Notes on the Orbit Method and Quantization

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Les Diablerets, March 1997

Contents

Orbit method and quantization

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$G =$ Lie group (infinite-dimensional group, quantum group. . .)

Category of unitary representations of G

Objects: continuous homomorphisms $T\colon G \to$ U($\mathcal{H}$) ($\mathcal{H}$ a Hilbert space)

Morphism ("intertwining operator") from T_1 to T_2: continuous linear $A\colon \mathcal{H}_1 \to \mathcal{H}_2$

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{A} & \mathcal{H}_2 \\ {\scriptstyle T_1(g)}\downarrow & & \downarrow{\scriptstyle T_2(g)} \\ \mathcal{H}_1 & \xrightarrow{A} & \mathcal{H}_2 \end{array}$$

Example. $X = G$-manifold with G-invariant measure μ. Unitary representation on $L^2(X, \mu)$: $T(g)f(x) = f(g^{-1}x)$. Map $F\colon X_1 \to X_2$ induces intertwining map $F^*\colon L^2(X_2, \mu_2) \to L^2(X_1, \mu_1)$ (if μ_2 is absolutely continuous w.r.t. $F_*\mu_1$).

T is *indecomposable* if $T \neq T_1 \oplus T_2$ for nonzero T_1 and T_2. T is *irreducible* if does not have nontrivial invariant subspaces.

For unitary representation irreducible $\Longleftrightarrow$ indecomposable.

"Unirrep" = unitary irreducible representation.

Main problems of representation theory

1. Describe unitary dual:

$$\widehat{G} = \{\text{unirreps of } G\}/\text{equivalence}.$$

2. Decompose any T into unirreps:

$$T(g) = \int\limits_Y T_y(g) d\mu(y).$$

   Special cases: for $H < G$ closed ("little group"),

   (a) for $T \in \widehat{G}$ decompose *restriction* $\mathrm{Res}_H^G T$.

   (b) for $S \in \hat{H}$ decompose *induction* $\mathrm{Ind}_H^G S$.

3. Compute character of $T \in \widehat{G}$.

Ad 2b: let $S\colon H \to \mathsf{U}(\mathcal{H})$. Suppose G/H has G-invariant measure μ. $\mathsf{Ind}_H^G S = L^2$-sections of $G \times^H \mathcal{H}$. Obtained by taking space of functions $f\colon G \to \mathcal{H}$ satisfying $f(gh^{-1}) = S(h)f(g)$, and completing w.r.t. inner product

$$\langle f_1, f_2 \rangle = \int_{G/H} \langle f_1(x), f_2(x) \rangle_{\mathcal{H}} d\mu(x).$$

Ad 3: let $\phi \in C_0^\infty(G)$. Put

$$T(\phi) = \int_G \phi(g) T(g) dg.$$

With luck $T(\phi)\colon \mathcal{H} \to \mathcal{H}$ is of trace class and $\phi \mapsto \mathsf{Tr}\, T(\phi)$ is a distribution on G, the *character* of T.

Solutions proposed by orbit method

1. Let $\mathfrak{g}$ = Lie algebra of G. Coadjoint representation = (non-unitary) representation of G on $\mathfrak{g}^*$.

$$\boxed{\widehat{G} = \mathfrak{g}^*/G, \text{ the space of coadjoint orbits}}$$

2. Let $T_{\mathcal{O}}$ be unirrep corresponding to $\mathcal{O} \in \mathfrak{g}^*/G$. For $H < G$ have projection $\mathrm{pr}\colon \mathfrak{g}^* \to \mathfrak{h}^*$. Then

$$\mathrm{Res}_H^G T_{\mathcal{O}} = \int_{\substack{\mathcal{O}' \in \mathfrak{h}^*/H \\ \mathcal{O}' \subset \mathrm{pr}\,\mathcal{O}}} m(\mathcal{O}, \mathcal{O}') T_{\mathcal{O}'} \quad \text{for } \mathcal{O} \in \mathfrak{g}^*/G,$$

$$\mathrm{Ind}_H^G T_{\mathcal{O}'} = \int_{\substack{\mathcal{O}' \in \mathfrak{g}^*/G \\ \mathrm{pr}\,\mathcal{O} \supset \mathcal{O}'}} m(\mathcal{O}, \mathcal{O}') T_{\mathcal{O}} \quad \text{for } \mathcal{O}' \in \mathfrak{h}^*/H.$$

Same $m(\mathcal{O}, \mathcal{O}')$ (Frobenius reciprocity).

3. For $\mathcal{O} \in \mathfrak{g}^*/G$ let $\chi_{\mathcal{O}}$ = character of $T_{\mathcal{O}}$. Kirillov character formula: for $\xi \in \mathfrak{g}$

$$\sqrt{j(\xi)}\,\chi_{\mathcal{O}}(\exp \xi) = \int_{\mathcal{O}} e^{2\pi i \langle f, \xi \rangle} df,$$

Fourier transform of $\delta_{\mathcal{O}}$. (df = canonical measure on $\mathcal{O}$, $j = \sqrt{j_l j_r}$, where $j_{l,r}$ = derivative of left resp. right Haar measure w.r.t. Lebesgue measure.)

Theorem (Kirillov)**.** *Above is* exactly *right for connected simply connected nilpotent groups (where $j(\xi) = 1$).*

Examples

$G = \mathbb{R}^n$. Then $\mathfrak{g}^*/G = \mathfrak{g}^* = (\mathbb{R}^n)^*$. Unirrep corresponding to $\lambda \in (\mathbb{R}^n)^*$ is

$$T_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle} \qquad (\mathcal{H} = \mathbb{C})$$

(Fourier analysis).

Heisenberg group: $G =$ group of matrices

$$g = \begin{pmatrix} 1 & g_1 & g_3 \\ 0 & 1 & g_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Typical element of Lie algebra $\mathfrak{g}$ is

$$\xi = \begin{pmatrix} 0 & \xi_1 & \xi_3 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix}$$

Basis:

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note $[p, q] = z$, z generates centre of $\mathfrak{g}$.

Complete list of unirreps (Stone-von Neumann)

For $\hbar \neq 0$: $T_\hbar \colon G \to L^2(\mathbb{R})$ is generated by

$$p \longmapsto \hbar \frac{d}{dx}, \quad q \longmapsto ix, \quad z \longmapsto i\hbar,$$

i.e. $T_\hbar(e^{tp})f(x) = f(x + t\hbar)$, $T_\hbar(e^{tq})f(x) = e^{itx}f(x)$, $T_\hbar(e^{tz}) = e^{it\hbar}$. Note $[T_\hbar p, T_\hbar q] = T_\hbar z$ (uncertainty principle).

For α, $\beta \in \mathbb{R}$: $S_{\alpha,\beta} \colon G \to \mathbb{C}$ is generated by

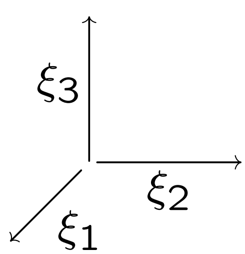
$$p \longmapsto i\alpha, \quad q \longmapsto i\beta, \quad z \longmapsto 0.$$

Description of $\mathfrak{g}/G$

Adjoint action:

$$g \cdot \xi = g \xi g^{-1} = \begin{pmatrix} 0 & \xi_1 & \xi_3 - g_2 \xi_1 + g_1 \xi_2 \\ 0 & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix}$$

Adjoint orbits:



Description of $\mathfrak{g}^*/G$

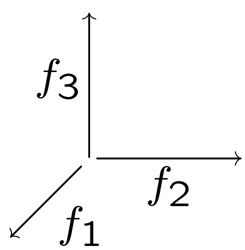
Identify $\mathfrak{g}^*$ with lower triangular matrices. Typical element is

$$f = \begin{pmatrix} 0 & 0 & 0 \\ f_1 & 0 & 0 \\ f_3 & f_2 & 0 \end{pmatrix}$$

Pairing $\langle f, \xi \rangle = \operatorname{Tr} f\xi = f_1\xi_1 + f_2\xi_2 + f_3\xi_3$.
Coadjoint action:

$$g{\cdot}f = \text{ lower triangular part of } gfg^{-1} =$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ f_1 + g_2 f_3 & 0 & 0 \\ f_3 & f_2 - g_1 f_3 & 0 \end{pmatrix}$$

Coadjoint orbits:



Two-dimensional orbits correspond to $T_\hbar$, zero-dimensional orbits to $S_{\alpha,\beta}$

“Explanation” for orbit method

Classical	*Quantum*
Symplectic manifold (M,ω)	Hilbert space $\mathcal{H} = Q(M)$ (or $\mathbb{P}\mathcal{H}$)
Observable (function) f	skew-adjoint operator $Q(f)$ on $\mathcal{H}$
Poisson bracket $\{f,g\}$	commutator $[Q(f),Q(g)]$
Hamiltonian flow of f	1-PS in $\mathrm{U}(\mathcal{H})$

Dirac’s “rules”: $Q(c) = ic$ (c constant), $f \mapsto Q(f)$ is linear, $[Q(f_1),Q(f_2)] = \hbar Q(\{f_1,f_2\})$.

I.e. $f \mapsto \hbar^{-1}Q(f)$ is a Lie algebra homomorphism $C^\infty(M) \to \mathfrak{u}(\mathcal{H})$.

So Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$ gives rise to unitary representation of G on $\mathcal{H}$.

Last “rule”: if G acts transitively, $Q(M)$ is a unirrep.

Hamiltonian actions

(M, ω) symplectic manifold on which G acts. Action is *Hamiltonian* if there exists G-equivariant map $\Phi\colon M \to \mathfrak{g}^*$, called *moment map* or *Hamiltonian*, such that

$$d\langle \Phi, \xi \rangle = \iota(\xi_M)\omega,$$

where ξ_M = vector field on M induced by $\xi \in \mathfrak{g}$.

If G connected, equivariance of Φ is equivalent to: transpose map $\phi\colon \mathfrak{g} \to C^\infty(M)$ defined by $\phi(\xi)(m) = \Phi(m)(\xi)$ is *homomorphism of Lie algebras*.

Triple (M, ω, Φ) is a *Hamiltonian G-manifold*.

Notation: $\Phi^\xi = \phi(\xi) =$ composite map $M \xrightarrow{\Phi} \mathfrak{g}^* \xrightarrow{\xi} \mathbb{R}$ (ξ-component of Φ).

Examples

1. Q = any manifold w. G-action $\rho\colon G \to \mathrm{Diff}(Q)$. $M = T^*Q$ with lifted action

$$\bar{\rho}(g)(q,p) = (\rho(g)q, \rho(g^{-1})^* p),$$

where $q \in Q$, $p \in T_q^*Q$. $\omega = -d\alpha$, where $\alpha_{(q,p)}(v) = p(\pi_* v)$; π = projection $M \to Q$. Moment map:

$$\Phi^{\xi}(q,p) = p(\xi_Q).$$

2. Poisson structure on $\mathfrak{g}^*$: for φ, $\psi \in C^{\infty}(\mathfrak{g}^*)$, $f \in \mathfrak{g}^*$,

$$\{\varphi, \psi\}(f) = \langle f, [d\varphi_f, d\psi_f] \rangle.$$

(Here $d\varphi_f$, $d\psi_f \in \mathfrak{g}^{**} \cong \mathfrak{g}$.)

Leaves: orbits for coadjoint action. For coadjoint orbit $\mathcal{O}$ moment map is *inclusion* $\mathcal{O} \to \mathfrak{g}^*$.

Theorem (Kirillov–Kostant–Souriau). *Let (M, ω, Φ) be homogeneous Hamiltonian G-manifold. Then $\Phi\colon M \to \mathfrak{g}^*$ is local symplectomorphism onto its image. Hence, if G compact, Φ is global symplectomorphism.*

Sketch proof. M homogeneous $\Rightarrow$ image of Φ is single orbit in $\mathfrak{g}^*$, and therefore a symplectic manifold.

Φ equivariant $\Rightarrow$ Φ is Poisson map. Conclusion: Φ preserves symplectic form.

If G compact all coadjoint orbits are simply connected. $\square$

Prequantization

First attempt: $Q(M) = L^2(M, \mu)$, where $\mu = \omega^n/n!$, Liouville volume element on M. For f function on M put

$$Q(f) = \hbar\,\Xi_f$$

skew-symmetric operator on L^2 (Ξ_f = Hamiltonian vector field of f).

Wrong: $Q(c) = 0$! Second try:

$$Q(f) = \hbar\,\Xi_f - if.$$

But then $[Q(f_1), Q(f_2)] = \cdots = \hbar^2\,\Xi_{f_3} + 2i\hbar f_3 \neq \hbar\,Q(f_3)$, where $f_3 = \{f_1, f_2\}$.

(Sign convention: $\{f, g\} = \omega(\Xi_f, \Xi_g) = -\Xi_f(g)$.)

Third attempt: suppose $\omega = -d\alpha$. Put

$$Q(f) = \hbar\,\Xi_f + i\Big(\alpha(\Xi_f) - f\Big).$$

Works! But: depends on α; and what if ω not exact? Note: first two terms are covariant differentation w.r.t. connection one-form $\alpha/\hbar$.

Definition (Kostant-Souriau). M is *prequantizable* if there exists a Hermitian line bundle L (*prequantum bundle*) with connection ∇ such that curvature is $\omega/\hbar$.

Prequantum Hilbert space is L^2-sections of L, and operator associated to $f \in C^\infty(M)$ is

$$Q(f) = \hbar\,\nabla_{\Xi_f} - if.$$

Example

$M = \mathbb{R}^{2n}$, $\omega = \sum_k dx_k \wedge dy_k$, $L = \mathbb{R}^{2n} \times \mathbb{C}$, $\alpha = -\sum_k x_k dy_k$. Inner product:

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}^{2n}} \varphi(x,y)\bar{\psi}(x,y)\, dx\, dy.$$

$\Xi_{x_k} = -\partial/\partial y_k$ and $\Xi_{y_k} = \partial/\partial x_k$ so

$$Q(x_k) = -\hbar \frac{\partial}{\partial y_k},$$
$$Q(y_k) = \hbar \frac{\partial}{\partial x_k} - i y_k.$$

Snag: prequantization is too big. For $n = 2$ get $L^2(\mathbb{R}^2)$. $\mathbb{R}^2$ is homogeneous space under Heisenberg group, but $L^2(\mathbb{R}^2)$ is not unirrep for this group.

Polarizations

Polarization on M = integrable Lagrangian subbundle of $T^{\mathbb{C}}M$, i.e. subbundle $\mathcal{P} \subset T^{\mathbb{C}}M$ s.t. $\mathcal{P}_m$ is Lagrangian in $T^{\mathbb{C}}_m M$ for all m, and vector fields tangent to $\mathcal{P}$ are closed under Lie bracket.

$\mathcal{P}$ is *totally real* if $\mathcal{P} = \bar{\mathcal{P}}$. $\mathcal{P}$ is *complex* if $\mathcal{P} \cap \bar{\mathcal{P}} = 0$.

Frobenius: real polarization $\Rightarrow$ Lagrangian foliation of M

Newlander-Nirenberg: complex polarization $\Rightarrow$ complex structure J on M s.t. $\mathcal{P}$ is spanned by $\partial/\partial z_k$ in holomorphic coordinates z_k.

$\mathcal{P}$ is *Kähler* if it is complex and $\omega(\cdot, J\cdot)$ is a Riemannian metric.

Section s of L is *polarized* if $\nabla_{\bar{v}} s = 0$ for all v tangent to $\mathcal{P}$.

Definition. $Q(M) = L^2$ polarized sections of L.

Problems

1. Existence of polarizations.

2. $Q(f)$ acts on $Q(M)$ only if Ξ_f preserves $\mathcal{P}$.

3. Polarized sections are constant along (real) leaves of $\mathcal{P}$. Square-integrability?!

4. M compact, $\mathcal{P}$ complex but not Kähler $\Rightarrow$ there are no polarized sections.

5. $Q(M)$ independent of $\mathcal{P}$?

Coadjoint orbits

$\mathcal{O}$ = coadjoint orbit through $f \in \mathfrak{g}^*$. Assume G simply connected, $(\mathcal{O}, \omega)$ prequantizable. G-action on $\mathcal{O}$ lifts to L. Infinitesimally,

$$\xi_L = \text{ lift of } \xi_{\mathcal{O}} + 2\pi \Phi^{\xi} \nu_L,$$

where $\xi \in \mathfrak{g}$, ν_L = generator of scalar S^1-action on L.

G-invariant polarization $\mathcal{P}$ of $\mathcal{O}$ is determined by $\mathfrak{p} \supset \mathfrak{g}_f^{\mathbb{C}}$, inverse image of $\mathcal{P}_f$ under $\mathfrak{g}^{\mathbb{C}} \to T_f^{\mathbb{C}}\mathcal{O}$.

$\mathcal{P}$ integrable $\iff$ $\mathfrak{p}$ subalgebra.

$\mathcal{P}$ Lagrangian $\iff$ $f|_{[\mathfrak{p},\mathfrak{p}]} = 0$ (i.e. $f|_{\mathfrak{p}}$ is infinitesimal character) and $2 \dim_{\mathbb{C}} \mathfrak{p} = \dim_{\mathbb{R}} G + \dim_{\mathbb{R}} G_f$.

$\mathcal{P}$ real $\iff$ $\mathfrak{p} = \mathfrak{p}_0^{\mathbb{C}}$ for $\mathfrak{p}_0 \subset \mathfrak{g}$. Let $P_0 =$ group generated by $\exp \mathfrak{p}_0$. Assume $f\colon \mathfrak{p}_0 \to \mathbb{R}$ exponentiates to character $S_f\colon P_0 \to S^1$; then

$$Q(M) = \mathrm{Ind}_{P_0}^{G} S_f.$$

If $\mathcal{P}$ complex, $Q(M)$ is *holomorphically* induced representation.

Example

G *compact* (and simply connected). Let $T =$ maximal torus, $\mathfrak{t}_+^* =$ positive Weyl chamber, $f \in \mathfrak{t}_+^*$. Then $\mathcal{O} = Gf$ integral $\iff$ f in integral lattice.

All invariant polarizations are complex and are determined by *parabolic* subalgebras $\mathfrak{p} \supset \mathfrak{g}_f^{\mathbb{C}}$. In fact, $\mathcal{O} = G/G_f \cong G^{\mathbb{C}}/P$, where $P = \exp \mathfrak{p}$.

$$\begin{aligned} Q(\mathcal{O}) &= \text{ holomorphic sections of } G^{\mathbb{C}} \times^{P} S_f \\ &= \text{ unirrep with highest weight } f. \end{aligned}$$

Character formula:

$$\sqrt{j(\xi)}\,\chi_{\mathcal{O}}(\exp\xi) = \int\limits_{\mathcal{O}} e^{2\pi i\langle f,\xi\rangle} df,$$

where

$$\sqrt{j(\xi)} = \prod_{\alpha>0} \frac{e^{\langle\alpha,\xi\rangle/2} - e^{-\langle\alpha,\xi\rangle/2}}{\langle\alpha,\xi\rangle}.$$

$\xi = 0$:

$$\dim Q(\mathcal{O}) = \mathrm{vol}(\mathcal{O}) = \prod_{\alpha>0} \frac{\langle\alpha, f\rangle}{\langle\alpha,\rho\rangle},$$

where $\rho = 1/2$ sum of positive roots. Compare Weyl dimension formula:

$$\dim Q(\mathcal{O}) = \prod_{\alpha>0} \frac{\langle\alpha, f+\rho\rangle}{\langle\alpha,\rho\rangle}$$

(ρ-shift).

Index theorem in symplectic geometry

Recall table:

Classical	*Quantum*
Symplectic manifold (M, ω)	Hilbert space $\mathcal{H} = Q(M)$ (or $\mathbb{P}\mathcal{H}$)
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Poisson bracket $\{f, g\}$	commutator $[Q(f), Q(g)]$
Hamiltonian flow of f	1-PS in $\mathrm{U}(\mathcal{H})$

Continuation:

Hamiltonian G-action on M	unitary representation on $Q(M)$
Moment polytope $\Delta(M)$	highest weights of irreducible components
Symplectic cross-section $\Phi^{-1}(\mathfrak{t}^*_+)$	highest-weight spaces
Symplectic quotients $\Phi^{-1}(\mathcal{O})/G$	isotypical components $\mathrm{Hom}\Big(Q(\mathcal{O}), Q(M)\Big)^G$

Lemma. $\ker d\Phi_m = T_m(Gm)^\omega$, *where* $Gm =$ G*-orbit through* m.

$\operatorname{im} d\Phi_m = \mathfrak{g}_m^0$, *where* $\mathfrak{g}_m = \{\xi : (\xi_M)_m = 0\}$.

Hence: if $f \in \mathfrak{g}^*$ *is regular value of* Φ, G_f *acts locally freely on* $\Phi^{-1}(f)$.

Theorem (Meyer, Marsden-Weinstein)**.** *If* f *is regular value of* Φ, *null-foliation of* $\omega|_{\Phi^{-1}(f)}$ *is equal to* G*-orbits of* G_f*-action. Hence the quotient* $M_f = \Phi^{-1}(f)/G_f = \Phi^{-1}(\mathcal{O}_f)/G$ *is a symplectic orbifold.*

Conjecture (Guillemin-Sternberg, "$[Q, R] = 0$")**.**

$$Q(M_0) = Q(M)^G.$$

(This implies $Q(M_\mathcal{O}) = \operatorname{Hom}(Q(\mathcal{O}), Q(M))^G$.)

Proved by Guillemin-Sternberg in Kähler case using geometric invariant theory.

In compact case can make life easier by changing definition of $Q(M)$: regard prequantum bundle L as element of $K_G(M)$. Let $\pi\colon M \to \bullet$ be map to a point. Define

$$Q(M) = \pi_*([L]),$$

regarded as element of $K_G(\bullet) = \mathrm{Rep}(G)$ (representation ring).

Disadvantages: works only for compact M and G; dimension can be negative; no natural inner product.

Advantages: by and large satisfies Dirac's rules; don't need polarization; can be computed by Atiyah-Segal-Singer Equivariant Index Theorem.

Definition of π_*: choose G-invariant compatible *almost* complex structure J. Splitting of de Rham complex $\Omega^p = \bigoplus_{k+l=p} \Omega^{kl}$.

Dolbeault operator $\bar\partial$ is $(0,1)$-part of d. $\bar\partial^2 \neq 0$ unless J integrable. With coefficients in L:

$$\bar\partial_L = \bar\partial \oplus 1 + 1 \otimes \nabla \colon \Omega^{0l}(L) \to \Omega^{0,l+1}.$$

Dolbeault-Dirac operator:

$$\partial\!\!\!/_L = \bar\partial_L + \bar\partial_L^* \colon \Omega^{0,\text{even}}(L) \to \Omega^{0,\text{odd}}.$$

Pushforward of L:

$$Q(M) = \pi_*([L]) = \ker \partial\!\!\!/_L - \operatorname{coker} \partial\!\!\!/_L,$$

a virtual G-representation.

$\mathrm{RR}(M, L)$, the *equivariant index* of M, is the character of $Q(M)$. Note $\mathrm{RR}(M, L)(0) = \operatorname{index} \partial\!\!\!/_L$.

$\mathrm{RR}(M, L)^G$ is by definition $\int_G \mathrm{RR}(M, L)(g)\, dg$, the multiplicity of 0 in $Q(M)$.

Theorem (Meinrenken, Guillemin, Vergne, . . .)**.** *If* 0 *regular value of* Φ*,*

$$\mathrm{RR}(M, L)^G = \mathrm{RR}(M_0, L_0).$$

(See [S] for attributions.)

Outline of proof for $G = S^1$ [DGMW]

Two ingredients:

Proposition. *If* $0 \notin \Phi(M)$*, then* $\mathrm{RR}(M, L)^G = 0$*. If* 0 *is minimum or maximum of* Φ*, then* $\mathrm{RR}(M, L)^G = \mathrm{RR}(M_0, L_0)$*.*

Theorem (gluing formula)**.**

$$\mathrm{RR}(M_{\le 0}, L_{\le 0}) + \mathrm{RR}(M_{\ge 0}, L_{\ge 0}) = \\ = \mathrm{RR}(M, L) + \mathrm{RR}(M_0, L_0).$$

(Cf. gluing formula for topological Euler characteristic.)

Here $(M_{\leq 0}, \omega_{\leq 0}, \Phi_{\leq 0})$, $(M_{\geq 0}, \omega_{\geq 0}, \Phi_{\geq 0})$ are Hamiltonian G-manifolds (orbifolds) such that

$$\Phi_{\leq 0}(M_{\leq 0}) = \Phi(M) \cap \mathbb{R}_{\leq 0},$$
$$\Phi_{\geq 0}(M_{\geq 0}) = \Phi(M) \cap \mathbb{R}_{\geq 0},$$

and $\Phi_{\leq 0}^{-1}(0)$ and $\Phi_{\geq 0}^{-1}(0)$ are symplectomorphic to M_0.

By Proposition,

$\mathrm{RR}(M_{\leq 0}, L_{\leq 0})^G = \mathrm{RR}(M_{\geq 0}, L_{\geq 0})^G = \mathrm{RR}(M_0, L_0)$.

Hence, taking G-invariants on both sides in gluing formula

$$2\,\mathrm{RR}(M_0, L_0) = \mathrm{RR}(M, L)^G + \mathrm{RR}(M_0, L_0),$$

Q.E.D.

Proposition and gluing formula follow from equivariant index theorem.

Definition of $M_{\leq 0}$ and $M_{\geq 0}$: *symplectic cutting* (Lerman). Roughly, $M_{\geq 0}$ is obtained by taking $\Phi^{-1}([0,\infty))$ and collapsing S^1-orbits on boundary $\Phi^{-1}(0)$. So $M_{\geq 0}$ = union of $M_{>0}$ and M_0.

$M_{\geq 0}$

M_0

$M_{\leq 0}$

Consider diagonal action of S^1 on $M \times \mathbb{C}$, which has moment map $\tilde{\Phi}(m, z) = \Phi(m) - \frac{1}{2}|z|^2$. Here $\mathbb{C}$ = is complex line w. standard cirle action and symplectic structure. Symplectic cut is symplectic quotient at 0,

$$M_{\geq 0} = (M \times \mathbb{C}) /\!\!/ S^1.$$

("$/\!\!/$" means symplectic quotient at 0.)

Embedding $\Phi^{-1}(0) \hookrightarrow \tilde{\Phi}^{-1}(0)$ defined by $m \mapsto (m, 0)$ descends to symplectic embedding $M_0 \hookrightarrow M_{\geq 0}$.

$M_{>0} = \Phi^{-1}((0, \infty))$ also embeds symplectically into $M_{\geq 0}$: define $M_{>0} \to \tilde{\Phi}^{-1}(0)$ by sending m to $\left(m, \sqrt{2\,\Phi(m)}\right)$.

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