= \frac{i}{2}\frac{d}{dt}\big((z+tw)^*\xi(z+tw)\big)\Big|_{t=0} \\ &= \frac{i}{2}(w^*\xi z + z^*\xi w). \end{aligned}$$

On the other hand,

$$\begin{aligned}\iota(\xi_{V,z})\omega(w) = \omega(\xi z, w) = \operatorname{Im}((\xi z)^* w) &= \frac{1}{2i}(z^*\xi^* w - (z^*\xi^* w)^*)\\ &= \frac{1}{2i}(z^*\xi^* w - w^*\xi z) = \frac{i}{2}(z^*\xi w + w^*\xi z),\end{aligned}$$

where we used that $\xi^* = -\xi$. Hence $d\langle \Phi_0, \xi\rangle = \iota(\xi_V)\omega$. The equivariance of Φ is left as an exercise for the reader.

A *real structure* on a Hamiltonian G-manifold X is a pair of smooth mappings

$$\sigma_G \colon G \longrightarrow G, \qquad \sigma_X \colon X \longrightarrow X$$

where σ_G is an *group involution* (i.e. $\sigma_G^2 = \mathrm{id}_G$ and $\sigma_G(gh) = \sigma_G(g)\sigma_G(h)$ for all g, $h \in G$), and where σ_X is a real structure on the symplectic manifold X. In addition, we require that the involution σ_X is compatible with the involution σ_G in the sense that

$$\sigma_X(g\cdot x) = \sigma_G(g)\cdot\sigma_X(x), \qquad \Phi(\sigma_X(x)) = -\sigma_G^*\Phi(x)$$

for all $g \in G$ and $x \in X$. Here $\sigma_G^* \colon \mathfrak{g}^* \to \mathfrak{g}^*$ is defined as the transpose of the Lie algebra involution $(\sigma_G)_* \colon \mathfrak{g} \to \mathfrak{g}$ induced by the group involution σ_G. We call the G-manifold X together with the additional data ω, Φ, σ_X and σ_G a *real Hamiltonian G-manifold*.

When no confusion can arise, we shall abuse the notation by writing σ for all four involutions σ_G, $(\sigma_G)_*$, σ_G^* and σ_X. We put

$$G^\sigma = \{\, g \in G \mid \sigma(g) = g \,\}.$$

1.4. **Lemma.** *Let X be a real Hamiltonian G-manifold. The real locus X^σ is invariant under the action of G^σ.*

Proof. Let $g \in G$ and $x \in X$. If $\sigma(g) = g$ and $\sigma(x) = x$, then $\sigma(g\cdot x) = \sigma(g)\cdot\sigma(x) = g\cdot x$, so $g\cdot x \in X^\sigma$. This proves that $G^\sigma \cdot X^\sigma \subseteq X^\sigma$. QED

1.5. **Example.** Let $V = \mathbf{C}^n$ with the standard symplectic structure and the Hamiltonian $\mathbf{U}(n)$-action of Example 1.3. Let σ_0 be the standard involution on $\mathbf{C}^n$. Define an involution σ_0 on $\mathbf{U}(n)$ by $\sigma_0(g) = \bar{g} = (g^{-1})^t$, the complex conjugate or inverse transpose of g. With these involutions on V and $\mathbf{U}(n)$, V is a real Hamiltonian $\mathbf{U}(n)$-manifold. Note that $V^\sigma = \mathbf{R}^n$ is a Lagrangian submanifold invariant under the subgroup $G^\sigma = \mathbf{O}(n)$, the orthogonal group.

1.6. **Example.** Let X be a sphere in $\mathbf{R}^3$ centred at the origin and let ω be the area form of X. The rotation group $G = \mathbf{SO}(3)$ acts on X and preserves the form ω. The Lie algebra of G is $\mathbf{R}^3$ (with the Lie bracket given by the cross product). With an identification $(\mathbf{R}^3)^* \cong \mathbf{R}^3$ given by a suitably normalized inner product, the inclusion map $\Phi \colon X \to \mathbf{R}^3$ is a moment map for the action. Let σ_X be the reflection in the xy-plane. Define σ_G by $\sigma_G(g) = \sigma_X \circ g \circ \sigma_X$; then $G^\sigma \cong \mathbf{SO}(2)$ is the group of rotations in the xy-plane. The involutions σ_X and σ_G define a structure of real Hamiltonian G-manifold on X. The real locus X^σ is a great circle (the equator) in X, which is preserved by $\mathbf{SO}(2)$.

1.7. **Example.** Let X, ω, G and Φ be as in Example 1.6. This time we choose as antisymplectic involution on X the antipodal map and as involution on G the identity map. This defines a new real structure on the Hamiltonian G-manifold X, for which the real locus X^σ is empty and the subgroup G^σ is equal to G.

Examples 1.6 and 1.7 are special cases of two large classes of examples: projective varieties defined over the real numbers and symmetric coadjoint orbits. The general case is as follows.

1.8. **Example.** Let G be a compact Lie group. Let σ_G be an involution on G and let $\phi\colon G \to \mathbf{U}(n)$ be a smooth homomorphism such that $\phi(\sigma_G(g)) = \overline{\phi(g)}$ for all $g \in G$. Let X be a nonsingular algebraic subvariety of the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$ which is invariant under the projective linear action of G given by ϕ. Let Ω be the Fubini-Study symplectic form of $\mathbf{P}^{n-1}(\mathbf{C})$ and let $\omega = \Omega \mid X$. As in Example 1.3, one can check that a moment map for the G-action on X is given by

$$\langle \Phi([z]), \xi \rangle = -\frac{1}{2\pi i}\frac{z^*\xi z}{z^* z}$$

for $z \in V \setminus \{0\}$ and $\xi \in \mathfrak{g}$. Here $V = \mathbf{C}^n$ and $[z]$ denotes the point in $\mathbf{P}^{n-1}(\mathbf{C})$ (i.e. the line through the origin) determined by z. We view $\xi \in \mathfrak{g}$ as an anti-selfadjoint matrix acting on vectors via the homomorphism $\phi_*\colon \mathfrak{g} \to \mathfrak{u}(n)$ induced by ϕ. Now assume that the variety X is *defined over the real numbers*. This means that the homogeneous ideal of X is generated by polynomials with real coefficients, in other words that X can be defined by real equations. Then the involution σ_G and the involution

$$\sigma_X([z]) = [\bar{z}]$$

define a real structure on the Hamiltonian G-manifold X. The real locus of X is $X^\sigma = X \cap \mathbf{P}^{n-1}(\mathbf{R})$.

1.9. **Example.** Let σ_G be an involution of G. As before, let $(\sigma_G)_*\colon \mathfrak{g} \to \mathfrak{g}$ be the Lie algebra involution induced by σ_G and let $\sigma_G^*\colon \mathfrak{g}^* \to \mathfrak{g}^*$ be the transpose of $(\sigma_G)_*$. Recall that the *coadjoint action* is the linear G-action on $\mathfrak{g}^*$ which is dual to the adjoint action on $\mathfrak{g}$ in the sense that $\langle \mathrm{Ad}^*(g)(\lambda), \xi \rangle = \langle \lambda, \mathrm{Ad}(g)^{-1}(\xi) \rangle$ for all $\lambda \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$. The coadjoint orbits (i.e. the orbits of the coadjoint action) are the leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ and each therefore carries a natural G-invariant symplectic form, known as the *Kirillov-Kostant-Souriau symplectic form*. The inclusion map $X \hookrightarrow \mathfrak{g}^*$ of a coadjoint orbit X is a moment map for the G-action. Let us call a coadjoint orbit X *symmetric* if it has the property that $(-\sigma_G^*)(X) = X$ and define

$$\sigma_X = -\sigma_G^* \mid X.$$

Since $(\sigma_G)_*$ is a Lie algebra involution, its transpose σ_G^* is an automorphism of the Poisson structure of $\mathfrak{g}^*$, and therefore σ_X is an antisymplectic involution of X. The involutions σ_G and σ_X define a structure of real Hamiltonian G-manifold on X. The real locus is $X^\sigma = X \cap \mathfrak{p}^*$, where

$$\mathfrak{p}^* = \{\, \lambda \in \mathfrak{g}^* \mid \sigma_G^*(\lambda) = -\lambda \,\}.$$

We conclude that the real locus is nonempty if and only if there exists $\lambda \in \mathfrak{p}^*$ such that $X = \mathrm{Ad}^*(G)(\lambda)$.

The notion of a real Hamiltonian G-manifold first arose (under a different name) in Duistermaat's paper [9] and was later taken up by O'Shea and me in [29] and by many others as well. See for instance the references [1, 4, 12, 16, 26].

Duistermaat discovered a remarkable convexity property of the moment map image of the real locus, which I will discuss in Lecture 2. He also found some cohomological properties, which I will survey in the remainder of this lecture.

Cohomology. There are many interesting relationships between the cohomology of a Hamiltonian G-manifold X and that of its fixed-point set X^G, such as the following theorem, the first part of which is due to Frankel [14]. The second part is due to Duistermaat, who showed in [9] that in the presence of a real structure a statement parallel to Frankel's is true for the real locus.

1.10. **Theorem.** *Let T be a torus (i.e. a Lie group isomorphic to a product of circles $\mathbf{U}(1) \times \mathbf{U}(1) \times \cdots \times \mathbf{U}(1)$) and let X be a compact Hamiltonian T-manifold. Let X^T be the fixed-point set of the T-action on X.*

(i) *Let $\mathbf{k}$ be a field. Then $\dim_{\mathbf{k}} H^*(X;\mathbf{k}) = \dim_{\mathbf{k}} H^*(X^T;\mathbf{k})$.*
(ii) *Let $\sigma_T(t) = t^{-1}$ and suppose X has a real structure σ_X compatible with σ_T. Let $\mathbf{k}$ be a field of characteristic 2. Then $\dim_{\mathbf{k}} H^*(X^\sigma;\mathbf{k}) = \dim_{\mathbf{k}} H^*(X^\sigma \cap X^T;\mathbf{k})$.*

Sketch of proof. We choose a generic element ξ of the Lie algebra $\mathfrak{t}$ of T. In the case of part (i) we put

$$A = X, \qquad B = X^T, \qquad f = \langle \Phi, \xi \rangle,$$

and in the case of part (ii) we put

$$A = X^\sigma, \qquad B = X^\sigma \cap X^T, \qquad f = \langle \Phi, \xi \rangle \mid A.$$

One shows that f is a Morse-Bott function on A (i.e. a function with nondegenerate critical manifolds in the sense of Bott [6]) and that B is the set of critical points of f. The inequality

$$\dim H^*(A;\mathbf{k}) \leq \dim H^*(B;\mathbf{k})$$

follows from the Morse-Bott inequalities, which are valid since the normal bundle of B in A is orientable over $\mathbf{k}$. The inequality

$$\dim H^*(A;\mathbf{k}) \geq \dim H^*(B;\mathbf{k})$$

is a result of Floyd [11], valid for any continuous action of the circle S^1 or of $\mathbf{Z}/p\mathbf{Z}$ (p prime) on a topological space A. QED

Equivariant cohomology. It turns out that similar facts are true in the theory of equivariant cohomology, which is a cohomology theory developed by Borel and others for the study of transformation groups of topological spaces. We recall some of the basic definitions. More details can be found in the references [2, 8, 19, 23, 27]. Let G be a compact Lie group and let E_G be a contractible topological space on which G acts freely and continuously. The quotient

$$B_G = E_G/G$$

is the *classifying space* of G, so called because principal G-bundles over a topological space X are classified up to isomorphism by homotopy classes of maps from X to B_G.

1.11. **Example.** Let $G = \mathbf{U}(1) = \{\, z \in \mathbf{C} \mid |z| = 1 \,\}$ be the unit circle and let E_G be the unit sphere in $l^2(\mathbf{C})$, the Hilbert space of square-summable complex sequences. Then G acts freely on E_G by scalar multiplication, E_G is contractible, and so $B_G = E_G/G = \mathbf{P}^\infty(\mathbf{C})$, the infinite-dimensional complex projective space.

1.12. **Example.** Let $G = \mathbf{Z}/2\mathbf{Z}$ and let E_G be the unit sphere in $l^2(\mathbf{R})$, the Hilbert space of square-summable real sequences. Then G acts freely on E_G by scalar multiplication, E_G is contractible, and so $B_G = E_G/G = \mathbf{P}^\infty(\mathbf{R})$, the infinite-dimensional real projective space.

Let X be a topological space on which G acts continuously. Then G acts diagonally on $E_G \times X$. Borel's *homotopy quotient* of X by G is defined by

$$X_G = (E_G \times X)/G.$$

The projection onto the first factor $E_G \times X \to E_G$ induces a map $X_G \to B_G$, which is a fibre bundle with fibre X. The projection onto the second factor $E_G \times X \to X$ induces a surjective map from the homotopy quotient X_G to the ordinary quotient or orbit space X/G. For any commutative ring $\mathbf{k}$ with identity we define the *equivariant cohomology ring* of X with coefficients in $\mathbf{k}$ to be

$$H^*_G(X;\mathbf{k}) = H^*(X_G;\mathbf{k}).$$

The following general properties follow immediately from the definition of equivariant cohomology.

1.13. **Lemma.** *Let X be a topological G-space.*

(i) *If the G-action on X is free, then the map $X_G \to X/G$ is a fibre bundle with contractible fibre E_G, so $H^*_G(X;\mathbf{k}) \cong H^*(X/G;\mathbf{k})$.*
(ii) *If X is a point, then $X_G = B_G$, so $H^*_G(X;\mathbf{k}) \cong H^*(B_G;\mathbf{k})$.*
(iii) *The map $X_G \to B_G$ induces a ring homomorphism*
$$H^*(B_G;\mathbf{k}) \to H^*_G(X;\mathbf{k}),$$
so $H^*_G(X;\mathbf{k})$ is an algebra over $H^*(B_G;\mathbf{k})$.
(iv) *If G acts trivially on X, then $X_G = B_G \times X$ and therefore*
$$H^*_G(X;\mathbf{k}) \cong H^*(B_G;\mathbf{k}) \otimes_{\mathbf{k}} H^*(X;\mathbf{k})$$
if $H^*(B_G;\mathbf{k})$ or $H^*(X;\mathbf{k})$ is a free $\mathbf{k}$-module.

1.14. **Example.** Let $G = \mathbf{U}(1)$. Example 1.11 gives $B_G = \mathbf{P}^\infty(\mathbf{C})$, so

$$H^*_G(\text{point};\mathbf{k}) = H^*(B_G;\mathbf{k}) = \mathbf{k}[x],$$

a polynomial algebra in one variable x of degree 2. (See for instance [21, Theorem 3.12].)

1.15. **Example.** Let $G = \mathbf{U}(1)$ and $X = S^2$, a two-dimensional sphere on which G acts by rotations about a fixed axis. Then $H^*_G(X;\mathbf{k}) = \mathbf{k}[x,y]/(x^2 - y^2)$, where x and y are variables of degree 2. Compare this with the ordinary cohomology, which is a truncated polynomial ring: $H^*(X;\mathbf{k}) = \mathbf{k}[y]/(y^2)$.

1.16. **Example.** Let $G = \mathbf{Z}/2\mathbf{Z}$. Example 1.12 gives $B_G = \mathbf{P}^\infty(\mathbf{R})$, so if the ring $\mathbf{k}$ has characteristic 2, then

$$H^*_G(\text{point};\mathbf{k}) = H^*(B_G;\mathbf{k}) = \mathbf{k}[u],$$

a polynomial algebra in one variable u of degree 1.

Here is an analogue of Theorem 1.10 in equivariant cohomology. The proof, which we omit, is very similar in spirit. The first part is due to Kirwan [24] and the second part is due to Biss, Guillemin and Holm [4].

1.17. **Theorem.** *Let T be a torus and let X be a compact Hamiltonian T-manifold. Let $\mathcal{F}$ be the set of connected components of X^T.*

(i) *Let $\mathbf{k}$ be a field. There exist a family of positive even integers $(d(F))_{F\in\mathcal{F}}$ and an isomorphism of $\mathbf{k}$-vector spaces*
$$H_T^*(X;\mathbf{k}) \cong \bigoplus_{F\in\mathcal{F}} H_T^{*-d(F)}(F;\mathbf{k}).$$

(ii) *Let $\sigma_T(t) = t^{-1}$ and suppose X has a real structure σ_X compatible with σ_T. Let the integers $d(F)$ be as in part* (i) *and let $\mathbf{k}$ be a field of characteristic* 2*. There is an isomorphism of $\mathbf{k}$-vector spaces*
$$H_{T^\sigma}^*(X^\sigma;\mathbf{k}) \cong \bigoplus_{F\in\mathcal{F}} H_T^{*-\frac{1}{2}d(F)}(F^\sigma;\mathbf{k}).$$

We will take up the theme of equivariant cohomology again in Lecture 3, but first we turn to a quite different, but equally striking property of the moment map.

2. The moment polytope of a Hamiltonian action

A polytope associated with a T-algebra. Let T be a torus and let $A = \bigoplus_{r=0}^\infty A_r$ be a commutative graded algebra over the field of complex numbers $\mathbf{C}$. We make the following assumptions on A.

(i) A is finitely generated.

(ii) A has no zero divisors.

(iii) The torus T acts on A by graded algebra endomorphisms. That is to say, the action is linear,
$$t\cdot(c_1a_1 + c_2a_2) = c_1(t\cdot a_1) + c_2(t\cdot a_2)$$
for all $t \in T$, c_1, $c_2 \in \mathbf{C}$ and a_1, $a_2 \in A$; multiplicative,
$$t\cdot(a_1a_2) = (t\cdot a_1)(t\cdot a_2)$$
for all $t \in T$ and a_1, $a_2 \in A$; and preserves the grading,
$$t\cdot a \in A_r$$
for all $t \in T$ and $a \in A_r$.

(iv) For all r, the action of T on A_r is continuous.

(v) $A_0 = \mathbf{C}$, the trivial one-dimensional representation of T.

Notice that, by assumptions (i) and (iii), each of the summands A_r is a finite-dimensional T-module. Therefore assumption (iv) makes sense; it simply means, by definition, that the action of T on A_r is given by a continuous homomorphism from T to the matrix group $\mathbf{GL}(A_r)$. These assumptions enable us to refine the grading of A into a bigrading by weight and degree. Let $\mathcal{X}(T) = \mathrm{Hom}(T, \mathbf{U}(1))$ be the character group of T. Define $A_{\lambda,r}$ to be the collection of all $a \in A_r$ such that $t\cdot a = \lambda(t)a$ for all $t \in T$. Then
$$A = \bigoplus_{(\lambda,r)\in\mathcal{X}(T)\times\mathbf{N}} A_{\lambda,r}.$$
Assumption (iii) implies

(vi) If $a \in A_{\lambda,r}$ and $b \in A_{\mu,s}$, then $ab \in A_{\lambda+\mu,r+s}$.

Let $\Sigma(A)$ be the set of all $(\lambda, r) \in \mathcal{X}(T) \times \mathbf{N}$ for which the direct summand $A_{\lambda,r}$ is nonzero.

2.1. **Lemma.** *$\Sigma(A)$ is a finitely generated submonoid of $\mathcal{X}(T) \times \mathbf{N}$.*

Proof. Assumption (ii) and assertion (vi) imply that $\Sigma(A)$ is closed under addition. Assumption (v) implies that $(0,0) \in \Sigma(A)$. Therefore $\Sigma(A)$ is a submonoid. By assumption (i), the algebra A is finitely generated. Let $a_1 \in A_{\lambda_1,r_1}$, $a_2 \in A_{\lambda_2,r_2}, \dots, a_k \in A_{\lambda_k,r_k}$ be a set of homogeneous generators. Then it follows from assumption (iii) that every $(\lambda, r) \in \Sigma(A)$ can be written in the form $(\lambda, r) = \sum_{l=1}^{k} n_l(\lambda_l, r_l)$ with $n_l \in \mathbf{N}$. Thus $\Sigma(A)$ is finitely generated as a monoid. QED

The monoid $\Sigma(A)$ can be quite complicated. It is often far from being a saturated submonoid of $\mathcal{X}(T) \times \mathbf{N}$. Somewhat easier to understand is its "classical limit",

$$\mathcal{P}(A) = \left\{ \frac{\lambda}{r} \;\middle|\; (\lambda, r) \in \Sigma(A),\, r > 0 \right\},$$

which is a subset of the $\mathbf{Q}$-vector space $\mathcal{X}(T)_{\mathbf{Q}} = \mathcal{X}(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Recall that a *convex polytope* in a vector space V over an ordered field is a subset of V which is the convex hull of a finite subset of V.

2.2. **Lemma.** *$\mathcal{P}(A)$ is a convex polytope in $\mathcal{X}(T)_{\mathbf{Q}}$.*

Proof. Let λ/r, $\mu/s \in \mathcal{P}(A)$. Then it follows from assumption (ii) and assertion (vi) that $(\lambda+\mu)/(r+s) \in \mathcal{P}(A)$. This implies that $\mathcal{P}(A)$ is convex. In fact, if $(\lambda_1, r_1), (\lambda_2, r_2), \dots, (\lambda_k, r_k)$ are generators of $\Sigma(A)$, then every element of $\mathcal{P}(A)$ is a convex combination of $\lambda_1/r_1, \lambda_2/r_2, \dots, \lambda_k/r_k$. Thus $\mathcal{P}(A)$ is a convex polytope. QED

We call $\Sigma(A)$ the *weight monoid* and $\mathcal{P}(A)$ the *weight polytope* of A.

First application: projective varieties. Let $\phi\colon T \to \mathbf{U}(n)$ be a Lie group homomorphism. This homomorphism defines an action of T on the vector space $V = \mathbf{C}^n$ by unitary transformations and an action on the projective space $\mathbf{P}^{n-1}(\mathbf{C})$ by projective unitary transformations. Let X be an irreducible algebraic subvariety of $\mathbf{P}^{n-1}(\mathbf{C})$ preserved by the action of T. (We do not need to assume that X is nonsingular.) We put

$$\begin{aligned} S &= \mathbf{C}[x_1, x_2, \dots, x_n], && \text{the algebra of polynomial functions on } \mathbf{C}^n, \\ I(X) &= \{\, f \in S \mid f \mid X = 0 \,\}, && \text{the homogeneous ideal of } X, \\ A(X) &= S/I(X), && \text{the homogeneous coordinate ring of } X. \end{aligned}$$

Then $A(X)$ is a graded algebra and T acts on $A(X)$ by algebra endomorphisms. Because X is a variety, $A(X)$ is finitely generated. Because X is irreducible, $A(X)$ has no zero divisors. Let $\mathcal{P}(X) = \mathcal{P}(A(X))$ be the weight polytope of $A(X)$.

The interpretation of this polytope is as follows. Recall from Example 1.8 that the projective space $\mathbf{P}^{n-1}(\mathbf{C})$ is a Hamiltonian T-manifold equipped with the Fubini-Study symplectic form and the moment map $\Phi\colon \mathbf{P}^{n-1}(\mathbf{C}) \to \mathfrak{t}^*$ given by

$$\langle \Phi([z]), \xi \rangle = -\frac{1}{2\pi i} \frac{z^* \xi z}{z^* z}$$

for $z \in V \setminus \{0\}$ and $\xi \in \mathfrak{t}$. The Lie algebra of $\mathbf{U}(1)$ is the imaginary axis $i\mathbf{R}$, so the differential λ_* of a character $\lambda \in \mathcal{X}(T)$ is a linear map $\mathfrak{t} \to i\mathbf{R}$, that is to say, an element of $i\mathfrak{t}^*$. The map $\mathcal{X}(T) \to \mathfrak{t}^*$ which sends a character λ to the real-valued functional $(2\pi i)^{-1}\lambda_*$ is an embedding of $\mathcal{X}(T)$ onto a lattice in $\mathfrak{t}^*$ and, as is common practice, we will identify $\mathcal{X}(T)$ with its image in $\mathfrak{t}^*$ under this embedding. Similarly, we will regard the $\mathbf{Q}$-vector space $\mathcal{X}(T)_{\mathbf{Q}}$ as a (dense) subset of $\mathfrak{t}^*$, the set of *rational points* of $\mathfrak{t}^*$. With these identifications we have inclusions

$$\mathcal{P}(X) \subseteq \mathcal{X}(T)_{\mathbf{Q}} \subseteq \mathfrak{t}^*.$$

The following theorem says that the weight polytope $\mathcal{P}(X)$ determines the moment map image $\Phi(X)$ and, conversely, $\Phi(X)$ determines $\mathcal{P}(X)$. It is due in various forms to Atiyah [3], Guillemin and Sternberg [18], Mumford [28] and Brion [7]. See also the monograph [17] and the survey paper [30].

2.3. **Theorem.** *Let X be an irreducible subvariety of $\mathbf{P}^{n-1}(\mathbf{C})$ invariant under the T-action.*

(i) $\Phi(X) = \overline{\mathcal{P}(X)}$.
(ii) $\mathcal{P}(X) = \Phi(X) \cap \mathcal{X}(T)_{\mathbf{Q}}$.
(iii) $\Phi(X)$ *is a convex polytope with rational vertices in the vector space* $\mathfrak{t}^*$.

The polytope $\Phi(X)$ is called the *moment polytope* of X.

Second application: real projective varieties. Let $\sigma_T\colon T \to T$ be an involution of T, for instance $\sigma_T(t) = t^{-1}$, and suppose that σ_T is compatible with the standard involution on $V = \mathbf{C}^n$. Then, as we saw in Example 1.8, the projective space $\mathbf{P}^{n-1}(\mathbf{C})$ is a real Hamiltonian T-manifold. Suppose that the irreducible subvariety X is defined over $\mathbf{R}$. What can we say about the moment map image of the real locus X^σ? Since $\Phi(\sigma(x)) = -\sigma(\Phi(x))$, we have $\Phi(x) = -\sigma(\Phi(x))$ for $x \in X^\sigma$. In other words,

$$\Phi(X^\sigma) \subseteq \mathfrak{p}^* = \{\, \lambda \in \mathfrak{g}^* \mid \sigma(\lambda) = -\lambda \,\},$$

and hence $\Phi(X^\sigma) \subseteq \Phi(X) \cap \mathfrak{p}^*$. The following theorem, which is due to Duistermaat [9], states that the reverse inclusion also holds.

2.4. **Theorem.** *If X^σ contains a nonsingular point of X, then $\Phi(X^\sigma) = \Phi(X) \cap \mathfrak{p}^*$. In particular, $\Phi(X^\sigma)$ is a convex polytope with rational vertices.*

See [1, 13, 29] for various generalizations.

2.5. **Example.** Let G be a compact connected Lie group with maximal torus T and let $\lambda \in \mathfrak{t}^*$ be an integral point, i.e. $\lambda \in \mathcal{X}(T)$. Then the coadjoint orbit $X = \mathrm{Ad}^*(G)(\lambda)$ of λ is an integral symplectic manifold. We will view X as a Hamiltonian T-manifold and compute its moment polytope. It is known that the Kostant-Kirillov-Souriau symplectic form is Kählerian (for a suitable choice of complex structure on X), so it follows from Kodaira's theorem that X is a complex projective variety. (The facts concerning coadjoint orbits used here can be found in [10, Chapter 4] and [17, Chapter 4].) It follows from the Borel-Weil theorem that

$$A(X) = \bigoplus_{r=0}^{\infty} V_{r\lambda}.$$

Here V_μ denotes the irreducible G-module of highest weight μ, where μ is any dominant weight. As a T-module, the representation V_μ decomposes into weight spaces. A basic result of representation theory says that the weights occurring in

V_μ are exactly those $\nu \in \mathcal{X}(T)$ such that $\nu \in \operatorname{conv}_{\mathbf{Q}}(W \cdot \mu)$ and $\nu - \mu$ is in the root lattice. (Here $\operatorname{conv}_{\mathbf{Q}}(A)$ denotes the convex hull of a set A contained in a $\mathbf{Q}$-vector space, and $W \cdot \mu$ denotes the Weyl group orbit of $\mu \in \mathfrak{t}^*$.) It follows from this that $\mathcal{P}(X) = \operatorname{conv}_{\mathbf{Q}}(W \cdot \lambda)$, so by Theorem 2.3 we conclude that

$$\Phi(X) = \operatorname{conv}_{\mathbf{R}}(W \cdot \lambda).$$

Now suppose G is equipped with an involution σ_G which preserves the maximal torus T. Also assume that the element λ is contained in $\mathfrak{p}^*$. Then the orbit X is symmetric in the sense of Example 1.9 and X^σ is nonempty, so Theorem 2.4 gives

$$\Phi(X^\sigma) = \operatorname{conv}_{\mathbf{R}}(W \cdot \lambda) \cap \mathfrak{p}^*.$$

These results were first obtained by Kostant [25] by entirely different methods.

3. Conjugation spaces

The Betti numbers of a space and its real locus. In Lecture 1, particularly Theorems 1.10 and 1.17, we noticed a close analogy between the cohomology of a real Hamiltonian G-manifold X and that of its real locus X^σ. This analogy arises from the fact that every component of the moment map is a Morse-Bott function on X and that its restriction is a Morse-Bott function on X^σ. However, in certain cases the analogy goes even further. To avoid problems related to orientability, in this lecture we shall only consider cohomology with coefficients in the field of two elements $\mathbf{F}_2$.

3.1. **Example.** The real projective space $\mathbf{P}^n(\mathbf{R})$ is the real locus of the complex projective space $\mathbf{P}^n(\mathbf{C})$ for a suitable antisymplectic involution on $\mathbf{P}^n(\mathbf{C})$. (See Example 1.8.) The Betti numbers (over $\mathbf{F}_2$) of $\mathbf{P}^n(\mathbf{C})$ are

$$\dim_{\mathbf{F}_2} H^k(\mathbf{P}^n(\mathbf{C}); \mathbf{F}_2) = \begin{cases} 1 & \text{if } k \text{ even, } 0 \le k \le 2n, \\ 0 & \text{otherwise,} \end{cases}$$

whereas the Betti numbers of $\mathbf{P}^n(\mathbf{R})$ are

$$\dim_{\mathbf{F}_2} H^k(\mathbf{P}^n(\mathbf{R}); \mathbf{F}_2) = \begin{cases} 1 & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\dim_{\mathbf{F}_2} H^{2k}(\mathbf{P}^n(\mathbf{C}); \mathbf{F}_2) = \dim_{\mathbf{F}_2} H^k(\mathbf{P}^n(\mathbf{R}); \mathbf{F}_2)$ for all k.

Borel and Haefliger [5] observed that the equality

$$\dim_{\mathbf{F}_2} H^{2k}(X; \mathbf{F}_2) = \dim_{\mathbf{F}_2} H^k(X^\sigma; \mathbf{F}_2)$$

between Betti numbers holds more generally for a certain class of complex projective varieties defined over $\mathbf{R}$. In this lecture I will survey some recent work of Hausmann, Holm and Puppe [22], Franz and Puppe [15] and van Hamel [20] related to this equality. But first let me give an example to show that it is not always true.

3.2. **Example.** Let $V = \mathbf{H}^n$ be the space of column vectors with n quaternionic components, viewed as a right vector space over the division algebra of the quaternions $\mathbf{H}$. The map $J\colon V \to V$ defined by left multiplication by j,

$$J(q_1, q_2, \ldots, q_n)^t = (jq_1, jq_2, \ldots, jq_n)^t,$$

is an $\mathbf{H}$-linear map. The map $\mathbf{C}^{2n} \to V$ defined by

$$(z_1, w_1, z_2, w_2, \dots, z_n, w_n)^t \longmapsto (z_1 + w_1 j, z_2 + w_2 j, \dots, z_n + w_n j)^t$$

(where we regard a complex number $a + bi$ as a quaternion $a + bi + 0j + 0k$) is a complex linear isomorphism, which we shall use to identify $\mathbf{C}^{2n}$ with V. Let $X = \mathrm{Gr}(2, 2n, \mathbf{C})$ be the Grassmannian of complex 2-planes in V. Define an involution σ of X by $\sigma(W) = J(W)$. There exists a symplectic structure ω on X such that σ is antisymplectic. Thus X is a real symplectic manifold. A point $W \in X$ (i.e. a complex 2-plane in V) is fixed under σ if and only if $J(W) = W$, which is the case if and only if W is a one-dimensional quaternionic subspace of V. Thus the real locus of X is $X^\sigma = \mathbf{P}^{n-1}(\mathbf{H})$, the $n-1$-dimensional quaternionic projective space. The cohomology ring $H^*(X; \mathbf{F}_2)$ is generated by classes of degree 2, but the cohomology ring $H^*(X; \mathbf{F}_2)$ is generated by a class of degree 4. Thus it is not true that $\dim_{\mathbf{F}_2} H^{2k}(X; \mathbf{F}_2) = \dim_{\mathbf{F}_2} H^k(X^\sigma; \mathbf{F}_2)$ for all k.

For the remainder of this lecture, all cohomology groups will be understood to have coefficients in $\mathbf{F}_2$ and we will drop the coefficient group from the notation.

Review: fundamental classes. We need to review some useful facts from algebraic topology. Let A be a topological space, let B be a closed subspace and let $\iota\colon B \to A$ be the inclusion map. Then we have the long exact sequence of the pair $(A, A \setminus B)$,

$$\cdots \longrightarrow H^k(A, A \setminus B) \longrightarrow H^k(A) \longrightarrow H^k(A \setminus B) \longrightarrow H^{k+1}(A, A \setminus B) \longrightarrow \cdots.$$

Let us assume that B has a *tubular neighbourhood* in A, which means a pair (N, ι_N) consisting of a real vector bundle $\pi\colon N \to B$ and a homeomorphism ι_N from N onto an open neighbourhood of B in A such that $\iota = \iota_N \circ \zeta$, where $\zeta\colon B \to N$ is the zero section of N. (For instance, this is the case if A is a smooth manifold and B a closed submanifold by the tubular neighbourhood theorem of differential topology.) We call N the *normal bundle* of B in A. The *punctured* normal bundle is the space $N^\times = N \setminus \zeta(B)$. The map

$$\iota_N^*\colon H^k(A, A \setminus B) \longrightarrow H^k(N, N^\times)$$

is an isomorphism by the excision property. Let $\Theta_N \in H^d(N, N^\times)$ be the Thom class of N, i.e. the unique class which for every point $b \in B$ restricts to the generator of

$$H^d(N_b, N_b^\times) \cong H^d(\mathbf{R}^d, \mathbf{R}^d \setminus \{0\}) \cong \mathbf{F}_2.$$

Here $N_b = \pi^{-1}(b)$ is the fibre of N over b and $d = \dim(N_b)$ is the rank of N. The Thom isomorphism theorem says that the map

$$\mathrm{Th}_N\colon H^{k-d}(B) \longrightarrow H^k(N, N^\times)$$

defined by $c \mapsto \pi^*(c) \cdot \Theta_N$ is an isomorphism for all k. (See for instance [21, Section 4.D].) In fact, the Thom class (over $\mathbf{F}_2$) can also be characterized as the unique class Θ_N for which Th_N is an isomorphism. The *Gysin homomorphism* $\iota_*\colon H^{k-d}(B) \to H^k(A)$ associated with ι is by definion the composition of the maps

$$H^{k-d}(B) \xrightarrow{\mathrm{Th}_N} H^k(N, N^\times) \xrightarrow{(\iota_N^*)^{-1}} H^k(A, A \setminus B) \longrightarrow H^k(A).$$

Inserting this into the long exact sequence we obtain the long exact *Gysin sequence* of the pair (A, B),

$$\cdots \longrightarrow H^{k-d}(B) \xrightarrow{\iota_*} H^k(A) \longrightarrow H^k(A \setminus B) \longrightarrow H^{k-d+1}(B) \longrightarrow \cdots .$$

The class $[B] = \iota_*(1) \in H^d(A)$ is called the *fundamental class*, or also the *orientation class*, of B in A.

Frequently we identify $H^k(A, A \setminus B)$ with $H^k(N, N^\times)$ through the isomorphism ι_N^* and consider the Thom class Θ_N as an element of $H^k(A, A \setminus B)$. With this identification, the fundamental class $[B]$ is the image of the Thom class under the natural map from $H^k(A, A \setminus B)$ to $H^k(A)$.

More about projective space. Let us now have a closer look at Example 3.1. Let $X = \mathbf{P}^n(\mathbf{C})$, let σ be the involution defined by complex conjugation and let $X^\sigma = \mathbf{P}^n(\mathbf{R})$ be the real locus of X. Let $\Gamma = \mathrm{Gal}(\mathbf{C}/\mathbf{R})$ be the Galois group of $\mathbf{C}$ over $\mathbf{R}$. The involution σ on X can be thought of as a Γ-action and the real locus is then the fixed point set of the action. Recall from Example 1.12 the space E_Γ, which is the unit sphere in $l^2(\mathbf{R})$, and the classifying space $B_\Gamma = \mathbf{P}^\infty(\mathbf{R})$. The homotopy quotient $X_\Gamma = (E_\Gamma \times X)/\Gamma$ is a fibre bundle over B_Γ, whose fibre over a fixed basepoint is X. Because Γ acts trivially on X^σ, its homotopy quotient is $(X^\sigma)_\Gamma = B_\Gamma \times X^\sigma$. In this section we will contemplate the commutative diagram

$$\begin{tikzcd} X_\Gamma & B_\Gamma \times X^\sigma \arrow[l, hook', "i_\Gamma"'] \\ X \arrow[u, hook, "j"] & X^\sigma, \arrow[l, hook', "i"] \arrow[u, hook, "j_\sigma"'] \end{tikzcd}$$

where i is the inclusion of the real locus, i_Γ is its equivariant counterpart, and j and j_σ are the inclusions of the fibre. The associated diagram in cohomology is

$$\begin{tikzcd} H_\Gamma^*(X) \arrow[r, "i_\Gamma^*"] \arrow[d, shift left, "j^*"] & H^*(X^\sigma) \otimes H^*(B_\Gamma) \arrow[d] \\ H^*(X) \arrow[u, dotted, shift left, "s"] \arrow[r, "i^*"] \arrow[r, dotted, shift right=2, "\kappa"'] & H^*(X^\sigma), \end{tikzcd}$$

where the maps κ and s are to be defined below. For $d = 1, 2, \ldots, n$ let Z_d be the projective linear subspace $\mathbf{P}^{n-d}(\mathbf{C})$ of X. These subspaces have the following properties:

(i) $\sigma(Z_d) = Z_d$ and $Z_d^\sigma = \mathbf{P}^{n-d}(\mathbf{R})$;
(ii) the codimension of Z_d in X is $2d$ and the codimension of Z_d^σ in X^σ is d;
(iii) the collection of fundamental classes

$$\{\, [Z_d] \in H^{2d}(X) \mid d = 1, 2, \ldots, n \,\}$$

is a basis of $H^*(X)$ and the collection of fundamental classes

$$\{\, [Z_d^\sigma] \in H^d(X^\sigma) \mid d = 1, 2, \ldots, n \,\}$$

is a basis of $H^*(X^\sigma)$.

This enables us to define a linear map

$$\kappa\colon H^{2*}(X) \longrightarrow H^*(X^\sigma)$$

by $\kappa([Z_d]) = [Z_d^\sigma]$. This map does not preserve the degrees, but divides them in half. To define the map s, consider the homotopy quotient $(Z_d)_\Gamma = (E_\Gamma \times Z_d)/\Gamma$, which is a submanifold of codimension $2d < \infty$ in X_Γ. It has a tubular neighbourhood in X_Γ and therefore a well-defined fundamental class

$$[Z_d]_\Gamma \in H^{2d}(X_\Gamma) = H^{2d}_\Gamma(X).$$

The intersection of $(Z_d)_\Gamma$ with the fibre X is equal to Z_d, so $j^*([Z_d]_\Gamma) = [Z_d]$. In other words, the linear map

$$s\colon H^*(X) \longrightarrow H^*_\Gamma(X)$$

defined on basis elements by $s([Z_d]) = [Z_d]_\Gamma$ is a *section* (right inverse) of the restriction map to the fibre: $j^* \circ s = \mathrm{id}_{H^*(X)}$. We call $[Z_d]_\Gamma$ a Γ-*equivariant extension* of the class $[Z_d]$. What can we say about the restriction of $[Z_d]_\Gamma$ to the real locus, that is the image of $[Z_d]_\Gamma$ under the restriction map

$$i^*_\Gamma\colon H^*_\Gamma(X) \longrightarrow H^*_\Gamma(X^\sigma)$$

induced by the inclusion $X^\sigma \hookrightarrow X$? Since $(X^\sigma)_\Gamma = B_\Gamma \times X^\sigma$, it follows from the Künneth theorem that

$$H^*_\Gamma(X^\sigma) \cong H^*(X^\sigma) \otimes \mathbf{F}_2[u] \cong H^*(X^\sigma)[u],$$

a polynomial ring over $H^*(X^\sigma)$ in a variable u of degree 1. Therefore we can express the class $i^*_\Gamma([Z_d]_\Gamma)$ of degree $2d$ as a polynomial in u,

$$i^*_\Gamma([Z_d]_\Gamma) = \sum_{k=0}^{2d} c_k u^k,$$

with coefficients $c_k \in H^{2d-k}(X^\sigma)$. It was observed by van Hamel [20] that all the terms above degree d of this polynomial vanish, and he also found an expression for the leading coefficient c_d.

3.3. **Theorem.** $c_k = 0$ *for* $k > d$ *and* $c_d = [Z_d^\sigma] = \kappa([Z_d])$.

Proof. By definition, the fundamental class $[Z_d]_\Gamma \in H^{2d}_\Gamma(X)$ is the image of the Thom class $\Theta_N \in H^{2d}_\Gamma(X, X \setminus Z_d)$ of the normal bundle N of $(Z_d)_\Gamma$ in X_Γ. For all l the following diagram commutes:

$$\begin{tikzcd}
H^l_\Gamma(X, X \setminus Z_d) \arrow[r] \arrow[d, "i^*_\Gamma"'] & H^l_\Gamma(X) \arrow[d, "i^*_\Gamma"] \\
H^l_\Gamma(X^\sigma, X^\sigma \setminus Z_d^\sigma) \arrow[r] & H^l_\Gamma(X^\sigma).
\end{tikzcd}$$

By the Thom isomorphism theorem, $H^l_\Gamma(X^\sigma, X^\sigma \setminus Z_d^\sigma) \cong H^{l-d}_\Gamma(Z_d^\sigma)$, which is 0 for $l < d$. Thus the coefficients of the polynomial $i^*_\Gamma([Z_d]_\Gamma) = \sum_{k=0}^{2d} c_k u^k$ vanish for $2d - k < d$.

To compute c_d we take another look at the Thom isomorphism

$$\mathrm{Th}_N\colon H^{*-d}_\Gamma(Z_d) \xrightarrow{\cong} H^*_\Gamma(X, X \setminus Z_d),$$

which is an isomorphism of modules over the ring $H^*(B_\Gamma) \cong \mathbf{F}_2[u]$. Just like the fundamental class $[Z_d]_\Gamma$, the restriction of the Thom class can be written as a polynomial

$$i_\Gamma^*(\Theta_N) = \sum_{k=0}^{2d} \Theta_k u^k$$

with coefficients $\Theta_k \in H^{2d-k}(X^\sigma, X^\sigma \setminus Z_d^\sigma)$. The coefficients Θ_k vanish in degrees $k > d$ for the same reason as for $[Z_d]$. The famous localization theorem of equivariant cohomology says that for any Γ-space Y the restriction map $H_\Gamma^*(Y) \to H_\Gamma^*(Y^\Gamma)$ becomes an *isomorphism* after inverting the variable u. Thus the Thom isomorphism gives rise to an isomorphism

$$\mathrm{Th}_N \colon H_\Gamma^{*-d}(Z_d^\sigma)_u \xrightarrow{\cong} H_\Gamma^*(X^\sigma, X^\sigma \setminus Z_d^\sigma)_u,$$

where $V_u = \mathbf{F}_2[u, u^{-1}] \otimes_{\mathbf{F}_2} V$ is the localization of an $\mathbf{F}_2$-vector space V at u. It follows from this that multiplication by the top degree part $\Theta_d \in H^d(X^\sigma, X^\sigma \setminus Z_d^\sigma)$ of $i_\Gamma^*(\Theta_N)$ is an isomorphism

$$H^{*-d}(Z_d^\sigma) \xrightarrow{\cong} H^*(X^\sigma, X^\sigma \setminus Z_d^\sigma).$$

This implies Θ_d is the Thom class of the normal bundle of Z_d^σ in X^σ. Thus c_d, which is the image in $H^d(X^\sigma)$ of Θ_d, is the fundamental class of Z_d^σ. QED

Because the classes $[Z_d]$ form a basis of $H^*(X)$, we obtain from this fact the so-called *conjugation equation*.

3.4. **Theorem.** *For every $a \in H^{2d}(X)$ there exist classes $c_k \in H^{2d-k}(X^\sigma)$ for $k = 0, 1, \dots, d-1$ such that $i_\Gamma^* s(a) = \kappa(a)u^d + c_{d-1}u^{d-1} + \cdots + c_1 u + c_0$.*

Frames. This example leads to the following two definitions. The first was stated in a more general form in [20]. Let X be an arbitrary symplectic manifold with a real structure σ. A *geometric frame* is a family of closed symplectic submanifolds $(Z_\alpha)_{\alpha \in A}$ such that

(i) $\sigma(Z_\alpha) = Z_\alpha$;
(ii) the collection $\{\, [Z_\alpha] \mid \alpha \in A \,\}$ is a basis of $H^*(X)$ and the collection $\{\, [Z_\alpha^\sigma] \mid \alpha \in A \,\}$ is a basis of $H^*(X^\sigma)$.

Note that the definition implies that $H^k(X) = 0$ when k is odd.

The next definition was given in [22] (and preceded the definition of a geometric frame). Assume that $H^k(X) = 0$ when k is odd. A *cohomological frame* is a pair (s, κ), where

$$s \colon H^*(X) \longrightarrow H_\Gamma^*(X)$$

is a section of the restriction map $j^* \colon H_\Gamma^*(X) \to H^*(X)$ and

$$\kappa \colon H^{2*}(X) \longrightarrow H^*(X^\sigma)$$

is an additive isomorphism which divides the degrees in half. These two maps are required to satisfy the *conjugation equation*: for each $a \in H^{2d}(X)$ there exist $c_k \in H^{2d-k}(X^\sigma)$ for $k = 0, 1, \dots, d-1$ such that

$$i_\Gamma^* s(a) = \kappa(a)u^d + c_{d-1}u^{d-1} + \cdots + c_1 u + c_0.$$

If a cohomological frame exists, we call the involution σ a *conjugation* and the manifold X a *conjugation space*.

Given a geometric frame $(Z_\alpha)_{\alpha\in A}$ one defines $\kappa([Z_\alpha]) = [Z_\alpha^\sigma]$ and $s([Z_\alpha]) = [Z_\alpha]_\Gamma$ and proves as in Theorem 3.4 that these two maps define a cohomological frame. Thus:

3.5. **Theorem** ([20]). *A geometric frame gives rise to a naturally defined cohomological frame.*

We mention a few examples of conjugation spaces. Many more are given in [22].

3.6. **Example.** Let T be a torus with involution $\sigma_T(t) = t^{-1}$ and let X be a real Hamiltonian T-manifold. Suppose the fixed point set X^T equipped with the involution $\sigma \mid X^T$ is a conjugation space. Then X is a conjugation space.

3.7. **Example.** Let G be a compact connected Lie group. A *Chevalley involution* of G is an involution satisfying $\sigma(g) = g^{-1}$ for g in some maximal torus T of G and $\sigma(\alpha) = -\alpha$ for all roots α of G. It is known that Chevalley involutions exist for all G and are unique up to conjugation. For instance, the Chevalley involution of $\mathbf{U}(n)$ is given by $\sigma(g) = \bar{g}$ and the Chevalley involution of $\mathbf{U}(n, \mathbf{H})$ is given by $\sigma(g) = -J\bar{g}J$, where $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. With respect to the Chevalley involution, every coadjoint orbit X is symmetric, and it is proved in [22] that X is a conjugation space.

Conjugation spaces have many other surprising properties. We finish with a few sample results.

3.8. **Theorem** ([22]). *Let X be a conjugation space.*

(i) *The real locus X^σ is nonempty. If X is connected, then so is X^σ.*
(ii) *The cohomological frame (s, κ) of X is unique. Both s and κ are ring homomorphisms.*

The next result says that not only the leading coefficient, but also the lower order coefficients in the conjugation equation can be expressed in terms of the class a. (See [21, Section 4.L] for a discussion of the Steenrod squaring operations.)

3.9. **Theorem** ([15]). *Let X be a conjugation space. For each $a \in H^{2d}(X)$ the coefficients $c_k \in H^k(X^\sigma)$ in the conjugation equation are uniquely determined by a, namely $c_k = \mathrm{Sq}^{d-k}(\kappa(a))$, the $d-k$th Steenrod square of $\kappa(a)$.*

3.10. **Corollary** ([15]). $\kappa(a)^2 = i^*(a)$.

Proof. For a class c of degree k we have $\mathrm{Sq}^k(c) = c^2$. Therefore, since $\kappa(a)$ has degree d, $c_0 = \mathrm{Sq}^d(\kappa(a)) = \kappa(a)^2$. Moreover, c_0 is the constant term in the polynomial, i.e. the restriction of the equivariant class $i_\Gamma^* s(a)$ to the fibre X^σ, so $c_0 = j_\sigma^* i_\Gamma^* s(a) = i^* j^* s(a) = i^*(a)$. QED

Thus the homomorphism κ is a "square root" of the restriction map to the real locus $i^*\colon H^*(X) \to H^*(X^\sigma)$.

References

1. A. Alekseev, E. Meinrenken, and C. Woodward, *Linearization of Poisson actions and singular values of matrix products*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 6, 1691–1717, arXiv:math/0012112.
2. C. Allday and V. Puppe, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics, vol. 32, Cambridge University Press, Cambridge, 1993.

3. M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15.
4. D. Biss, V. Guillemin, and T. Holm, *The mod 2 cohomology of fixed point sets of anti-symplectic involutions*, Adv. Math. **185** (2004), no. 2, 370–399, arXiv:math/0107151.
5. A. Borel and A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89** (1961), 461–513, available at `http://www.numdam.org/`.
6. R. Bott, *Nondegenerate critical manifolds*, Ann. of Math. (2) **60** (1954), 248–261, available at `http://www.jstor.org/`.
7. M. Brion, *Sur l'image de l'application moment*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986) (M.-P. Malliavin, ed.), Lecture Notes in Mathematics, vol. 1296, Springer-Verlag, Berlin, 1987, pp. 177–192.
8. ———, *Equivariant cohomology and equivariant intersection theory*, Representation theories and algebraic geometry (Montréal, PQ, 1997) (A. Broer and A. Daigneault, eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, notes by A. Rittatore, pp. 1–37, arXiv:math/9802063.
9. J. J. Duistermaat, *Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution*, Trans. Amer. Math. Soc. **275** (1983), no. 1, 417–429, available at `http://www.jstor.org/`.
10. J. J. Duistermaat and J. Kolk, *Lie groups*, Springer-Verlag, Berlin, 2000.
11. E. Floyd, *On periodic maps and the Euler characteristics of associated spaces*, Trans. Amer. Math. Soc. **72** (1952), 138–147, available at `http://www.jstor.org/`.
12. P. Foth, *A note on Lagrangian loci of quotients*, Canad. Math. Bull. **48** (2005), no. 4, 561–575, arXiv:math/0303322.
13. P. Foth and M. Otto, *A symplectic approach to van den Ban's convexity theorem*, Doc. Math. **11** (2006), 407–424, arXiv:math/0505063.
14. T. Frankel, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. (2) **70** (1959), 1–8, available at `http://www.jstor.org/`.
15. M. Franz and V. Puppe, *Steenrod squares on conjugation spaces*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 3, 187–190, arXiv:math/0510157.
16. R. Goldin and T. Holm, *Real loci of symplectic reductions*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4623–4642, arXiv:math/0209111.
17. V. Guillemin and R. Sjamaar, *Convexity properties of Hamiltonian group actions*, CRM Monograph Series, vol. 26, American Mathematical Society, Providence, RI, 2005, available at `http://www.math.cornell.edu/~sjamaar`.
18. V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513, available at `http://www.gdz-cms.de/`.
19. ———, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer-Verlag, Berlin, 1999.
20. J. van Hamel, *Geometric cohomology frames on Hausmann-Holm-Puppe conjugation spaces*, Proc. Amer. Math. Soc. **135** (2007), no. 5, 1557–1564, arXiv:math/0509498.
21. A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002, available at `http://www.math.cornell.edu/~hatcher`.
22. J.-C. Hausmann, T. Holm, and V. Puppe, *Conjugation spaces*, Algebr. Geom. Topol. **5** (2005), 923–964, arXiv:math/0412057.
23. W.-Y. Hsiang, *Cohomology theory of topological transformation groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 85, Springer-Verlag, Berlin, 1975.
24. F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, vol. 31, Princeton University Press, Princeton, N.J., 1984.
25. B. Kostant, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. sci. École norm. sup. (4) **6** (1973), 413–455, available at `http://www.numdam.org/`.
26. B. Krötz and M. Otto, *Lagrangian submanifolds and moment convexity*, Trans. Amer. Math. Soc. **358** (2006), no. 2, 799–818, arXiv:math/0311179.
27. M. Libine, *Lecture notes on equivariant cohomology*, preprint, Yale University, 2007, arXiv:0709.3615.
28. L. Ness, *A stratification of the null cone via the moment map*, Amer. J. Math. **106** (1984), no. 6, 1281–1329, with an appendix by D. Mumford, available at `http://www.jstor.org/`.
29. L. O'Shea and R. Sjamaar, *Moment maps and Riemannian symmetric pairs*, Math. Ann. **317** (2000), no. 3, 415–457, arXiv:math/9902059.
30. R. Sjamaar, *The moment polytope of a Kähler G-manifold*, Mat.: Enseñ. Univ. (N.S.), to appear, available at `http://www.math.cornell.edu/~sjamaar`.

E-mail address: `sjamaar@math.cornell.edu`

Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA