THE MOMENT POLYTOPE OF A KÄHLER G-MANIFOLD

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Dedicated to the memory of Chepe Escobar

ABSTRACT. This is the transcript of an introductory talk on the Kirwan convexity theorem for Kähler manifolds and some of its generalizations.

1. INTRODUCTION

This expository note contains a sketch of the proof of the convexity theorem for moment maps on Kähler manifolds, which is due independently to Guillemin and Sternberg [9] and to Mumford [18, Appendix], with improvements due to Brion [3]. It also explains how the idea behind this proof has been extended and generalized in various ways. This material is based on lectures I gave at the International Conference in Memory of José Escobar in Cali, Colombia, and at the Séminaire Itinérant de Géométrie et Physique in Hà Nội, Việt Nam. I thank my hosts at the Universidad del Valle in Cali and at the Hà Nội University of Education for their hospitality.

2. FROM ALGEBRAS TO POLYTOPES

Let T be a torus (compact connected abelian real Lie group) and let $A = \bigoplus_{r=0}^{\infty} A_r$ be a commutative graded algebra over the field of complex numbers $\mathbf{C}$. We make the following assumptions on A.

(i) A is finitely generated.
(ii) A has no zero divisors.
(iii) The torus T acts on A. The action is linear,

$$t \cdot (c_1 a_1 + c_2 a_2) = c_1(t \cdot a_1) + c_2(t \cdot a_2)$$

for all $t \in T$, c_1, $c_2 \in \mathbf{C}$ and a_1, $a_2 \in A$; multiplicative,

$$t \cdot (a_1 a_2) = (t \cdot a_1)(t \cdot a_2)$$

for all $t \in T$ and a_1, $a_2 \in A$; and preserves the grading,

$$t \cdot a \in A_r$$

for all $t \in T$ and $a \in A_r$.
(iv) For all r, the action of T on A_r is continuous.
(v) $A_0 = \mathbf{C}$, the trivial one-dimensional representation of T.

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Notice that, by assumptions (i) and (iii), each of the summands A_r is a finite-dimensional T-module. Therefore assumption (iv) makes sense; it simply means, by definition, that the action of T on A_r is given by a continuous homomorphism from T to the matrix group $\mathbf{GL}(A_r)$. These assumptions enable us to refine the grading of A into a bigrading by weight and degree. Let $\mathcal{X}(T) = \mathrm{Hom}(T, \mathbf{U}(1))$ be the character group of T, where $\mathbf{U}(1) = \{\, z \in \mathbf{C} \mid |z| = 1 \,\}$ denotes the unit circle. Define $A_{\lambda,r}$ to be the collection of all $a \in A_r$ such that $t \cdot a = \lambda(t)a$ for all $t \in T$. Then

$$A = \bigoplus_{(\lambda,r)\in\mathcal{X}(T)\times\mathbf{N}} A_{\lambda,r}.$$

Assumption (iii) implies

(v) If $a \in A_{\lambda,r}$ and $b \in A_{\mu,s}$, then $ab \in A_{\lambda+\mu,r+s}$.

Let $\Sigma(A)$ be the set of all $(\lambda, r) \in \mathcal{X}(T) \times \mathbf{N}$ for which the direct summand $A_{\lambda,r}$ is nonzero.

2.1. Lemma. *$\Sigma(A)$ is a finitely generated submonoid of $\mathcal{X}(T) \times \mathbf{N}$.*

Proof. Assumption (ii) and assertion (v) imply that $\Sigma(A)$ is closed under addition. Assumption (v) implies that $(0,0) \in \Sigma(A)$. Therefore $\Sigma(A)$ is a submonoid. By assumption (i), the algebra A is finitely generated. Let $a_1 \in A_{\lambda_1,r_1}$, $a_2 \in A_{\lambda_2,r_2}$, …, $a_k \in A_{\lambda_k,r_k}$ be a set of homogeneous generators. Then it follows from assumption (iii) that every $(\lambda, r) \in \Sigma(A)$ can be written in the form $(\lambda, r) = \sum_{l=1}^{k} n_l(\lambda_l, r_l)$ with $n_l \in \mathbf{N}$. Thus $\Sigma(A)$ is finitely generated as a monoid. QED

The monoid $\Sigma(A)$ can be quite complicated. Somewhat easier to understand is its "classical limit",

$$\mathcal{P}(A) = \left\{ \frac{\lambda}{r} \;\middle|\; (\lambda, r) \in \Sigma(A),\ r > 0 \right\},$$

which is a subset of the $\mathbf{Q}$-vector space $\mathcal{X}(T)_{\mathbf{Q}} = \mathcal{X}(T) \otimes_{\mathbf{Z}} \mathbf{Q}$. Recall that a *convex polytope* in a vector space V over an ordered field is a subset of V which is the convex hull of a finite subset of V.

2.2. Lemma. *$\mathcal{P}(A)$ is a convex polytope in $\mathcal{X}(T)_{\mathbf{Q}}$.*

Proof. Let λ/r, $\mu/s \in \mathcal{P}(A)$. Then it follows from assumption (ii) and assertion (v) that $(\lambda+\mu)/(r+s) \in \mathcal{P}(A)$. Hence $\mathcal{P}(A)$ is convex. In fact, if (λ_1, r_1), (λ_2, r_2), …, (λ_k, r_k) are generators of $\Sigma(A)$, then every element of $\mathcal{P}(A)$ is a convex combination of λ_1/r_1, λ_2/r_2, …, λ_k/r_k. Thus $\mathcal{P}(A)$ is a convex polytope. QED

An example of an algebra A satisfying assumptions (i)–(v) is the homogeneous coordinate ring of an irreducible projective variety equipped with a linear action of the torus T. More generally, we will consider the following situation. Let M be a compact connected complex manifold and let L be a positive holomorphic line bundle over M. Let G be a compact connected Lie group with maximal torus T. Suppose that G acts holomorphically on M and that this action lifts to an action on L by holomorphic bundle transformations. Then the space of global holomorphic sections $\Gamma(M, L)$ of L is a G-module. Likewise, G acts naturally on the global sections of the tensor powers $L^{\otimes r}$ and hence on the graded algebra

$$S(M, L) = \bigoplus_{r=0}^{\infty} \Gamma(M, L^{\otimes r}).$$

It follows from a theorem of Bochner and Montgomery (see e.g. [15, Ch. III]) that the G-actions on M and L extend uniquely to holomorphic $G_{\mathbf{C}}$-actions, where $G_{\mathbf{C}}$ denotes the complexification of G. Therefore the algebra $S(M, L)$ carries a natural action of the complex reductive group $G_{\mathbf{C}}$. Now we choose a maximal unipotent subgroup N of $G_{\mathbf{C}}$ which normalizes $T_{\mathbf{C}}$ and we define $A(M, L) = S(M, L)^N$, the subalgebra of $S(M, L)$ consisting of N-invariant sections.

2.3. **Proposition.** *The algebra* $A(M, L)$ *satisfies assumptions* (i)–(v).

Proof. It follows from the Kodaira embedding theorem that M can be embedded into a complex projective space. Serre's GAGA theorems [21] imply that L is an algebraic line bundle and that the action of $G_{\mathbf{C}}$ on M and L is algebraic. The results of [20] now imply that the algebra $S(M, L)$ is finitely generated. The fact that the subalgebra $A(M, L)$ of $S(M, L)$ is finitely generated then follows from a theorem of Hadziev and Grosshans; see e.g. [17, Section III.3.2]. It follows from the identity principle for holomorphic functions that $S(M, L)$ has no zero divisors, and therefore the subalgebra $A(M, L)$ has no zero divisors. The subgroup N of $G_{\mathbf{C}}$ normalizes T and therefore T acts on N by conjugation. Hence, if $s \in A(M, L)$, $t \in T$ and $g \in N$,

$$g \cdot (t \cdot s) = t \cdot \big((t^{-1} g t) \cdot s\big) = t \cdot s,$$

so $t \cdot s \in A(M, L)$. In other words, T acts on $A(M, L)$. The remaining assumptions are now easy to check. QED

We will denote the convex polytope $\mathcal{P}(A(M, L))$ associated with the algebra $A(M, L)$ by $\mathcal{P}(M, L)$. Let $\mathfrak{t}$ be the Lie algebra of the torus T. The Lie algebra of $\mathbf{U}(1)$ is the imaginary axis $i\mathbf{R}$, so the differential λ_* of a character $\lambda \in \mathcal{X}(T)$ is a linear map $\mathfrak{t} \to i\mathbf{R}$, that is to say, an element of $i\mathfrak{t}^*$. The map $\mathcal{X}(T) \to \mathfrak{t}^*$ which sends a character λ to the real-valued functional $(2\pi i)^{-1}\lambda_*$ is an embedding of $\mathcal{X}(T)$ onto a lattice in $\mathfrak{t}^*$ known as the *weight lattice*. As is common practice, we will identify $\mathcal{X}(T)$ with the weight lattice by means of this embedding. Similarly, we will regard the $\mathbf{Q}$-vector space $\mathcal{X}(T)_{\mathbf{Q}}$ as a (dense) subset of $\mathfrak{t}^*$, the set of *rational points* of $\mathfrak{t}^*$.

Let C denote the closed Weyl chamber in $\mathfrak{t}^*$ which is positive with respect to N. Recall that a character $\lambda \in \mathcal{X}(T)$ is *dominant* if $\lambda \in C$. The following fundamental result of representation theory, often called the Cartan-Weyl theorem, says that there is a bijective correspondence between dominant characters and (isomorphism classes of) irreducible complex representations of G. See e.g. [2, Ch. 9.7], [5, Ch. 4], or [17, Section III.1.5] for a proof.

2.4. **Theorem.** (i) *Let* V *be an irreducible* G*-module. Then* $\dim V^N = 1$. *Let* $\lambda \in \mathcal{X}(T)$ *be the weight of the* T*-action on* V^N. *Then* $\lambda \in C$.

(ii) *For each* $\lambda \in \mathcal{X}(T) \cap C$ *there exists an irreducible* G*-module* V *such that the* T*-action on* V^N *has weight* λ*. This* G*-module* V *is unique up to isomorphism.*

The character λ in part (i) is called the *highest weight* of V (because it is the largest weight of T occurring in V with respect to a certain natural partial ordering on $\mathfrak{t}^*$). This theorem implies the following statement.

2.5. **Proposition.** *The polytope* $\mathcal{P}(M, L)$ *is contained in* C.

Proof. Let $\mu \in \mathcal{P}(M, L)$. Choose $\lambda \in \mathcal{X}(T)$ and a positive integer r such that $\mu = \lambda/r$ and such that there exists a nonzero element s of $A(M, L)_{\lambda,r}$. Choose an irreducible G-submodule V of S(M, L) containing s. Then $s \in V^N$, so, by the Cartan-Weyl theorem, the weight λ of s is dominant. Hence $\mu = \lambda/r$ is contained in C. QED

3. Example: quadratic forms

To illustrate Propositions 2.3 and 2.5, we let G be the unitary group $\mathbf{U}(n)$ and $T \cong \mathbf{U}(1)^n$ its standard maximal torus, which consists of all diagonal matrices

$$\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \tag{1}$$

with $|t_1| = |t_2| = \cdots = |t_n| = 1$. Then $G_{\mathbf{C}}$ is the complex general linear group $\mathbf{GL}(n, \mathbf{C})$ and $T_{\mathbf{C}}$ is the subgroup of all matrices (1) with $t_1, t_2, \ldots, t_n \in \mathbf{C}\setminus\{0\}$. A convenient choice for N is the subgroup consisting of all unipotent upper triangular matrices

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

A basis of the Lie algebra $\mathfrak{t}$ of T is given by

$$\varepsilon_1 = \begin{pmatrix} 2\pi i & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \; \varepsilon_2 = \begin{pmatrix} 0 & & & \\ & 2\pi i & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \ldots, \; \varepsilon_n = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 2\pi i \end{pmatrix}.$$

The dual vectors $\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_n^*$ are a basis of the weight lattice $\mathcal{X}(T)$. The closed positive chamber C is the set of all $\lambda \in \mathfrak{t}^*$ such that $\langle \lambda, \varepsilon_j \rangle \geq \langle \lambda, \varepsilon_{j+1} \rangle$ for $j = 1, 2, \ldots, n$, where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$. Let V be a finite-dimensional G-module, let $M = \mathbf{P}(V)$, the space of lines in V, and let $L = \mathcal{O}(1)$, the canonical quotient bundle over M. This is by definition the line bundle dual to the tautological bundle

$$\{ (l, v) \in M \times V \mid v \in l \}$$

over M, from which one sees that the G-action on M lifts naturally to a G-action on L. The space of sections $\Gamma(M, L)$ is naturally isomorphic as a G-module to the dual V^* of V and, similarly, $\Gamma(M, L^{\otimes r})$ is naturally isomorphic to its rth symmetric power

$S^r(V^*)$. Thus $S(M, L) = S(V^*)$, the algebra of polynomials on V. To take a specific example, let $V = S^2(\mathbf{C}^n)^*$, the space of quadratic forms in n variables, on which G acts by substitution. If we think of elements of V as symmetric $n \times n$-matrices, the action is given by $g \cdot v = {}^t g^{-1} v g^{-1}$ for $g \in G$ and $v \in V$. For $1 \le r \le n$ define the polynomial function a_r on V by

$$a_r(v) = \begin{vmatrix} v_{1,1} & \cdots & v_{1,r} \\ \vdots & & \vdots \\ v_{1,r} & \cdots & v_{r,r} \end{vmatrix},$$

the principal $r \times r$-minor of v. Then a_r has degree r and one checks by direct computation that a_r is N-invariant and transforms under T with weight

$$\lambda_r = 2(\varepsilon_1^* + \varepsilon_2^* + \cdots + \varepsilon_r^*).$$

Thus $a_r \in A(M, L)_{\lambda_r, r}$. By [24, Proposition 1], the algebra $A(M, L) = S(M, L)^N$ is freely generated by $a_1, a_2, \dots, a_n$,

$$A(M, L) = \mathbf{C}[a_1, a_2, \dots, a_n],$$

and therefore $\mathcal{P}(M, L)$ is the $n-1$-simplex with vertices $\lambda_1, \frac{1}{2}\lambda_2, \dots, \frac{1}{n}\lambda_n$. This example is taken from [3], where one can find several further examples.

4. The moment map

Let M and L be as in Section 2. It was discovered by Guillemin and Sternberg that the polytope $\mathcal{P}(M, L)$ has an alternative interpretation in terms of Kähler geometry. Their discovery was inspired by the theory of geometric quantization and Kirillov's orbit method, an introduction to which can be found in [11]. A key element of their argument is the Borel-Weil theorem, which establishes a correspondence between irreducible G-modules and integral coadjoint orbits of G, and can be regarded as a geometric version of the Cartan-Weyl theorem, Theorem 2.4. This section is a summary of Guillemin and Sternberg's argument, which follows in large part Brion's exposition in [3].

Because the line bundle L is positive, there exist a G-invariant Hermitian bundle metric $\|\cdot\|$ on L and a G-invariant Hermitian connection ∇ such that the curvature form $\omega \in \Omega^2(M)$ is positive definite and hence a Kähler form. Each Lie algebra element $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$ defines two first-order differential operators on the space of holomorphic sections $\Gamma(M, L)$: covariant differentiation ∇_ξ and the Lie derivative

$$\mathcal{L}_\xi(s) = \frac{d}{dt} \exp(t\xi) \cdot s \Big|_{t=0}.$$

It was observed by Kostant [16, Theorem 4.3.1] that ∇_ξ and $\mathcal{L}_\xi$ have the same principal symbol and that the zeroth order operator $\mathcal{L}_\xi - \nabla_\xi$ is multiplication by an imaginary-valued function. Thus we have a map $\phi\colon \mathfrak{g} \to C^\infty(M, \mathbf{R})$ defined by

$$\phi(\xi) = \frac{1}{2\pi i}(\mathcal{L}_\xi - \nabla_\xi). \tag{2}$$

Kostant showed that ϕ is G-equivariant with respect to the adjoint action on $\mathfrak{g}$ and the action on $C^\infty(M, \mathbf{R})$ dual to the action on M. Moreover, it satisfies $d\phi(\xi) = \iota(\xi_M)\omega$, where ξ_M denotes the vector field on M induced by ξ. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $\mathfrak{g}$ and its dual. The map $\Phi\colon M \to \mathfrak{g}^*$ defined by $\langle \Phi(m), \xi \rangle = \phi(\xi)(m)$ is known as the *moment map* for the G-action on the Kähler G-manifold (M, ω). By construction, it is G-equivariant (with respect to the given action on M and the coadjoint action on $\mathfrak{g}^*$) and therefore its image $\Phi(M)$ is a G-invariant subset of $\mathfrak{g}^*$. The chamber C in $\mathfrak{t}^*$ is a fundamental domain for the coadjoint action, so the image $\Phi(M)$ is completely determined by its intersection with C. The following theorem goes back to [9] and [18, Appendix].

4.1. **Theorem.** (i) $\mathcal{P}(M, L)$ *is the set of rational points in* $\Phi(M) \cap C$.
(ii) $\Phi(M) \cap C = \overline{\mathcal{P}(M, L)}$.
(iii) $\Phi(M) \cap C$ *is a rational convex polytope in* $\mathfrak{t}^*$.

Outline of proof. Let $\mu \in \mathcal{P}(M, L)$. Then there exist a positive integer r and $\lambda \in \mathcal{X}(T)$ such that $\mu = \lambda/r$ and $A(M, L)_{\lambda,r} \neq \{0\}$. Choose a nonzero $s \in A(M, L)_{\lambda,r}$ and let U be the complement in M of the zero locus of s. Define $f\colon U \to \mathbf{R}$ by $f(x) = (4\pi)^{-1} \log\|s(x)\|^2$. A calculation using (2) shows that

$$\frac{d}{dt} f\big(\exp(it\xi) \cdot x\big)\Big|_{t=0} = -\lambda(\xi) + r\langle \Phi(x), \xi \rangle \tag{3}$$

for all $\xi \in \mathfrak{g}$. (See e.g. [8, Section 3].) Now let $x \in U$ be a point at which f attains its maximum. Then the left-hand side of (3) vanishes, so $\Phi(x) = \lambda/r = \mu$. This shows that $\mathcal{P}(M, L)$ is contained in $\Phi(M) \cap C$. Conversely, let $\mu \in C$ be a rational point. To finish the proof of (i) we need to show that $\mu \in \Phi(M) \cap C$ implies $\mu \in \mathcal{P}(M, L)$. First assume that $\mu = 0$. An important fact (for which see [10, 12, 18, 22]) is that every $x \in \Phi^{-1}(0)$ is *semistable* in the sense that there exists a section $s \in \Gamma(M, L^r)^G$ such that $s(x) \neq 0$. A G-invariant section is an N-invariant section of weight 0. Thus $0 \in \Phi(M) \cap C$ implies $0 \in \mathcal{P}(M, L)$. The case of an arbitrary μ can be reduced to the case $\mu = 0$ by the "shifting trick", which consists in replacing the triple M, L, Φ with a new triple M', L', Φ' such that

$$\mu \in \Phi(M) \cap C \iff 0 \in \Phi'(M') \cap C, \tag{4}$$

$$\mu \in \mathcal{P}(M, L) \iff 0 \in \mathcal{P}(M', L'). \tag{5}$$

To do this, we write $\mu = \lambda/n$, where $\lambda \in \mathcal{X}(T) \cap C$ and n is a positive integer. The coadjoint orbit $O_{-\lambda} = \mathrm{Ad}^*(G)(-\lambda)$ is equipped with a standard G-invariant symplectic form $\omega_{-\lambda}$, known as the Kirillov-Kostant-Souriau form. Moreover, it possesses a G-invariant complex structure such that $\omega_{-\lambda}$ is Kähler. These facts are true for any coadjoint orbit, but since λ is integral, there is in addition a G-equivariant Hermitian holomorphic line bundle $L_{-\lambda}$ on $O_{-\lambda}$ whose curvature form is equal to $\omega_{-\lambda}$. (See e.g. [11].) The moment map for the G-action on $O_{-\lambda}$ is the inclusion $O_{-\lambda} \hookrightarrow \mathfrak{g}^*$. We define $M' = M \times O_{-\lambda}$ and $L' = \mathrm{pr}_1^* L^{\otimes n} \oplus \mathrm{pr}_2^* L_{-\lambda}$, where $\mathrm{pr}_1\colon M' \to M$ and $\mathrm{pr}_2\colon M' \to O_{-\lambda}$ denote the projections onto the respective factors. The curvature form of L' is equal to $n\omega \oplus \omega_{-\lambda}$, and the moment map Φ' for the diagonal G-action on M' is given by $\Phi'(x, \xi) = n\Phi(x) + \xi$, from which one easily obtains (4). The algebra associated with the pair (M', L') is

$$S(M', L') = \bigoplus_{r=0}^{\infty} \Gamma\bigl(M', (L')^{\otimes r}\bigr) = \bigoplus_{r=0}^{\infty} \Gamma\bigl(M', (\mathrm{pr}_1^* L^{\otimes n} \oplus \mathrm{pr}_2^* L_{-\lambda})^{\otimes r}\bigr)$$
$$\cong \bigoplus_{r=0}^{\infty} \Gamma(M, L^{\otimes rn}) \otimes \Gamma\bigl(O_{-\lambda}, L_{-\lambda}^{\otimes r}\bigr).$$

The Borel-Weil theorem says that $\Gamma(O_{-\lambda}, L_{-\lambda}) \cong V(\lambda)^*$, the dual of the irreducible G-module with highest weight λ. (See e.g. [11] or [5, Ch. 4].) The orbits $O_{-\lambda}$ and $O_{-r\lambda}$ are isomorphic as holomorphic G-manifolds and under this isomorphism the line bundles $L_{-\lambda}^{\otimes r}$ and $L_{-r\lambda}$ are G-equivariantly equivalent. Therefore

$$S(M', L') \cong \bigoplus_{r=0}^{\infty} \Gamma(M, L^{\otimes rn}) \otimes V(r\lambda)^* \cong \bigoplus_{r=0}^{\infty} \mathrm{Hom}\bigl(V(r\lambda), \Gamma(M, L^{\otimes rn})\bigr),$$

so in particular

$$S(M', L')^G \cong \bigoplus_{r=0}^{\infty} \mathrm{Hom}\bigl(V(r\lambda), \Gamma(M, L^{\otimes rn})\bigr)^G.$$

It follows from this that $0 \in \mathcal{P}(M', L')$ if and only if, for some $r > 0$, $\Gamma(M, L^{\otimes rn})$ contains a copy of the irreducible G-module $V(r\lambda)$. By Theorem 2.4, this is the case if and only if $\mu = r\lambda/rn \in \mathcal{P}(M, L)$, which proves (5). This completes our sketch of the proof of part (i). For the proof of part (ii) we refer to [18, Appendix] or, for an alternative proof, [8, Section 4]. Finally, part (iii) follows immediately from (i) and (ii). QED

5. Generalizations

Hamiltonian G-manifolds. In their paper [9], Guillemin and Sternberg also conjectured the following more general statement, which does away with the Kähler and integrality hypotheses, and which was shortly afterwards proved by Kirwan [13].

5.1. Theorem. *Let M be a compact Hamiltonian G-manifold with moment map $\Phi\colon M \to \mathfrak{g}^*$. Then $\Phi(M) \cap C$ is a rational convex polytope.*

This result was subsequently extended in many different ways. Below I will list a few of these generalizations and give some pointers to the literature. I will concentrate on those versions that rely on the "Kähler" argument given in Section 4 (which is different from the argument used by Kirwan).

Complex subvarieties. Let M and L be as in Section 2. Let X be a G-invariant closed irreducible complex subvariety of M. (Such a variety is necessarily $G_{\mathbf{C}}$-invariant.) Defining $A(X, L) = S(M, L)^N / I(X, L)^N$, where $I(X, L)$ is the ideal of $S(M, L)$ consisting of all sections vanishing on X, one checks as in the proof of Proposition 2.3 that the algebra $A(X, L)$ satisfies all the assumptions imposed in Section 2. (That $A(X, L)$ has no zero divisors follows from the irreducibility of X.) The arguments of Section 4 go through with a few trivial changes and the upshot is the following generalization of Theorem 4.1, due to Brion [3].

5.2. **Theorem.** $\mathcal{P}(X, L) = \Phi(X) \cap C \cap \mathcal{X}(T)_{\mathbf{Q}}$ *and* $\Phi(X) \cap C = \overline{\mathcal{P}(X, L)}$. *Hence* $\Phi(X) \cap C$ *is a rational convex polytope.*

Here Φ denotes the moment map for the G-action on the ambient Kähler manifold M. The most interesting kind of subvariety to which this result applies is perhaps that of the closure of a $G_{\mathbf{C}}$-orbit in M.

More recently, it was noted in [8] that for Theorem 5.2 to hold it is not even necessary for X to be G-invariant, but that it suffices for X to be B-invariant, where $B = T_{\mathbf{C}}N$ is the Borel subgroup containing N. Examples of such varieties are closures of B-orbits in M.

Real subvarieties. Let M and L be as in Section 2. Let γ be an anti-Kähler involution of M, i.e. an antiholomorphic map $M \to M$ satisfying $\gamma^2 = \mathrm{id}_M$ and $\gamma^*\omega = -\omega$. The fixed-point set $X = M^\gamma$, if nonempty, is a totally real Lagrangian submanifold of M, which we call a *real form* of M. Suppose that γ is compatible with the G-action in the sense that $\gamma(g \cdot x) = \sigma(g) \cdot \gamma(x)$ for all $g \in G$ and $x \in M$, where σ is an involution of G. The submanifold X is not G-invariant, but K-invariant, where $K = G^\sigma$ is the subgroup fixed under σ. Suppose that γ lifts to a map $L \to L$ which is conjugate linear on the fibres. Then one can show that $\Phi(\gamma(x)) = -\sigma^*(\Phi(x))$ for all $x \in M$, where $\sigma^* \colon \mathfrak{g}^* \to \mathfrak{g}^*$ denotes the transpose of the derivative $\sigma_* \colon \mathfrak{g} \to \mathfrak{g}$ of σ. This implies that the image of X under Φ is contained in the subspace $\mathfrak{p}^*$ of $\mathfrak{g}^*$ consisting of all $\lambda \in \mathfrak{g}^*$ satisfying $\sigma^*(\lambda) = -\lambda$. (This subspace can be naturally identified with the dual of $\mathfrak{p} = \{\, \xi \in \mathfrak{g} \mid \sigma_*(\xi) = -\xi \,\}$, the tangent space of the symmetric space $P = G/K$.) A result of [19] asserts the following.

5.3. **Theorem.** $\Phi(X) \cap C = \Phi(M) \cap C \cap \mathfrak{p}^*$. *Hence* $\Phi(X) \cap C$ *is a rational convex polytope.*

(For this to be true, the maximal torus T and the chamber C must be chosen "correctly" with respect to the involution, but this can always be done.) If G is a torus, this is a special case of the main theorem of [4]. The proof of this result combines the argument behind Theorem 4.1 with the following two facts: the real form X is dense in M with respect to the complex Zariski topology (and therefore any nonzero global section of L is not identically zero on X); and for all $x \in X$ and $\xi \in \mathfrak{p}$ the curve $\exp(it\xi) \cdot x$ is contained in X.

Affine G-varieties. Let V be a finite-dimensional unitary G-module and let X be a G-invariant closed irreducible affine subvariety of V. Let $A(X) = S(X)^N$, where $S(X)$ is the affine coordinate ring of X. The algebra $A(X)$ has all the properties listed in Section 2, except that it has in general no grading. But we can still decompose it into weight spaces, $A(X) = \bigoplus_{\lambda \in \mathcal{X}(T)} A(X)_\lambda$, where $A(X)_\lambda$ consists of all $a \in A$ such that $t \cdot a = \lambda(t)a$ for all $t \in T$. Hence we can define

$$\Sigma(X) = \bigl\{\, \lambda \in \mathcal{X}(T) \bigm| A(X)_\lambda \neq \{0\} \,\bigr\},$$

a finitely generated submonoid of $\mathcal{X}(T)$. Let us define $\mathcal{P}(X)$ to be the convex hull in $\mathcal{X}(T)_{\mathbf{Q}}$ of the monoid $\Sigma(X)$. This is a convex polyhedral cone in $\mathcal{X}(T)_{\mathbf{Q}}$, contained in the chamber C. Let $\Phi \colon V \to \mathfrak{g}^*$ be the moment map for the flat Kähler form on V, which is given by $\langle \Phi(v), \xi \rangle = \frac{i}{2}(\xi_V v, v)$, where $\xi_V \in \mathfrak{gl}(V)$ is the linear map defined by $\xi \in \mathfrak{g}$ and $(\cdot, \cdot)$ is the inner product on V. One of the results of [23] is as follows.

5.4. Theorem. *$\mathcal{P}(X) = \Phi(X) \cap C \cap \mathcal{X}(T)_{\mathbf{Q}}$ and $\Phi(X) \cap C = \overline{\mathcal{P}(X)}$. Hence $\Phi(X) \cap C$ is a rational convex polyhedral cone.*

Another fact proved in [23] is that every isotropic G-orbit in an arbitrary Hamiltonian G-manifold has an invariant neighbourhood which is isomorphic, as a Hamiltonian G-manifold, to a germ (in the classical topology) of an affine G-variety. By means of a local-to-global argument, this fact can be combined with Theorem 5.4 to give an alternative proof of Kirwan's convexity theorem; see [23, Section 6]. (By an analogous reasoning it is possible to prove "real forms" of Theorems 5.4 and 5.1; see [19].)

6. Conclusion

The Kirwan convexity theorem is a perennial favourite of symplectic geometers and in these few pages I cannot possibly do justice to all the people who have contributed to it. But I would be remiss not to mention its close relationship to various problems of linear algebra concerning matrices and their eigenvalues, such as Horn's problem on sums of Hermitian matrices. In such eigenvalue problems one is often concerned with finding an explicit minimal set of inequalities for a special moment polytope. For more on this matter, and for an ample bibliography, see [6, 14, 7].

This note has focussed on the algebro-geometric approach to the convexity theorem. This approach seems to hold the best promise of obtaining explicit inequalities for moment polytopes and also, as far as I am aware, yields the only available proof of the real form and various other versions of Kirwan's theorem. But other valid approaches exist, in particular to the local-to-global principle, which give rise to different generalizations of the convexity theorem. For this I refer to the monograph [7] and to the recent preprints [25] (in which convexity is cast in terms of affine geometry) and [1].

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