$L_{(2)}$-COHOMOLOGY OF ORBIT SPACES

REYER SJAMAAR

Abstract. Suppose that a compact Lie group acts on a smooth compact manifold and that the manifold is equipped with an invariant Riemannian metric. This metric induces a metric on the open dense stratum of the quotient space. Then the $L_{(2)}$-cohomology of the open dense stratum with respect to the induced metric is isomorphic to the intersection cohomology with upper middle perversity of the quotient space.

AMS Subject Classification: Primary 57S15; Secondary 55N35
compact Lie group actions, orbit spaces, intersection homology

Introduction

Let M be a smooth compact manifold and let G be a compact Lie group acting smoothly on M. A smooth differential form ω on M is called *invariant* with respect to the action of G if $A_g^*(\omega) = \omega$ for all $g \in G$, where $A_g : M \to M$ denotes the action of the element g. It is called *horizontal* with respect to the action if $\iota(\xi_M)\omega = 0$ for all ξ in the Lie algebra of G, where $\iota(\xi_M)\omega$ denotes the interior product of ω with the vector field on M induced by ξ. If a form is both invariant and horizontal, it is called *basic*. The graded algebra of all basic forms is denoted by $\Omega^{\cdot}(M/G)$. It follows from the infinitesimal homotopy formula, $\mathcal{L}_{\xi_M}\omega = \iota(\xi_M)\, d\omega + d\,\iota(\xi_M)\omega$, that the exterior derivative d maps $\Omega^{\cdot}(M/G)$ into itself. Hence $(\Omega^{\cdot}(M/G), d)$ is a cochain complex. Koszul [10] has proved that the cohomology of $\Omega^{\cdot}(M/G)$ is isomorphic to the singular cohomology with real coefficients of the orbit space M/G. Here it is not assumed that the action is free. (Note, by the way, that the singular cohomology of M/G coincides with its Čech cohomology, since M/G is compact and locally contractible.) His argument involves slices for the action of G on M, a G-equivariant Poincaré lemma and G-invariant partitions of unity on M. If one interprets the basic forms on M as being the differential forms on the singular space M/G, one can read Koszul's theorem as a de Rham theorem for quotient spaces. The purpose of this note is to sketch the proof of a similar theorem:

Theorem 1. *There is a natural isomorphism* $H^{\cdot}_{(2)}(M^{\mathrm{princ}}/G) \cong I_{\bar{n}}H^{\cdot}(M/G)$.

Here $H^{\cdot}_{(2)}(M^{\mathrm{princ}}/G)$ is the $L_{(2)}$-cohomology of the open dense (or 'principal') stratum of the orbit space with respect to a metric induced by a G-invariant metric on M. From the compactness of M it follows that $H^{\cdot}_{(2)}(M^{\mathrm{princ}}/G)$ does not depend on the choice of an invariant metric on M. On the right-hand side, $\bar{n}$ denotes the upper middle perversity, which associates to a stratum of codimension k the

Date: Mathematisch Instituut der Rijksuniversiteit te Utrecht
PO Box 80.010, 3508 TA Utrecht, The Netherlands

June 1989; revised February 1991.

integer $[(k-1)/2]$, and $I_{\bar{n}}H^{\cdot}(M/G)$ is the intersection cohomology of M/G with real coefficients and with respect to the perversity $\bar{n}$. As a consequence of the theorem, $H^{\cdot}_{(2)}(M^{\mathrm{princ}}/G)$ is even a topological invariant of the orbit space M/G. The proof hinges on the fact that the metric on the quotient space is conical in the sense of Cheeger [5], [6].

Section 1 is a review the definitions of $L_{(2)}$-cohomology and harmonic forms. Section 2 contains an outline of the proof of the $L_{(2)}$ de Rham Theorem and some comments, in particular on generalized Poincaré duality. Intersection cohomology with real coefficients and with respect to the upper middle perversity $\bar{n}$ of a space Y will be denoted by $I_{\bar{n}}H^{\cdot}(Y)$, and its singular cohomology with real coefficients will be denoted by $H^{\cdot}(Y)$.

I would like to thank Eduard Looijenga and my thesis advisor Hans Duistermaat for useful discussions. This paper is a version of a chapter of my PhD thesis [14].

1. $L_{(2)}$-Cohomology and Harmonic Forms

This section is a cursory introduction to $L_{(2)}$-cohomology. See [5] and [15] for more information and further references.

Let Y be an m-dimensional Riemannian manifold with metric ν. This metric induces in a natural way inner products on the fibres of the exterior powers of the cotangent bundle of Y. If α and β are differential forms of the same degree on Y, their inner product, regarded as a function on Y, will be denoted by (α, β), and $(\alpha, \alpha)^{1/2}$ will be abbreviated to $|\alpha|$. For a k-form β the form $*\beta$ is defined as the unique $(m-k)$-form such that for all k-forms α one has $\alpha \wedge *\beta = (\alpha, \beta)\, d\nu$, where $d\nu$ is the volume form defined by the metric. (There is a little problem here if Y is not orientable. The solution is to regard $d\nu$ and $*\alpha$ as forms with coefficients in the orientation bundle of Y; cf. de Rham [12].) Define the global inner product $\langle \alpha, \beta \rangle$ and norm $\|\alpha\|$ by

$$\langle \alpha, \beta \rangle = \int_Y \alpha \wedge *\beta = \int_Y (\alpha, \beta)\, d\nu, \qquad \|\alpha\|^2 = \langle \alpha, \alpha \rangle = \int_Y \alpha \wedge *\alpha,$$

if these integrals are absolutely convergent. For a k-form α the $(k+1)$-form $d^*\alpha$ is defined to be $d^*\alpha = (-1)^{m(k+1)+1} * d * \alpha$. The operator d^* is the 'formal adjoint' of d in the sense that for all k-forms α and $(k+1)$-forms β such that either α or β has compact support we have the equality

$$\langle d^*\alpha, \beta \rangle = \langle \alpha, d\beta \rangle. \tag{1}$$

Let $L^i_{(2)}(Y)$ be the space of i-forms α with measurable coefficients such that $\int_Y (\alpha, \alpha)\, d\nu$ is absolutely convergent. Define the domains $\operatorname{dom} d$ and $\operatorname{dom} d^*$ of the operators d and d^* by

$$\operatorname{dom} d^i = \left\{ \alpha \text{ is a } \textit{smooth } i\text{-form } : \alpha \in L^i_{(2)}(Y) \text{ and } d\alpha \in L^{i+1}_{(2)}(Y) \right\},$$
$$\operatorname{dom} (d^*)^i = \left\{ \alpha \text{ is a } \textit{smooth } i\text{-form } : \alpha \in L^i_{(2)}(Y) \text{ and } d^*\alpha \in L^{i-1}_{(2)}(Y) \right\}.$$

Then $\operatorname{dom} d$ is a cochain complex, whose cohomology is denoted by $H^{\cdot}_{(2)}(Y)$, or by $H^{\cdot}_{(2),\nu}(Y)$ if we want to stress its dependence on the metric ν. Also, $\operatorname{dom} d^*$ is a chain complex and the Hodge $*$-operator induces an isomorphism $H_i(\operatorname{dom} d^*) \simeq H^{m-i}_{(2)}(Y)$, where m is the dimension of Y. Formula (1) implies that the operators d

and d^* with the domains defined above have functional-analytic closures, denoted by respectively $\bar{d}$ and $\bar{d}^*$. By means of a regularization argument one can show that the inclusion $\operatorname{dom} d^\cdot \hookrightarrow \operatorname{dom} \bar{d}^\cdot$ induces an isomorphism on cohomology. (See Cheeger [5].) In other words,

$$H^i_{(2)}(Y) = \ker d^i / \operatorname{im} d^{i-1} \cong \ker \bar{d}^i / \operatorname{im} \bar{d}^{i-1}.$$

A (distributional) differential form α is called *harmonic* if $d\alpha = d^*\alpha = 0$ in the weak (i.e. distributional) sense. The ellipticity of the Laplacian $\Delta = dd^* + d^*d$ implies that harmonic forms are smooth. The group of harmonic $L_{(2)}$-forms is denoted by $\mathfrak{H}^\cdot_{(2)}(Y)$. One has

$$\mathfrak{H}^i_{(2)}(Y) = \ker d^i \cap \ker(d^*)^i = \ker \bar{d}^i \cap \ker(\bar{d}^*)^i.$$

It is not hard to show that the Hodge $*$-operator restricts to give an isomorphism $\mathfrak{H}^i_{(2)}(Y) \cong \mathfrak{H}^{m-i}_{(2)}(Y)$, if Y is orientable. There is a natural map $\mathfrak{H}^\cdot_{(2)}(Y) \to H^\cdot_{(2)}(Y)$ from harmonic forms to $L_{(2)}$-cohomology, which is in general neither injective nor surjective.

A diffeomorphism Φ from Y to another Riemannian manifold Y' with metric ν' is called a *quasi-isometry* if there is a positive real number C such that for all $p \in Y$ and $u, v \in T_pY$

$$C^{-1}\nu'(\Phi_*u, \Phi_*v) \le \nu(u, v) \le C\nu'(\Phi_*u, \Phi_*v).$$

It is easy to see that a quasi-isometry Φ induces an isomorphism $\Phi^* : H^\cdot_{(2),\nu'}(Y') \xrightarrow{\cong} H^\cdot_{(2),\nu}(Y)$. If $Y = Y'$, the metrics ν and ν' are said to be *quasi-isometric*, if the identity mapping is a quasi-isometry from Y with metric ν to Y with metric ν'.

Definition 2. The *metric cylinder* on Y is the cartesian product $Y \times (0, 1)$ equipped with the product Riemannian metric. The *metric cone* CY on Y is the product $CY = Y \times (0, 1)$ with metric $r^2\nu \oplus dr \otimes dr$, where r is the standard coordinate on $(0, 1)$.

Theorem 3 (Cheeger [5])**.** *The projection on the first factor $p : Y \times (0, 1) \to Y$ induces an isomorphism $H^\cdot_{(2)}(Y) \xrightarrow{\cong} H^\cdot_{(2)}(Y \times (0, 1))$.*

Theorem 4 (loc. cit.)**.** a. *The projection on the first factor $p : CY \to Y$ induces an isomorphism $H^i_{(2)}(Y) \xrightarrow{\cong} H^i_{(2)}(CY)$ for $i \le [m/2]$ (where m is the dimension of Y).*
b. *$H^i_{(2)}(CY) = 0$ for $i > (m + 1)/2$.*
c. *If m is odd, $i = (m + 1)/2$, and the image of $\bar{d}^{i-1}_Y$ is closed, then $H^i_{(2)}(CY) = 0$.*

2. An $L_{(2)}$ de Rham Theorem for Orbit Spaces

Let (M, μ) be a compact Riemannian manifold and let G be a compact Lie group acting on M by isometries. Let π denote the projection from M onto the orbit space M/G. Pick a point m in M. Let H be the stabilizer of m and let $V = (T_m(G\cdot m))^\perp$ denote the orthogonal complement in T_mM (with respect to the metric μ) of the tangent space $T_m(G\cdot m)$ of the orbit. Then V is an orthogonal representation space for H and the normal bundle of the orbit $G\cdot m$ is the vector bundle $G \times_H V$ associated to the principal fibre bundle $G \to H$. The bundle $G \times_H V$ is a G-space (where

G acts by left translations on the first factor) and the exponential map defined by the metric provides a G-equivariant diffeomorphism from a neighbourhood of the zero section of $G \times_H V$ to a G-invariant neighbourhood of the orbit $G \cdot m$. (This is equivalent to the familiar slice theorem, see e.g. Koszul [10] or Bredon [4].) For a subgroup H of G denote by $M_{(H)}$ the set of all points whose stabilizer is conjugate to H,

$$M_{(H)} = \{\, m \in M : G_m \text{ is conjugate to } H \,\}.$$

By virtue of the slice theorem the set $M_{(H)}$ is a smooth submanifold of M (possibly consisting of components of different dimensions), called the manifold of *orbit type* (H), and the quotient $M_{(H)}/G$ is also a smooth manifold. Thus we have a decomposition $M/G = \coprod_{H<G} M_{(H)}/G$ of the orbit space M/G into a disjoint union of manifolds, and it is easy to see that this decomposition is a stratification of M/G in the sense of [7]. If V^H denotes the subspace of fixed vectors in V and W the orthogonal complement $(V^H)^\perp$, then the point $\pi(m) \in M/G$ has a neighbourhood O of the form $O = B(V^H) \times B(W)/H$. Here $B(V^H)$ denotes a ball round the origin in V^H and $B(W)$ a ball round the origin in W. A neighbourhood of this form will be called a *distinguished neighbourhood*.

For convenience we shall assume that the orbit space is connected. This implies that there exists a maximal orbit type (K), in the sense that for all orbit types (H) occurring in the G-manifold M the group K is conjugate to a subgroup of H (see Bredon [4]). The stratum $M_{(K)}/G$ is connected, open and dense in M/G. It follows from the slice theorem that $M_{(K)}$ consists precisely of those points m where the representation of $H = G_m$ on V is trivial, or equivalently, where the vector space $W = (V^H)^\perp$ is 0. The open dense strata of M and M/G are called *principal*, and they are denoted by respectively M^{princ} and M^{princ}/G. For each open part U of M, resp. M/G, define $U^{\mathrm{princ}} = U \cap M^{\mathrm{princ}}$, resp. $U^{\mathrm{princ}} = U \cap (M^{\mathrm{princ}}/G)$.

The metric μ induces a Riemannian metric on M^{princ}/G, denoted by $\pi_*\mu$. If m is a principal point in M, then the restriction of π_* to the slice V at m is an isometry $V \xrightarrow{\cong} T_{\pi(m)}(M^{\mathrm{princ}}/G)$. Note that all metrics obtained in this way from a G-invariant metric on M are quasi-isometric to one another. Hence $H^{\cdot}_{(2)}(M^{\mathrm{princ}}/G)$ does not depend on the metric μ. Let us describe the behaviour of the metric $\pi_*\mu$ up to quasi-isometry near the singularities of M/G. Let $m \in M$ and let $O = B(V^H) \times B(W)/H \subset M/G$ be a distinguished neighbourhood of the point $\pi(m) \in M/G$. The principal stratum of O comes equipped with two Riemannian metrics: the restriction of $\pi_*\mu$ and the metric $\tilde{\mu}_m$ induced by the inner product μ_m on T_mM. If we choose O small enough, these metrics are obviously quasi-isometric. So in order to study the metric $\pi_*\mu$ near $\pi(m)$ and the local $L_{(2)}$-cohomology of M/G at $\pi(m)$, we can turn our attention to O with the metric $\tilde{\mu}_m$. Let $S(W)$ denote the unit sphere in the vector space W. If m is not a principal point in M, then $W \neq \{0\}$ and the map

$$\Psi : B(V^H) \times S(W)^{\mathrm{princ}} \times (0,1) \to B(V^H) \times B(W)^{\mathrm{princ}}$$

sending (y, z, r) to (y, rz) is an H-equivariant diffeomorphism. It becomes an isometry as well, if we endow the product $S(W)^{\mathrm{princ}} \times (0,1)$ with the conical metric $r^2\nu \oplus dr \otimes dr$, where ν is the metric $\mu_m|_{S(W)^{\mathrm{princ}}}$ and r the standard coordinate on $(0,1)$. So Ψ descends to an isometry

$$B(V^H) \times S(W)^{\mathrm{princ}}/H \times (0,1) \xrightarrow{\cong} B(V^H) \times B(W)^{\mathrm{princ}}/H = O^{\mathrm{princ}},$$

where, again, $S(W)^{\mathrm{princ}}/H \times (0,1)$ is equipped with a conical metric. The orbit space $S(W)/H$ is called the *link* of the point $\pi(m)$.

There are two naturally defined complexes of sheaves on the orbit space.

Definition 5. For any open subset U of M/G, let $\Omega^{\cdot}(U)$ be the set of all smooth basic differential forms on the G-invariant subset $\pi^{-1}(U)$ of M. (See the Introduction for the definition of basic forms.) The *de Rham complex* of the orbit space is the complex of sheaves $\Omega^{\cdot}$ assigning to U the complex of vector spaces $\Omega^{\cdot}(U)$. The *$L_{(2)}$-complex* is the complex of sheaves $\mathcal{L}^{\cdot}$ generated by the complex of presheaves $U \mapsto \operatorname{dom} d^{\cdot}_{U^{\mathrm{princ}}}$. So an element of $\mathcal{L}^i(U)$ is an i-form ω on U^{princ} with the following property: For all $x \in U$ there is an open neighbourhood $O \subset U$ of x such that $\omega|_{O^{\mathrm{princ}}} \in \operatorname{dom} d^i_{O^{\mathrm{princ}}}$.

If ω is an element of $\mathcal{L}^{\cdot}(U)$, we shall say that it is a form on U^{princ} such that ω and $d\omega$ are 'square-integrable locally on U'. The complex of vector spaces $\mathcal{L}^{\cdot}(U)$ is not necessarily the same as the complex $\operatorname{dom} d^{\cdot}$ of the Riemannian manifold U^{princ}, but may depend on the inclusion $U^{\mathrm{princ}} \hookrightarrow U$. But since M/G is compact, $\mathcal{L}^{\cdot}(M/G) = \operatorname{dom} d^{\cdot}_{(M^{\mathrm{princ}}/G)}$, and the cohomology $H^{\cdot}(\mathcal{L}^{\cdot}(M/G))$ of the complex of global sections of $\mathcal{L}^{\cdot}$ is equal to the $L_{(2)}$-cohomology of the Riemannian manifold M^{princ}/G with metric $\pi_*\mu$.

Proposition 6. a. *For all open subsets U of M/G, $\Omega^{\cdot}(U)$ is contained in $\mathcal{L}^{\cdot}(U)$.*
b. *The $L_{(2)}$-complex $\mathcal{L}^{\cdot}$ is a module over the de Rham complex $\Omega^{\cdot}$.*
c. *The complex $\mathcal{L}^{\cdot}$ is fine.*

Proof. a. Let U be an open subset of the orbit space and let ω be a smooth basic form on $\pi^{-1}(U)$. We want to show that the forms $\omega|_{\pi^{-1}(U)^{\mathrm{princ}}}$ and $d\omega|_{\pi^{-1}(U)^{\mathrm{princ}}}$, which can be regarded in a natural way as differential forms on U^{princ}, are square-integrable locally on U. Let $x \in U$. It follows from the fact that the metric $\pi_*\mu$ is conical near the singularities that x has a neighbourhood $O \subset U$ such that O^{princ} has finite volume. Moreover, it is easy to see from the fact that ω is a smooth form on the whole of $\pi^{-1}(U)$, that x also has a neighbourhood $O' \subset U$ such that $|\omega|$ and $|d\omega|$ are bounded on O'. Hence, $\int_{O\cap O'} |\omega|^2\, d(\pi_*\mu)$ and $\int_{O\cap O'} |d\omega|^2\, d(\pi_*\mu)$ are finite.
b. If $U \subset M/G$ is open and $\omega \in \mathcal{L}^{\cdot}(U)$ and $\alpha \in \Omega^{\cdot}(U)$, then on U^{princ} one has

$$\begin{aligned} |\alpha \wedge \omega| &= |\alpha| \cdot |\omega| \\ |d(\alpha \wedge \omega)| &\leq |d\alpha| \cdot |\omega| + |\alpha| \cdot |d\omega|. \end{aligned}$$

From the fact that $|\alpha|$ and $|d\alpha|$ are bounded locally on U (see the proof of statement 1), it now follows that $\alpha \wedge \omega$ and $d(\alpha \wedge \omega)$ are square-integrable locally on U.
c. The sheaf Ω^0 is fine, because there exist G-invariant smooth partitions of unity subordinate to any cover of M with G-invariant open subsets. (These can be obtained e.g. by averaging an arbitrary partition of unity subordinate to the given cover with respect to the Haar measure on G.) Hence, by a standard theorem of sheaf theory (see e.g. Bredon [3, Chapter II]), the module $\mathcal{L}^{\cdot}$ over Ω^0 is also fine. □

Sketch proof of Theorem 1. Let S_k denote the union of all k-dimensional strata of the orbit space M/G, let n be the dimension of M/G, and let $U_0 \subset U_1 \subset U_2 \subset \cdots$ be the increasing sequence of open subsets defined by putting $U_0 = \emptyset$ and $U_{k+1} = U_k \cup S_{n-k}$. Then $U_1 = S_n = M^{\mathrm{princ}}/G$. If $\mathcal{A}$ is a sheaf on M/G, its restriction to U_k will be written as $\mathcal{A}_k$. Let i_k be the inclusion $U_k \hookrightarrow U_{k+1}$. Goresky and MacPherson have given a sheaf-theoretic characterization of intersection cohomology in [8]. I will quote this in a version due to Borel [2]. According to [2, Theorem V 2.3] the hypercohomology of a complex of sheaves $\mathcal{A}^{\cdot}$ is isomorphic to the intersection cohomology $I_{\bar{n}}H^{\cdot}(M/G)$ with respect to the upper middle perversity $\bar{n}$ if the following statements hold:

1. The complex $\mathcal{A}^{\cdot}$ is bounded, $\mathcal{A}^{\cdot} = 0$ for $i < 0$, and $\mathcal{A}_1^{\cdot}$ is a resolution of the constant sheaf $\mathcal{R}$ on the manifold S_n;
2. The derived sheaves $\mathcal{H}^i(\mathcal{A}_{k+1}^{\cdot})$ on U_{k+1} are zero for $i > [(k-1)/2]$, where $k = 1$, $2, \ldots, n$;
3. The attachment map $\alpha_k : \mathcal{A}_{k+1}^{\cdot} \longrightarrow Ri_{k*}\mathcal{A}_k^{\cdot}$ induces an isomorphism of derived sheaves on U_{k+1}, $\alpha_k : \mathcal{H}^i(\mathcal{A}_{k+1}^{\cdot}) \xrightarrow{\cong} \mathcal{H}^i(Ri_{k*}\mathcal{A}_k^{\cdot})$, for $i \leq [(k-1)/2]$, where $k = 1$, $2, \ldots, n$.

Remark 7. Goresky and MacPherson have defined intersection cohomology in [7] only for stratified spaces whose stratum of codimension one is empty. The orbit space M/G, however, may have a codimension one stratum. King [9] has extended the definition of intersection cohomology to arbitrary stratified spaces and shown that it is a topological invariant. The upper middle perversity index of the codimension one strata is defined to be $\bar{n}(1) = 0$. Actually, a codimension one stratum of the orbit space always consists of boundary points. To see this, consider a point x in a codimension one stratum with distinguished neighbourhood $B(V^H) \times B(W)/H$. Then $B(V^H)$ has codimension one in M/G, and so the link $S(W)/H$ must be zero-dimensional. If $\dim W > 1$, then $S(W)$ is connected, so the link must be a point. If $\dim W = 1$, then the action of H must interchange the two points of $S(W)$, so $S(W)/H$ is again a point. It follows that x has a neighbourhood homeomorphic to a half-space $\mathbf{R}^l \times \mathbf{R}_{\geq 0}$.

It is now fairly straightforward to check axioms 1–3 for the complex $\mathcal{L}^{\cdot}$. Axiom 1 follows from the fact that the complex $\mathcal{L}_1^{\cdot}$ is just the de Rham complex of the manifold S_n, which is a resolution of $\mathcal{R}$ by the Poincaré lemma. Axioms 2 and 3 follow from Theorems 3 and 4 of Cheeger and from the observation above that the metric is conical near the singularities. An outline of the argument can be found in [11]. This leads to the conclusion that the hypercohomology of the complex of sheaves $\mathcal{L}^{\cdot}$ is isomorphic to $I_{\bar{n}}H^{\cdot}(M/G)$. But this complex is fine by Proposition 6, so its hypercohomology is equal to $H^{\cdot}(\mathcal{L}^{\cdot}(M/G)) \cong H_{(2)}^{\cdot}(M^{\mathrm{princ}}/G)$. This proves Theorem 1. $\square$

By virtue of Remark 7, Theorem 1 implies that the group $H_{(2)}^{\cdot}(M^{\mathrm{princ}}/G)$ is a topological invariant of the orbit space M/G.

Remark 8. Assume that the group G is finite. Let m be a point in M and let W be the orthogonal complement in T_mM to the subspace of H-fixed vectors, where H is the stabilizer of m. Then it follows from Bredon [3, Theorem 19.1] that

$$H^i(S(W)/H) \cong H^i(S(W))^H = \begin{cases} \mathbf{R} & \text{if } i = 0; \\ 0 & \text{if } 0 < i < \dim S(W). \end{cases}$$

(If i equals the dimension of $S(W)$, one has $H^i(S(W))^H = \mathbf{R}$, resp. $= 0$, if the action of H on W does, resp. does not, preserve an orientation of W. Therefore, M/G is a rational homology manifold if and only if for all $m \in M$ the action of H on T_mM is orientation-preserving.) Using this, one can easily show that the sheaf $\Omega^{\cdot}$ of invariant forms on M/G satisfies the axioms of intersection cohomology. Hence, the inclusion $\Omega^{\cdot} \hookrightarrow \mathcal{L}^{\cdot}$ induces an isomorphism $H^{\cdot}(M/G) \xrightarrow{\cong} H^{\cdot}_{(2)}(M^{\text{princ}}/G)$. Note that we do not require the action to be orientation-preserving. An analogous statement is true for (compact) V-manifolds (i.e., spaces that are locally modelled on quotient spaces of manifolds by finite group actions, see Satake [13]).

Remark 9. One of the most important properties of intersection cohomology with respect to the upper middle perversity is that it satisfies Poincaré duality for a large class of singular spaces. A necessary condition for this to hold was stated in [5] and [8]. For orbit spaces it reads as follows: For any m in M such that the image $\pi(m) \in M/G$ lies in a stratum of *odd* codimension k and for all distinguished neighbourhoods $O \approx B(V^H) \times B(W)/H$ of $\pi(m)$, the intersection cohomology (or, equivalently, the $L_{(2)}$-cohomology) of the link $S(W)/H$ in middle dimension vanishes:

$$H^{(k-1)/2}_{(2)}(S(W)^{\text{princ}}/H) \cong I_{\bar{n}}H^{(k-1)/2}(S(W)/H) = 0. \tag{2}$$

If this is the case, we have the generalized Poincaré isomorphism

$$H^i_{(2)}(M^{\text{princ}}/G) \cong H^{n-i}_{(2)}(M/G^{\text{princ}}, \mathcal{O})^*,$$

where $\mathcal{O}$ denotes the orientation bundle of M^{princ}/G (with a constant fibre metric). Cheeger [5] has shown that in this case we also have a Hodge-de Rham isomorphism $H^i_{(2)}(M^{\text{princ}}/G) \cong \mathfrak{H}^i_{(2)}(M^{\text{princ}}/G)$. Of course, condition (2) is fulfilled at all points $m \in M$ if there are no strata of odd codimension in M/G. For finite G one can easily deduce from Remark 8 that (2) holds if and only if the orbit space has no boundary points. So for V-manifolds without boundary we find a Hodge-de Rham theorem, which was already known to Baily [1]. The following proposition gives a necessary and sufficient condition for (2) to hold for orbit spaces of circle actions. It would be interesting to have an analogous result on actions of higher-dimensional groups. In the proof we use the notion of a regular point.

Definition 10. A point m in M is called *regular* with respect to the action of G if all orbits near m are of the same dimension as the orbit through m. Equivalently, m is regular if the kernel of the slice representation $H \to Gl(V)$ has finite index in H.

All points in M^{princ} are regular, but a regular point is not necessarily principal.

Proposition 11. *Assume that G is the circle. Then (2) holds if and only if*

a. *M/G has no boundary, and*

b. *The projection $\pi(M^G) \subset M/G$ of the fixed point set M^G has no connected components of codimension* $1 \bmod 4$ *in M/G.*

Proof. Let k be an odd integer, let m be a point in M such that $\pi(m)$ lies in a stratum of codimension k in M/G, and let $O \approx B(V^H) \times B(W)/H$ be a distinguished neighbourhood of $\pi(m)$. We may assume that H acts effectively on W. (Otherwise replace it by its image under the homomorphism $H \to Gl(W)$.) Therefore, $\dim W = k + \dim H$. The origin in W is the only fixed point, so m is regular if and only if H is finite. Since G is the circle, every point in M is either regular or a fixed point.
Case 1: Suppose the point m is regular. Then H is finite, $k = \dim W$ and Remark 8 implies that

$$I_{\bar{n}}H^{(k-1)/2}(S(W)/H) \cong H^{(k-1)/2}(S(W)/H) \cong H^{(k-1)/2}(S(W))^H. \tag{3}$$

If $k > 1$, this cohomology group is 0. If $k = 1$, the group H must be $\mathbf{Z}/2\mathbf{Z}$ acting by reflection on the one-dimensional space W, and $S(W)$ consists of two points. Hence $H^0(S(W))^H = \mathbf{R}$.
Case 2: Suppose m is a fixed point. Then $H \cong S^1$. The action of H on $S(W)$ has no fixed points, hence each point on $S(W)$ has finite stabilizer. Therefore $S(W)/H$ is a V-manifold. So, by Remark 8, its intersection cohomology is the same as its ordinary cohomology.

The vector space W is a direct sum of two-dimensional subspaces on each of which H acts by rotations with a certain angular velocity. So one can introduce complex coordinates $(z_1, \dots, z_r)$ on W (where $r = (k-1)/2$) such that $H \cong S^1 = \{\, t \in \mathbf{C} : |t| = 1 \,\}$ acts by $t \cdot (z_1, \dots, z_r) = (t^{w_1} z_1, \dots, t^{w_r} z_r)$, with $w_1, \dots, w_r \in \mathbf{Z}_{>0}$. If $w_1 = \dots = w_r = 1$, then $S(W)/H$ is the complex projective space $\mathbf{P}(W)$ of W with homogeneous coordinates $(z_1 : \dots : z_r)$, and $S(W) \to \mathbf{P}(W)$ is the Hopf fibration. Now let $w_1, \dots, w_r$ be arbitrary positive integers. There is an action of the product $R := \mathbf{Z}/w_1\mathbf{Z} \times \dots \times \mathbf{Z}/w_r\mathbf{Z}$ on $\mathbf{P}(W)$, defined by multiplication of the homogeneous coordinate z_i with the w_i-th roots of unity. The quotient $\mathbf{P}(W)/R$ is the weighted projective space with weights $w_1, \dots, w_r$. It is easy to see that the mapping $S(W) \to \mathbf{P}(W)/R$, assigning

$$(z_1, \dots, z_r) \mapsto (\sqrt[w_1]{z_1} : \dots : \sqrt[w_r]{z_r}) \bmod R,$$

descends to a homeomorphism of $S(W)/H$ onto $\mathbf{P}(W)/R$. Moreover, all (rational) cohomology classes of $\mathbf{P}(W)$ are invariant under the action of R. Therefore,

$$\begin{aligned} H^i(S(W)/H) &\cong H^i(\mathbf{P}(W)/R) \cong H^i(\mathbf{P}(W))^R \\ &= H^i(\mathbf{P}(W)) \cong \begin{cases} \mathbf{R} & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Consequently, $H^{(k-1)/2}(S(W)/H) \neq 0$ if and only if $k = 4l + 1$ with $l \in \mathbf{N}$.

It follows from this and (3) that condition (2) is fulfilled at the point $m \in M$ if and only if either of the following two conditions holds:
a. m is regular and $\pi(m)$ lies in a stratum of codimension $k \neq 1$, or
b. m is not regular (so it must be a fixed point) and $\pi(m)$ lies in a stratum of codimension $k \neq 1 \bmod 4$.
This finishes the proof of the Proposition. Note that we did not assume M to be orientable or the action to be orientation-preserving. □

Example 12. Consider the product $\mathbf{R} \times \mathbf{C}^n$ with the circle action defined by $e^{i\theta} \cdot (t, z) = (t, e^{i\theta} z)$. The action restricts to the sphere $S^{2n} \subset \mathbf{R} \times \mathbf{C}^n$, where it has two fixed points, the north pole $(1,0)$ and the south pole $(-1,0)$. The orbit space $X := S^{2n}/S^1$ is $(2n-1)$-dimensional, so by Proposition 11 the intersection complex of X is 'self-dual' if and only if n is even. The principal stratum X^{princ} is a product of the complex projective space $\mathbf{C}P^{n-1}$ and an open interval, say $(0,1)$. The metric is a 'Riemannian suspension', $\sin^2 \pi r \cdot \mu \oplus dr \otimes dr$, where μ is the real part of the Fubini-Study metric on $\mathbf{C}P^{n-1}$. The space X^{princ} has an open cover consisting of two open subsets, $\mathbf{C}P^{n-1} \times (0, 2/3)$ and $\mathbf{C}P^{n-1} \times (1/3, 1)$, which are both quasi-isometric to the metric cone on $\mathbf{C}P^{n-1}$. Their intersection is quasi-isometric to the metric cylinder on $\mathbf{C}P^{n-1}$. A computation using the Mayer-Vietoris sequence for intersection cohomology, or, alternatively, the $L_{(2)}$-version of the Mayer-Vietoris sequence (see Cheeger [5]) yields:

$$\begin{array}{rcll} I_{\bar{n}}H^i(X) & = & H^i(\mathbf{C}P^{n-1}) & \text{for } i < n; \\ I_{\bar{n}}H^n(X) & = & 0; & \\ I_{\bar{n}}H^i(X) & = & H^{i-1}(\mathbf{C}P^{n-1}) & \text{for } i > n. \end{array}$$

We see that Poincaré duality is violated for odd n.

We can find harmonic representatives in $L^{\cdot}_{(2)}(X^{\text{princ}})$ for all intersection cohomology classes as follows. If α is any k-form on X^{princ}, we can write $\alpha = \alpha_1 + \alpha_2 \wedge dr$, with $\alpha_2 = \iota(\partial/\partial r)\alpha$ and $\alpha_1 = \alpha - \alpha_2 \wedge dr$. Let $*$ be the Hodge operator on X^{princ}. A straightforward computation yields:

$$*\alpha = \sin^{2(n-k-1)}(\pi r)(*'\alpha_1) \wedge dr + (-1)^{k-1} \sin^{2(n-k)}(\pi r)(*'\alpha_2), \tag{4}$$

where $*'$ is the Hodge operator on $\mathbf{C}P^{n-1}$. Similarly, if α is square-integrable, we find:

(5)
$$\|\alpha\|^2 = \int_0^1 \left\{ (\sin \pi r)^{2(n-k-1)} \|i_r^* \alpha\|^2_{\mathbf{C}P^{n-1}} + (\sin \pi r)^{2(n-k)} \|i_r^*(\iota(\partial/\partial r)\alpha)\|^2_{\mathbf{C}P^{n-1}} \right\} dr.$$

Now let ω be the imaginary part of the Fubini-Study metric on $\mathbf{C}P^{n-1}$. Then ω is a harmonic two-form, as are its powers ω^i. The ω^i are harmonic representatives for a basis in $H^{\cdot}(\mathbf{C}P^{n-1})$. Let $p : X^{\text{princ}} = \mathbf{C}P^{n-1} \times (0,1) \to \mathbf{C}P^{n-1}$ be the cartesian projection. Put $\zeta = p^*(\omega)$. Formula (4) implies that the following forms are harmonic on X^{princ}: ζ^i and $*(\zeta^i)$ for $i = 0, \ldots, n-1$; and also $\zeta^{(n-1)/2} \wedge dr$ if n is odd. It follows from (5) that $\zeta^{(n-1)/2} \wedge dr \in L^n_{(2)}$; $\zeta^i \in L^{2i}_{(2)}$ iff $i < n$; and $*(\zeta^i) \in L^{2(n-i)-1}_{(2)}$ iff $i > n$. In particular, we see that, for n odd, $\mathfrak{H}^n_{(2)}(X^{\text{princ}}) \neq 0$, while $H^n_{(2)}(X^{\text{princ}}) = 0$, so that the Hodge-de Rham isomorphism breaks down.

References

[1] W. Baily, The Decomposition Theorem for V-Manifolds, *Amer. J. of Math.* **78** (1956), 862–888.

[2] A. Borel et alii, *Intersection Cohomology*, Birkhäuser, Boston, 1984.

[3] G. Bredon, *Sheaf Theory*, McGraw-Hill, New York, 1967.

[4] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[5] J. Cheeger, On the Hodge Theory of Riemannian Pseudomanifolds, in: *Proc. of Symposia in Pure Mathematics* **36** (1980), 91–146.

[6] J. Cheeger, M. Goresky and R. MacPherson, $L_{(2)}$-Cohomology and Intersection Homology for Singular Algebraic Varieties, in: *Proceedings of year in differential geometry*, editor S. Yau, Ann. of Math. Studies, Princeton University Press, Princeton, 1981.

[7] M. Goresky and R. MacPherson, Intersection Homology Theory, *Topology* **19** (1980), 135–162.

[8] M. Goresky and R. MacPherson, Intersection Homology II, *Invent. Math.* **71** (1983), 77–129.

[9] H. C. King, Topological Invariance of Intersection Homology without Sheaves, *Topology and its Applications* **20** (1985), 149–160.

[10] J. L. Koszul, Sur certains groupes de transformations de Lie, dans: *Colloque de Géométrie Différentielle*, Colloques du CNRS **71** (1953), 137–141.

[11] M. Nagase, Sheaf-theoretic $L_{(2)}$-Cohomology, *Complex-analytic Singularities*, Adv. Stud. in Pure Math. **8**, 273–279, North Holland, Amsterdam-New York, 1987.

[12] G. de Rham, *Variétés différentiables*, Hermann, Paris, 1955.

[13] I. Satake, On a Generalization of the Notion of Manifold, *Proc. Nat. Acad. Sc.* **42** (1956), 359–363.

[14] R. Sjamaar, *Singular Orbit Spaces in Riemannian and Symplectic Geometry*, PhD thesis, University of Utrecht, 1990.

[15] S. Zucker, $L_{(2)}$-Cohomology of Warped Products and Arithmetic Groups, *Invent. Math.* **70** (1982), 169–218.