RADICIAL SUBGROUPS AND WEIGHT VARIETIES

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ABSTRACT. Let G be a compact connected Lie group and T a maximal torus of G. In this note I compute the orbit type stratification of the flag manifold G/T viewed as a T-manifold. This is an easy exercise, which has already been done by others, but which I could not find recorded in the literature, and which offers an interesting perspective on the classification of the connected maximal rank subgroups of G.

CONTENTS

1. RADICIAL SUBGROUPS

Let G be a compact connected Lie group. A subgroup C of G is *radicial* if there exist a maximal torus T of G and a subset R' of the root system $R(G,T) \subseteq \mathrm{Hom}(T, \mathbf{U}(1))$ such that $C = \bigcap_{\alpha \in R'} \ker \alpha$. We may without loss of generality assume the subset R' to be *closed* (i.e. if $\alpha, \beta \in R'$ and $\alpha + \beta \in R(G,T)$ then $\alpha + \beta \in R'$) and *symmetric* (i.e. $-R' = R'$). (We follow for the most part the terminology and notation of Bourbaki [2].)

1.1. **Example.** Maximal tori of G are radicial.

1.2. **Example.** The centre $C(G)$ of G is radicial: take any maximal torus T; then $C(G) = \bigcap_{\alpha \in R(G,T)} \ker \alpha$.

1.3. **Example.** Let H be a closed connected subgroup of maximal rank of G. Then $C(H)$ is a radicial subgroup of G: take any maximal torus T of G contained in H; then $R(H,T)$, the root system of H with respect to T, is contained in $R(G,T)$ and $C(H) = \bigcap_{\alpha \in R(H,T)} \ker \alpha$. This example contains as special cases both previous examples.

Now fix a maximal torus T of G and denote by R or R_G the root system $R(G,T)$. Let $\mathscr{P}$ be the (finite) collection of closed symmetric subsets of R; let $\mathscr{R}$ be the collection of radicial subgroups of G which are contained in T; and let $\mathscr{M}$ be the collection of closed connected subgroups of G which contain T. The three collections $\mathscr{P}$, $\mathscr{R}$ and $\mathscr{M}$ are partially ordered by inclusion. Define a map $\mathscr{P} \to \mathscr{R}$ by $R' \mapsto \bigcap_{\alpha \in R'} \ker \alpha$. Define a map $\mathscr{R} \to \mathscr{M}$ by $C \mapsto Z_G(C)_0$, the identity component

Date: 2009-06-09.

of the centralizer $Z_G(C)$ of C in G. Define a map $\mathscr{M} \to \mathscr{P}$ by $H \mapsto R_H = R(H, T)$. See [1] and also [2, § IX.4.7, Proposition 12, Remarque 2] for the following result.

1.4. **Proposition.** *The maps*

$$\begin{tikzcd} & \mathscr{P} \arrow[dl, "\sim"'] & \\ \mathscr{R} \arrow[rr, "\sim"] & & \mathscr{M} \arrow[ul, "\sim"'] \end{tikzcd}$$

defined above are bijections and the diagram commutes. The inverse of $\mathscr{R} \to \mathscr{M}$ is the map $\mathscr{M} \to \mathscr{R}$ defined by $H \mapsto C(H)$. The bijection $\mathscr{M} \to \mathscr{P}$ is increasing; the bijections $\mathscr{P} \to \mathscr{R}$ and $\mathscr{R} \to \mathscr{M}$ are decreasing (with respect to the given partial orderings).

To compare the various radicial subgroups to one another, we introduce the *Pontryagin dual* or *character group*

$$\mathscr{X}^*(A) = \operatorname{Hom}(A, \mathbf{U}(1))$$

and the *cocharacter group*

$$\mathscr{X}_*(A) = \operatorname{Hom}(\mathbf{U}(1), A)$$

of an abelian Lie group A. These are abelian Lie groups, whose group laws are usually written additively. The evaluation map $\mathscr{X}^*(A) \times A \to \mathbf{U}(1)$ is a $\mathbf{Z}$-bilinear pairing, which induces an isomorphism $A \to \mathscr{X}^*\mathscr{X}^*(A)$ (Pontryagin duality, [3, § II.1.5, Théorème 2]). In particular, the groups A and $\mathscr{X}^*(A)$ determine each other. If A is finite or isomorphic to $\mathbf{R}^n$, A is (noncanonically) isomorphic to $\mathscr{X}^*(A)$. An integer-valued bilinear pairing $\mathscr{X}^*(A) \times \mathscr{X}_*(A) \to \mathbf{Z}$ is defined by $\langle \chi, \lambda \rangle = \deg(\chi \circ \lambda)$, the degree of the endomorphism $\chi \circ \lambda$ of $\mathbf{U}(1)$. This pairing induces a map $\mathscr{X}^*(A) \to \operatorname{Hom}_{\mathbf{Z}}(\mathscr{X}_*(A), \mathbf{Z})$, which is an isomorphism if A is compact and connected.

For a $\mathbf{Z}$-module M we denote by $M_{\mathbf{R}}$ the real vector space $\mathbf{R} \otimes_{\mathbf{Z}} M$.

The root system R_G is contained in $\mathscr{X}^*(T)$. The *root lattice* is the subgroup $Q = Q_G = \mathbf{Z} \cdot R_G$ generated by R_G. (It is a lattice in its linear span $\mathbf{R} \cdot R_G \subseteq \mathscr{X}^*(T)_{\mathbf{R}}$, and it depends only on the root system.) The dual root system $R_G^\vee$ is contained in $\mathscr{X}_*(T)$. The *group of coweights* $P^\vee = P_G^\vee$ is the set of those $\lambda \in \mathscr{X}_*(T)_{\mathbf{R}}$ satisfying $\langle \alpha, \lambda \rangle \in \mathbf{Z}$ for all $\alpha \in R_G$. (If G is not semisimple, the coweight group is not discrete. But the intersection $P_G^\vee \cap \mathbf{R} \cdot R_G^\vee$ is a lattice in $\mathbf{R} \cdot R_G^\vee$, called the *coweight lattice*, which depends only on the root system.) Dually, we have the *coroot lattice* $Q^\vee = Q_G^\vee = \mathbf{Z} \cdot R_G^\vee \subseteq \mathscr{X}_*(T)$ and the *weight group*

$$P = P_G = \{\, \chi \in \mathscr{X}^*(T)_{\mathbf{R}} \mid \langle \chi, R_G^\vee \rangle \subseteq \mathbf{Z} \,\},$$

which is the group of all $\tilde{T}$-weights of all representations $\tilde{G} \to \mathbf{U}(n)$ of the universal covering group $\phi\colon \tilde{G} \to G$ with respect to the maximal torus $\tilde{T} = \phi^{-1}(T)$. Similarly, for each $H \in \mathscr{M}$ we have the corresponding data $R_H = R(H, T)$, $R_H^\vee$, Q_H, $P_H^\vee$, $Q_H^\vee$, and P_H.

We identify $\mathfrak{u}(1) = \operatorname{Lie}(\mathbf{U}(1))$ with $i\mathbf{R}$ in such a way that the exponential map $\mathfrak{u}(1) \to \mathbf{U}(1)$ is given by $\exp x = e^x$; then the kernel of exp is $2\pi i\mathbf{Z}$. The map $\mathscr{X}_*(T) \to \mathfrak{t} = \operatorname{Lie}(T)$ which sends a cocharacter $\lambda\colon \mathbf{U}(1) \to T$ to the element $d_1\lambda(2\pi i) \in \mathfrak{t}$ is an isomorphism from $\mathscr{X}_*(T)$ onto the kernel of the exponential map $\exp_T\colon \mathfrak{t} \to T$. This isomorphisms extends to an isomorphism $\mathscr{X}_*(T)_{\mathbf{R}} \cong \mathfrak{t}$, which maps $P_G^\vee$ onto $\exp_T^{-1}(C(G))$. Dually, the restriction map from $\mathscr{X}^*(T)$ to $\mathscr{X}^*(C(G))$ is surjective and has kernel Q_G. Thus we have natural isomorphisms

$$P_G^\vee/\mathscr{X}_*(T) \xrightarrow{\cong} C(G), \qquad \mathscr{X}^*(T)/Q_G \xrightarrow{\cong} \mathscr{X}^*(C(G)). \tag{1}$$

Similarly, by using the long exact homotopy sequence of the fibre bundle $G \to G/T$ we find canonical isomorphisms

$$\mathscr{X}_*(T)/Q_G^\vee \xrightarrow{\cong} \pi_1(G), \qquad P_G/\mathscr{X}^*(T) \xrightarrow{\cong} \mathscr{X}^*(\pi_1(G)), \tag{2}$$

where we view $\pi_1(G)$ as a discrete abelian group. (See e.g. [2, § IX.4.6, § IX.4.9].)

1.5. **Lemma.** *Let $H \in \mathscr{M}$.*

(i) *The inclusion $C(G) \to C(H)$ induces isomorphisms*

$$C(H)/C(G) \cong P_H^\vee/P_G^\vee, \qquad \mathscr{X}^*\big(C(H)/C(G)\big) \cong Q_G/Q_H.$$

(ii) *The inclusion $H \to G$ induces isomorphisms*

$$\pi_2(G/H,\bar{1}) \cong Q_G^\vee/Q_H^\vee, \qquad \mathscr{X}^*\big(\pi_2(G/H,\bar{1})\big) \cong P_H/P_G.$$

(iii) *The following three conditions are equivalent: $C(H)$ has the same dimension as $C(G)$; $\pi_2(G/H,\bar{1})$ is finite; R and R_H span the same linear subspace of $\mathscr{X}^*(T)_{\mathbf{R}}$.*

(iv) *$C(H)$ has the same number of connected components as $C(G)$ if and only if Q_H is a direct summand of Q_G.*

Proof. Let $\Gamma = C(H)/C(G)$. The exact sequence

$$0 \longrightarrow C(G) \longrightarrow C(H) \longrightarrow \Gamma \longrightarrow 0$$

together with the first isomorphism of (1) (applied to the groups G and H) proves the first assertion of (i). Dualizing the sequence gives the sequence

$$0 \longrightarrow \mathscr{X}^*(\Gamma) \longrightarrow \mathscr{X}^*(C(H)) \longrightarrow \mathscr{X}^*(C(G)) \longrightarrow 0,$$

which is exact by [3, § II.1.7, Théorème 4]. Combining this with the second isomorphism of (1) proves the second assertion of (i). Let $\Pi = \pi_2(G/H,\bar{1})$. Since $\pi_2(G) = 0$ ([2, § IX.5, Exercice 2]) and $\pi_1(G/H,\bar{1}) = 0$ ([2, § IX.2.4, Proposition 3]), we obtain from the long exact homotopy sequence of the fibre bundle $G \to G/H$ an exact sequence

$$0 \longrightarrow \Pi \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow 0. \tag{3}$$

One now proves (ii) in the same way as (i), but using (2) instead of (1). One deduces (iii) from (i) and (ii) by using that an abelian Lie group A is finite if and only if $\mathscr{X}^*(A)$ is finite. One deduces (iv) from (i) by using that A is connected if and only if $\mathscr{X}^*(A)$ is torsion-free. QED

1.6. **Example.** Let G be an adjoint group of type **A**. Then $C(G) = 1$. The root system R_H of each $H \in \mathscr{M}$ is parabolic, with a basis consisting of a union of mutually orthogonal subsets of a basis of R_G. Therefore Q_H is a direct summand of Q_G. Hence every radicial subgroup of G is connected.

Since G/H is simply connected, by Hurewicz's theorem $\pi_2(G/H,\bar{1})$ is isomorphic to $H_2(G/H,\mathbf{Z})$ for all $H \in \mathscr{M}$.

1.7. **Example.** Let G be of type $\mathbf{G}_2$. Let R_0 be the set of the long roots, which is closed and symmetric and of type $\mathbf{A}_2$, let Q_0, P_0, $Q_0^\vee$ and $P_0^\vee$ be the corresponding lattices, let H_0 be the corresponding subgroup of G and $C_0 = C(H_0)$ its centre. Then $R_0^\vee$ is the subset of the short roots of $R^\vee$ (which is not closed in $R^\vee$), so $Q_0^\vee = Q^\vee$ and $\pi_2(G/H_0, \bar{1}) = 0$ (unsurprisingly since G/H_0 is the six-sphere). Since $\pi_1(G) = 0$, it follows from (3) that $\pi_1(H_0) = 0$ as well, so $H_0 \cong \mathbf{SU}(3)$. We have $C(G) = 1$ and $C(H_0) \cong \mathbf{Z}/3\mathbf{Z}$. An element of $P_0^\vee$ which exponentiates to a generator of C_0 is $\frac{1}{3}\varpi_1^\vee$. (Here $\varpi_1^\vee$ and $\varpi_2^\vee$ are the fundamental coweights of G, i.e. the generators of $P^\vee$ dual to the short simple root α_1 and the long simple root α_2, numbered as in [2, Planche IX].)

1.8. **Example.** Let G be of type $\mathbf{G}_2$ and let us use the notation of Example 1.7. Let $R_1 = \{\pm\alpha_0, \pm\alpha_1\}$, where $\alpha_0 = -3\alpha_1 - 2\alpha_2$ is the lowest root. Then R_1 is closed and symmetric and of type $\mathbf{A}_1 \times \mathbf{A}_1$. Let H_1 be the corresponding subgroup of G (which is semisimple just like H_0) and $C_1 = C(H_1)$ its centre. The coroot lattice of R_1 has index 2 in that of R, so $\pi_1(H_1) \cong Q^\vee/Q_1^\vee \cong \mathbf{Z}/2\mathbf{Z}$. A generator of $\pi_1(H)$ is the class of the loop generated by $\alpha_1^\vee + \alpha_2^\vee \in Q^\vee$. Since $\alpha_1^\vee + \alpha_2^\vee$ is the sum of the fundamental coweights of $R_1^\vee$, the fundamental group $\pi_1(H_1)$ is diagonally embedded in $\mathbf{SU}(2) \times \mathbf{SU}(2)$, the universal cover of H_1. Therefore $H_1 \cong \mathbf{SO}(4)$. The coweight lattice $P^\vee$ has index 2 in $P_1^\vee$, so $C_1 = C(H_1) \cong \mathbf{Z}/2\mathbf{Z}$. An element of $P_1^\vee$ which exponentiates to a generator of C_1 is $\frac{1}{2}\varpi_2^\vee$.

See [2, § VI.2, § IX.5] for the following facts and definitions. An element $g \in G$ is *regular* if $Z_G(g)_0$ is a maximal torus of G. We denote by G_r the set of regular points in G and by T_r its intersection with T. We put $\mathfrak{g}_r = \exp^{-1}(G_r)$ and $\mathfrak{t}_r = \mathfrak{g}_r \cap \mathfrak{t}$. An *alcove* of $\mathfrak{t} = \mathrm{Lie}(T)$ is a connected component of $\mathfrak{t}_r$. Every alcove A of $\mathfrak{t}$ is a product of open simplexes (one for each simple component of $\mathfrak{g}$) and a Euclidean space (namely the Lie algebra of $C(G)$). Thus it makes sense to speak of the *faces* of A.

Now assume that G is simply connected. Then the centralizer of every element of G is connected. An *alcove* of T is a connected component of T_r. An alcove of T is of the form $\exp(A)$, where A is an alcove of $\mathfrak{t}$. The restriction of the exponential map to an alcove A is a diffeomorphism onto its image; by a *face* of $\exp(A)$ we mean the image of a face of A. The W-action on T induces a simply transitive W-action on the collection of alcoves of T.

1.9. **Lemma.** *Assume that G is simply connected. An element $t \in T$ is contained in a finite radicial subgroup of T if and only if t is the vertex of an alcove of T.*

Proof. Let $H = Z_G(t) = Z_G(t)_0$ and $C = C(H)$. Choose an alcove A of T such that $t \in \bar{A}$ and let F be the face of A which contains t. The Lie algebra of C is the linear subspace of $\mathfrak{t}$ which is parallel to the affine subspace spanned by F. Therefore the dimension of C is equal to the dimension of F. Thus we are reduced to showing that t is contained in a finite radicial subgroup if and only if C is finite. The sufficiency of C being finite is clear, because $t \in C$ and C is radicial. Conversely, assume that t is contained in a finite $C' \in \mathscr{R}$. Let $H' = Z_G(C')_0$. Then $H' \leq H$, so $C' \geq C$ (Proposition 1.4), so C is finite. QED

2. ORBIT TYPES OF THE TORUS ACTION

There is a fourth set which is in a natural one-to-one correspondence with the three sets $\mathscr{P}$, $\mathscr{R}$ and $\mathscr{M}$. This is implicit in the results of [8, Ch. I] and [6, § 8]. Let X be the flag manifold of G (i.e. the manifold of maximal tori); identify as usual the homogeneous space X with the quotient G/T by identifying a coset gT with the maximal torus gTg^{-1}. Let $\pi\colon G \to X$ be the quotient map. The torus T acts on X and the stabilizer of $x \in X$ is

$$T_x = T \cap gTg^{-1},$$

where $g \in G$ is any element satisfying $\pi(g) = x$. Let $\mathscr{O}$ be the collection of orbit types of the T-action on X, i.e. the set of those subgroups of T which occur as stabilizers of points in X. Note that $C \geq C(G)$ for all $C \in \mathscr{O}$. For $C \in \mathscr{O}$ let

$$X_C = \{\, x \in X \mid T_x = C \,\}$$

be the stratum of orbit type C and $\bar{X}_C$ its closure. For orbit type strata X_C and $X_{C'}$ put $X_C \leq X_{C'}$ if $X_C \subseteq \bar{X}_{C'}$. This defines a partial ordering on the strata. It follows from the slice theorem ([2, § IX.9.3, Proposition 6]) that $X_C \leq X_{C'}$ implies $C \geq C'$.

Let $W = W_G = N_G(T)/T$ be the Weyl group of G with respect to T. The inclusion $N_G(T) \to G$ induces an injective map $W_G \to X$, the image of which is $X_T = X^T$, the fixed-point set of T. Next recall the Weyl group action on X, which is given by the formula $w \cdot gT = g\dot{w}^{-1}T$, where $\dot{w} \in N_G(T)$ is any element representing $w \in W_G$. This action is simply transitive on the fixed-point set X^T.

For each $H \in \mathscr{M}$ the set $H \cdot X^T$ is a closed H-stable submanifold of X, which is H-equivariantly diffeomorphic to a disjoint union of copies of the flag manifold of H. The Weyl group acts transitively on the set of the connected components of $H \cdot X^T$, and the subgroup stabilizing the component $H \cdot \bar{1}$ is the Weyl group $W_H = N_H(T)/T$ of H. Therefore $H \cdot X^T$ consists of $|W_G|/|W_H|$ copies of H/T.

It turns out that the submanifolds $H \cdot X^T$ (for H ranging over $\mathscr{M}$) are exactly the closures of the orbit type strata of X. The orbit types are the radicial subgroups $C(H)$, and the stratification is isomorphic as a partially ordered set to $\mathscr{M}$.

2.1. **Proposition.** (i) *$\mathscr{O} = \mathscr{R}$.*

(ii) *Let $H \in \mathscr{M}$. Then $\bar{X}_{C(H)} = H \cdot X^T$ and $X_{C(H)} = H \cdot X^T \setminus \bigcup_{K \in \mathscr{M},\, K<H} K \cdot X^T$.*

(iii) *Let $H, K \in \mathscr{M}$. Then $X_{C(K)} \leq X_{C(H)}$ if and only if $K \leq H$.*

(iv) *The principal orbit type is $C(G)$.*

Proof. This is based on the equality

$$\{\, x \in X \mid T_x \neq C(G) \,\} = \bigcup_{H \in \mathscr{M} \setminus \{G\}} H \cdot X^T, \tag{4}$$

which is proved in the same way as [2, § IX.4.7, Corollaire]: for $x \in X$ put $H(x) = Z_G(T_x)_0$, which is an element of $\mathscr{M}$. Choose any $g \in G$ such that $\pi(g) = x$. Then T and gTg^{-1} are maximal tori of $H(x)$, and so there exists $h \in H(x)$ such that $gTg^{-1} = hTh^{-1}$. Hence there is $n \in N_G(T)$ such that $g = hn$; in other words, $x \in H(x) \cdot X^T$. Now let Y be the right-hand side of (4). If $x \in X$ and $T_x \neq C(G)$, then $H(x) \neq G$, so $x \in H(x) \cdot X^T \subseteq Y$. Conversely, if $x \in Y$, then $x \in H \cdot X^T$ for some $H \in \mathscr{M}$ not equal to G. Then the tori T and gTg^{-1} are contained in H, so $T_x \supseteq C(H)$, and hence $T_x \supsetneq C(G)$ by Proposition 1.4. This proves (4). For $H \in \mathscr{M}$

put

$$X(H) = \{\, x \in H \cdot X^T \mid T_x = C(H) \,\}.$$

Then $X(H) \subseteq X_{C(H)}$ by definition. Also the sets $X(H)$ are pairwise disjoint, because if $H, K \in \mathscr{M}$, then $C(H) = C(K)$ implies $H = K$ by Proposition 1.4. By replacing G with H in (4) we find

$$\bigl(H \cdot X^T\bigr) \setminus X(H) = \bigcup_{K \in \mathscr{M},\, K < H} K \cdot X^T. \tag{5}$$

It follows from (4) and (5) that X is the disjoint union of the subsets $X(H)$ over all $H \in \mathscr{M}$. Therefore $X(H) = X_{C(H)}$. Since $\dim K \cdot X^T < \dim H \cdot X^T$ if $K < H$, it follows from (5) that $X_{C(H)}$ is an open and dense subset of $H \cdot X^T$. In particular, $X_{C(H)}$ is nonempty for all $H \in \mathscr{M}$, which proves (i), and $\bar{X}_{C(H)} = H \cdot X^T$, which proves (ii). (iii) follows from (ii). Taking $H = G$ we find that $X_{C(G)}$ is open and dense in X, which proves (iv). QED

2.2. **Corollary.** *The radicial subgroups of G are the subgroups of the form $S \cap T$, where S and T are maximal tori of G.*

Proof. This follows immediately from Proposition 2.1(i). QED

2.3. **Corollary.** *Let G be an adjoint group of type* **A**. *Then the stabilizer of every point in X is connected.*

Proof. This follows immediately from Example 1.6 and Proposition 2.1(i). QED

2.4. **Corollary.** *Let $t \in T$. There exists $x \in X$ such that T_x is finite and $t \in T_x$ if and only if t is of the form* $\exp \xi$, *where ξ is the vertex of an alcove of* $\mathfrak{t}$.

Proof. If G is not semisimple, the alcoves of $\mathfrak{t}$ have no vertices and every point in X has infinite stabilizer, so the assertion is vacuous. Now assume that G is semisimple. Then $g \in G$ is contained in a finite radicial subgroup of G if and only if $\tilde{g}$ is contained in a finite radicial subgroup of $\tilde{G}$, where $\tilde{G}$ is any covering group of G and $\tilde{g}$ an element of $\tilde{G}$ lying over g. Thus we may replace G by its universal covering group. But for simply connected G the result follows from Proposition 2.1 and Lemma 1.9. QED

3. Infinitesimal orbit types

The infinitesimal orbit type stratification of the flag manifold is rather simpler. A subalgebra $\mathfrak{c}$ of $\mathfrak{g} = \mathrm{Lie}(G)$ is *radicial* if there exist a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a closed and symmetric subset R' of the root system $R(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^*$ such that $\mathfrak{c} = \bigcap_{\alpha \in R'} \ker \alpha$. The subset R' corresponding to $\mathfrak{c}$ is *parabolic*, i.e. equal to the set of all roots orthogonal to a certain face F of a chamber of $\mathfrak{t}$; the subalgebra $\mathfrak{c}$ is then the linear span of F. The centralizer $Z_G(\mathfrak{c})$ is a *Levi subgroup* of G. It is connected and the Lie algebra of its centre is $\mathfrak{c}(Z_G(\mathfrak{c})) = \mathfrak{c}$.

Fix a maximal torus T of G and let $R = R(G, T)$ be the root system, which we shall as usual identify with the infinitesimal root system $R(\mathfrak{g}, \mathfrak{t})$, where $\mathfrak{t} = \mathrm{Lie}(T)$. Let $\mathscr{P}_0$ be the collection of all parabolic subsets of R; let $\mathscr{R}_0$ be the collection of all radicial subalgebras of $\mathfrak{g}$ which are contained in $\mathfrak{t}$; and let $\mathscr{M}_0$ be the collection of all Levi subgroups of G that contain T. The bijections of Proposition 1.4 restrict to bijections

$$\begin{tikzcd} & \mathscr{P}_0 \arrow[dl, "\sim"'] & \\ \mathscr{R}_0 \arrow[rr, "\sim"] & & \mathscr{M}_0. \arrow[ul, "\sim"'] \end{tikzcd}$$

The infinitesimal stabilizer of $x \in X$ is

$$\mathfrak{t}_x = \mathfrak{t} \cap \mathrm{Ad}(g)(\mathfrak{t}),$$

where $g \in G$ is any element satisfying $\pi(g) = x$. Let $\mathscr{O}_0$ be the collection of infinitesimal orbit types of the T-action on X, i.e. the set of those subalgebras of $\mathfrak{t}$ which occur as infinitesimal stabilizers of points in X. For $\mathfrak{c} \in \mathscr{O}_0$ let

$$X_{\mathfrak{c}} = \{\, x \in X \mid \mathfrak{t}_x = \mathfrak{c} \,\}$$

be the stratum of infinitesimal orbit type $\mathfrak{c}$ and $\bar{X}_{\mathfrak{c}}$ its closure. The *regular* infinitesimal orbit type is the unique maximal element of $\mathscr{O}_0$. The infinitesimal analogue of Proposition 2.1 is as follows; the proof is almost the same.

3.1. **Proposition.** (i) *$\mathscr{O}_0 = \mathscr{R}_0$.*

(ii) *Let $H \in \mathscr{M}_0$. Then $\bar{X}_{\mathfrak{c}(H)} = H \cdot X^T$ and $X_{\mathfrak{c}(H)} = H \cdot X^T \setminus \bigcup_{K \in \mathscr{M}_0,\, K<H} K \cdot X^T$.*

(iii) *Let $H, K \in \mathscr{M}_0$. Then $X_{\mathfrak{c}(K)} \leq X_{\mathfrak{c}(H)}$ if and only if $K \leq H$.*

(iv) *The regular infinitesimal orbit type is $\mathfrak{c}(G)$.*

4. SINGULARITIES OF WEIGHT VARIETIES

Choose a chamber $\mathscr{C}$ of $\mathfrak{t}^*$ and an element $\lambda \in \mathscr{C}$. The centralizer of λ is T, so the map $G \to \mathfrak{g}^*$ defined by $g \mapsto \mathrm{Ad}(g)^*(\lambda)$ descends to a G-equivariant diffeomorphism of X onto the coadjoint orbit of λ. We shall identify X with this orbit and in this way regard X as a Hamiltonian T-manifold, with the moment map $\Phi\colon X \to \mathfrak{t}^*$ being the restriction to X of the canonical projection $\mathfrak{g}^* \to \mathfrak{t}^*$. For $\mu \in \mathfrak{t}^*$, the symplectic quotient $X(\mu) = \Phi^{-1}(\mu)/T$ is the *weight variety* of *weight* μ of G. (See [8]. For background material on Hamiltonian group actions, see [4, Ch. VIII–X], [9].)

Let us call a point $\mu \in \mathfrak{t}^*$ a *very regular value* of Φ if the T-action on the fibre $\Phi^{-1}(\mu)$ is free. Recall that $x \in X$ is a regular point of Φ if and only if the stabilizer subalgebra $\mathfrak{t}_x$ is trivial. It follows that a very regular value of Φ is a regular value of Φ. By the symplectic reduction theorem, if μ is a very regular value, then $X(\mu)$ is a (possibly empty) smooth symplectic manifold, and if μ is a regular value, then $X(\mu)$ is a (possibly empty) symplectic orbifold. By the stratified symplectic reduction theorem ([12]), for general $\mu \in \mathfrak{t}^*$ the weight variety $X(\mu)$ is a stratified space, whose strata

$$X_C(\mu) = \big(X_C \cap \Phi^{-1}(\mu)\big)/T,$$

where $C \in \mathscr{O}$, are symplectic manifolds. For each μ there is a unique maximal element among the strata of $X(\mu)$, the *μ-principal* stratum, which is connected. Alternatively, we can stratify $X(\mu)$ into the sets

$$X_{\mathfrak{c}}(\mu) = \big(X_{\mathfrak{c}} \cap \Phi^{-1}(\mu)\big)/T,$$

where $\mathfrak{c} \in \mathscr{O}_0$, which are symplectic orbifolds ([11]). This stratification too has a unique maximal element, the *μ-regular* stratum, which is connected as well.

It follows from Proposition 2.1(ii) that the closure of each stratum $X_{C(H)}(\mu)$ of the weight variety is the disjoint union of at most $|W_G|/|W_H|$ copies of a weight variety of the group $H \in \mathscr{M}$.

For $\lambda \in \mathscr{C}$ and $H \in \mathscr{M}$ we denote by $\Delta_H(\lambda)$, or Δ_H, the convex hull of the orbit $W_H \cdot \lambda$. This is a closed convex polytope in $\mathfrak{t}^*$, whose vertex set is equal to $W_H \cdot \lambda$. We denote by $\Delta_H^\circ(\lambda)$, or Δ_H°, the relative interior of Δ_H, i.e. its interior relative to its affine span. We call the open polytopes $w(\Delta_H^\circ)$ for $w \in W_G$ and $H \in \mathscr{M}$ the *generalized faces* of Δ_G.

4.1. **Proposition.** *Let $\lambda \in \mathscr{C}$.*

(i) *Assume that $C(G) = 1$. The set of very regular values of Φ is the complement of $W_G \cdot \bigcup_{H \in \mathscr{M} \setminus \{G\}} \Delta_H$.*

(ii) *Assume that G is semisimple. The set of regular values of Φ is the complement of $W_G \cdot \bigcup_{H \in \mathscr{M}_0 \setminus \{G\}} \Delta_H$. Hence $X(\mu)$ is an orbifold for an open and dense subset of $\mu \in \Delta_G$.*

(iii) $\Phi(X_{C(G)}) = \Phi(X_{\mathfrak{c}(G)}) = \Delta_G^\circ$.

Proof. Let $H \in \mathscr{M}$. By Kostant's convexity theorem [10, Theorem 4.1], $\Phi(H \cdot 1) = \Delta_H$. The moment map is W_G-equivariant, so $\Phi(H \cdot X^T) = W_G \cdot \Delta_H$. By Proposition 2.1(ii), if G has trivial centre, T acts freely at $x \in X$ if and only if x is not in $H \cdot X^T$ for any $H < G$. Thus $\mu \in \mathfrak{t}^*$ is very regular unless $\mu \in W_G \cdot \Delta_H$ for some $H < G$. This proves (i). One proves (ii) in exactly the same way, but using Proposition 3.1 instead of Proposition 2.1. (iii) follows from Proposition 4.4 below. QED

4.2. **Example.** Let G be an adjoint group of type **A**. By Corollary 2.3, the stabilizer of every point in X is connected, so every regular value is very regular.

We can now determine which of the strata of a weight variety are nonempty.

4.3. **Corollary.** *Let $\lambda \in \mathscr{C}$ and $\mu \in \Delta_G$.*

(i) $\Phi(X_{C(H)}) = W_G \cdot \Delta_H^\circ$ *for all $H \in \mathscr{M}$. Therefore $X_{C(H)}(\mu)$ is nonempty if and only if $\mu \in W_G \cdot \Delta_H^\circ$.*

(ii) $\Phi(X_{\mathfrak{c}(H)}) = W_G \cdot \Delta_H^\circ$ *for all $H \in \mathscr{M}_0$. Therefore $X_{\mathfrak{c}(H)}(\mu)$ is nonempty if and only if $\mu \in W_G \cdot \Delta_H^\circ$.*

Proof. Replace G with H in Proposition 4.1(iii). QED

It is possible for a weight variety $X(\mu)$ to be a manifold (or orbifold) when μ is a singular value: namely when $X(\mu)$ contains a single orbit type stratum (resp. infinitesimal orbit type stratum). This may happen at various boundary faces of the polytope Δ_G.

The proof of Proposition 4.1 relies on the following general result.

4.4. **Proposition.** *Let M be a compact connected Hamiltonian T-manifold with moment map $\Psi\colon M \to \mathfrak{t}^*$, moment polytope $\Delta = \Psi(M)$, principal orbit type S, and regular infinitesimal orbit type $\mathfrak{s} = \mathrm{Lie}(S)$. Then $\Psi(M_S) = \Psi(M_{\mathfrak{s}}) = \Delta^\circ$. For all $\mu \in \Delta^\circ$, the μ-principal orbit type is S, the μ-regular infinitesimal orbit type is $\mathfrak{s}$, and the dimension of the fibre is* $\dim \Psi^{-1}(\mu) = \dim M - \dim T + \dim S$.

Proof. Let $x \in M$ and suppose $\mu = \Psi(x)$ is contained in $\Delta \setminus \Delta^\circ$. Choose nonzero $\xi \in \mathfrak{t}$ such that Δ° is contained in the complement of the affine hyperplane $\mu + \ker \xi$. Then the component $\langle \Psi, \xi \rangle$ of Ψ attains an extremal value at x, so x is fixed by the subtorus of T generated by ξ, and therefore $\xi \in \mathfrak{t}_x$. Let $\mathfrak{d}$ be the linear subspace of $\mathfrak{t}^*$ parallel to the affine span of the polytope Δ. Then $\mathfrak{d}$ is the subspace orthogonal to $\mathfrak{s} \subseteq \mathfrak{t}$, so ξ is not in $\mathfrak{s}$. Therefore $\mathfrak{t}_x \neq \mathfrak{s}$; in other words, x is not in $M_{\mathfrak{s}}$. It follows that $\Psi(M_{\mathfrak{s}}) \subseteq \Delta^\circ$. Let $x \in M$ and $\mu = \Psi(x)$. Let $K = T_x$ and $\mathfrak{k} = \operatorname{Lie} K$. Let V be the symplectic slice at x and let $\Psi_V \colon V \to \mathfrak{k}^*$ be the moment map for the K-action on V, normalized so that $\Psi_V(0) = 0$. Choose a K-invariant compatible complex structure on V. This choice turns V into a unitary K-module. Let $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathfrak{k}^*$ be the weights of the complex $\mathfrak{k}$-module V and let $\pi_1, \pi_2, \dots, \pi_k \colon V \to V$ be the projections onto the respective weight spaces. Let

$$\mathfrak{C} = \left\{ \sum_{j=1}^{k} a_j \lambda_j \;\middle|\; a_1, a_2, \dots, a_k \geq 0 \right\}$$

be the cone in $\mathfrak{k}^*$ spanned by the weights. By the equivariant Darboux theorem for Hamiltonian T-actions ([7, Ch. 2]), there exist a T-stable open neighbourhood U of x and an open neighbourhood O of μ such that

$$\Psi(U) = O \cap \bigl(\mu + p^{-1}(\mathfrak{C})\bigr), \tag{6}$$

where $p \colon \mathfrak{t}^* \to \mathfrak{k}^*$ is the canonical projection. Now assume that $\mu \in \Delta^\circ$. Then it follows from (6) that the origin of $\mathfrak{k}^*$ is an interior point of $\mathfrak{C}$, so there exist $a_j > 0$ such that $\sum_{j=1}^{k} a_j \lambda_j = 0$. This implies that there exists $v \in V$ such that

$$\pi_1(v) \neq 0, \quad \pi_2(v) \neq 0, \dots, \quad \pi_k(v) \neq 0, \qquad \Psi_V(v) = \sum_{j=1}^{k} \|\pi_j(v)\|^2 \lambda_j = 0.$$

If $\xi \in \mathfrak{k}$ is in the stabilizer subalgebra of such a point v, then we see from

$$0 = \xi_V(v) = \sum_{j=1}^{k} \lambda_j(\xi) \pi_j(v)$$

that $\lambda_j(\xi) = 0$ for all j. Therefore $\mathfrak{k}_v = \bigcap_j \ker \lambda_j$, which means that $v \in \Psi_V^{-1}(0)$ is in the regular stratum of the K-action on V. By another application of the equivariant Darboux theorem, this shows that there exists a point $y \in \Psi^{-1}(\mu)$ which is in the regular stratum of M. This proves that $\Psi(M_{\mathfrak{s}}) = \Delta^\circ$. Since the infinitesimal orbit type stratification of each fibre has a unique maximal element, it follows that the regular stratum of the fibre $\Psi^{-1}(\mu)$ over each $\mu \in \Delta^\circ$ is $M_{\mathfrak{s}} \cap \Psi^{-1}(\mu)$. Since $\Psi | M_{\mathfrak{s}}$ is a submersion onto Δ°, for each $\mu \in \Delta^\circ$ the dimension of $\Psi^{-1}(\mu)$ (i.e. the dimension of its regular stratum) is equal to

$$\dim M - \dim \Delta = \dim M - \dim T + \dim S.$$

Finally, let $\mu \in \Delta^\circ$ and suppose that $M_S \cap \Psi^{-1}(\mu)$ is empty. Then the μ-principal orbit type S' is strictly larger than S, so $\dim M_{S'} < \dim M$. On the other hand, $\Psi^{-1}(\mu)$ contains points of infinitesimal type $\mathfrak{s}$, so $\operatorname{Lie}(S') = \mathfrak{s}$. Choose $x \in M$ such that $\Psi(x) = \mu$ and $T_x = S'$. Let M' be the closure in M of the connected component of $M_{S'}$ which contains x. Then M' is a compact connected Hamiltonian T-manifold with moment map $\Psi' = \Psi | M'$, moment polytope $\Delta' = \Psi(M')$, and principal orbit type S'. By induction on the dimension of M we may assume that the proposition is true when we replace M with M'. Accordingly, $\mu = \Psi'(x) \in (\Delta')^\circ$ and

$$\dim\bigl(M_{S'} \cap \Psi^{-1}(\mu)\bigr) = \dim(\Psi')^{-1}(\mu) = \dim M' - \dim T + \dim S'$$
$$= \dim M' - \dim T + \dim S < \dim \Psi^{-1}(\mu),$$

which contradicts the assumption that S' is μ-principal. Therefore the μ-principal orbit type is S. We conclude that $\Psi(M_S) = \Delta^\circ$. QED

If H, $K \in \mathscr{M}$ and $H \leq K$, then $W_H \leq W_K$, so $\Delta_H \subseteq \Delta_K$. Therefore, when $C(G) = 1$, to determine the set of very regular values of Φ it is enough find the polytopes Δ_H for all *maximal* $H \in \mathscr{M} \setminus \{G\}$. By [1, Théorèmes 5, 6], such maximal elements are of the form $H = Z_G(x)_0$, where x is either (1) a non-central vertex of an alcove of $\mathfrak{t}$ or (2) a point on an edge connecting two central vertices of an alcove. In case (2) we have $H \in \mathscr{M}_0$ and $\dim C(H) = 1$, so $\dim \Delta_H = \dim \Delta_G - 1$ and Δ_H consists of singular values. In case (1) H is semisimple and $\dim \Delta_H = \dim \Delta_G$, so such an H gives rise to an open subset of regular values which are not very regular. Thus the set of very regular values is the complement in the set of regular values of all polytopes Δ_H, where H ranges over the semisimple maximal elements of $\mathscr{M} \setminus \{G\}$.

As pointed out to me by A. Knutson, the Dynkin diagrams of the semisimple maximal elements of $\mathscr{M} \setminus \{G\}$ are found by completing a single component of the Dynkin diagram of G and then deleting one vertex from that component. Repeating this process gives the lesser semisimple elements of $\mathscr{M}$; the algorithm stops once each component is of type $\mathbf{A}$. (To find the Dynkin diagrams of all elements of $\mathscr{M}$, one must delete one or more vertices at each step; this modified algorithm terminates at the empty diagram.)

If G is simple and not of type $\mathbf{A}$, the alcoves have non-central vertices and therefore Φ has regular values which are not very regular, as observed in [5, p. 260, footnote]. The set of very regular values may or may not be empty.

4.5. **Example.** Let G be an adjoint group of type $\mathbf{B}_2$. The generalized faces of the Kostant polytope Δ_G for a typical value of $\lambda \in \mathscr{C}$ are shown in Figure 1. The elements of $\mathscr{P}$ (aside from the empty root system and the full root system R) are the long root subsystem R_0 (which is isomorphic to $\mathbf{A}_1 \times \mathbf{A}_1$ and corresponds to the subgroup $\mathbf{SO}(4)$ of $\mathbf{SO}(5)$), and the subsystems of the form $\{\pm\alpha\}$ for any $\alpha \in R$. The subgroup H_0 of G corresponding to R_0 is semisimple and its Weyl group has index 2 in the Weyl group of G, so the image of the stratum X_{C_0} (where $C_0 = C(H_0) \cong P_0^\vee / P^\vee$) is the union of two rectangles, referred to as regular faces in Figure 1. The coweight lattice $P^\vee$ has index 2 in $P_0^\vee$, so the radicial subgroup C_0 which stabilizes the stratum X_{C_0} is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. The regular faces in Figure 1 are labelled with an element of $P_0^\vee$ which exponentiates to a generator of C_0, namely the element $\frac{1}{2}\varpi_2^\vee$. (Here $\varpi_1^\vee$ and $\varpi_2^\vee$ are the fundamental coweights of G, i.e. the generators of $P^\vee$ dual to the long simple root α_1 and the short simple root α_2.) The image of the principal stratum is the interior of the full polytope Δ_G, called the principal face in Figure 1. There are four open regions consisting of very regular values, at each of which the weight variety is a smooth manifold.

4.6. **Example.** A similar exercise for G of type $\mathbf{G}_2$ leads to Figure 2. The elements of $\mathscr{P}$ (other than $\varnothing$ and R itself) are

- the long root subsystem R_0 (see Example 1.7);

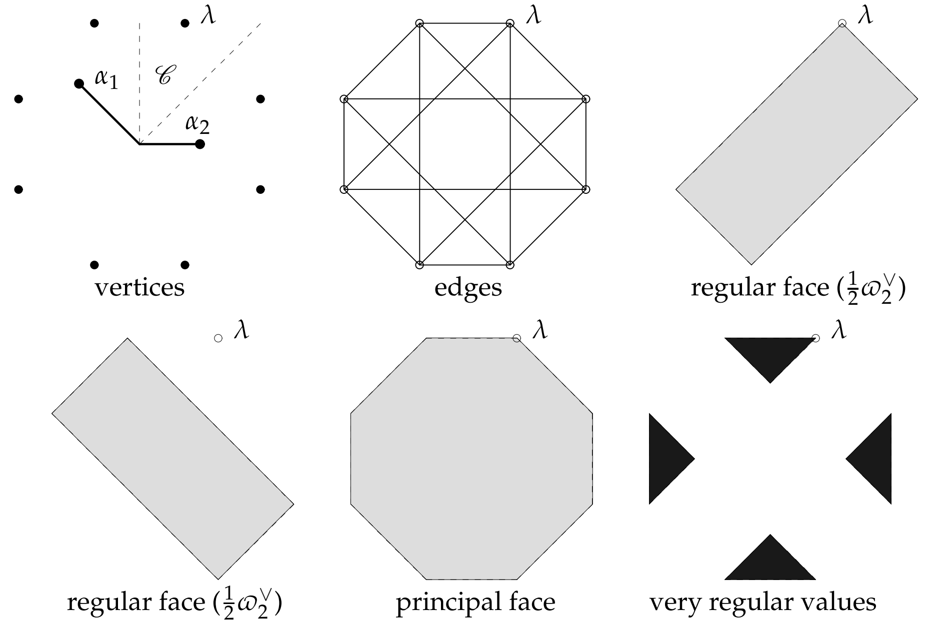

FIGURE 1. Generalized faces of Kostant polytope of type $\mathbf{B}_2$

- the subsystem $R_1 = \{\pm\alpha_0, \pm\alpha_1\}$ (see Example 1.8);
- two subsystems R_2 and R_3 that are W-conjugate to R_1;
- the subsystems of the form $\{\pm\alpha\}$ for any $\alpha \in R$.

For $0 \leq i \leq 3$, let Q_i, $P_i^\vee$, $Q_i^\vee$ and P_i be the various lattices associated with R_i, let H_i be the subgroup of G corresponding to R_i and $C_i = C(H_i) \cong P_i^\vee/P^\vee$ its centre. The Weyl group of H_0 has index 2 in the Weyl group of G. Thus the stratum X_{C_0} of orbit type $C_0 \cong \mathbf{Z}/3\mathbf{Z}$ has two connected components, which map to two hexagons. The Weyl group of H_1 has index 3 in the Weyl group of G, so the stratum X_{C_1} of orbit type $C_1 \cong \mathbf{Z}/2\mathbf{Z}$ has three connected components, which map to three rectangles. The two subsystems R_2 and R_3 are conjugate to R_1 and so give rise to two further families of three rectangles each. As in Figure 1, each regular face is labelled with an element of $P_i^\vee$ which exponentiates to a generator of the centre C_i. (The fundamental coweights $\varpi_1^\vee$ and $\varpi_2^\vee$ are numbered as in Example 1.7.) The union of the regular faces is the principal face, so there are *no* very regular values: every weight variety over the interior of the polygon has at least orbifold singularities (although the generic point of each weight variety is smooth).

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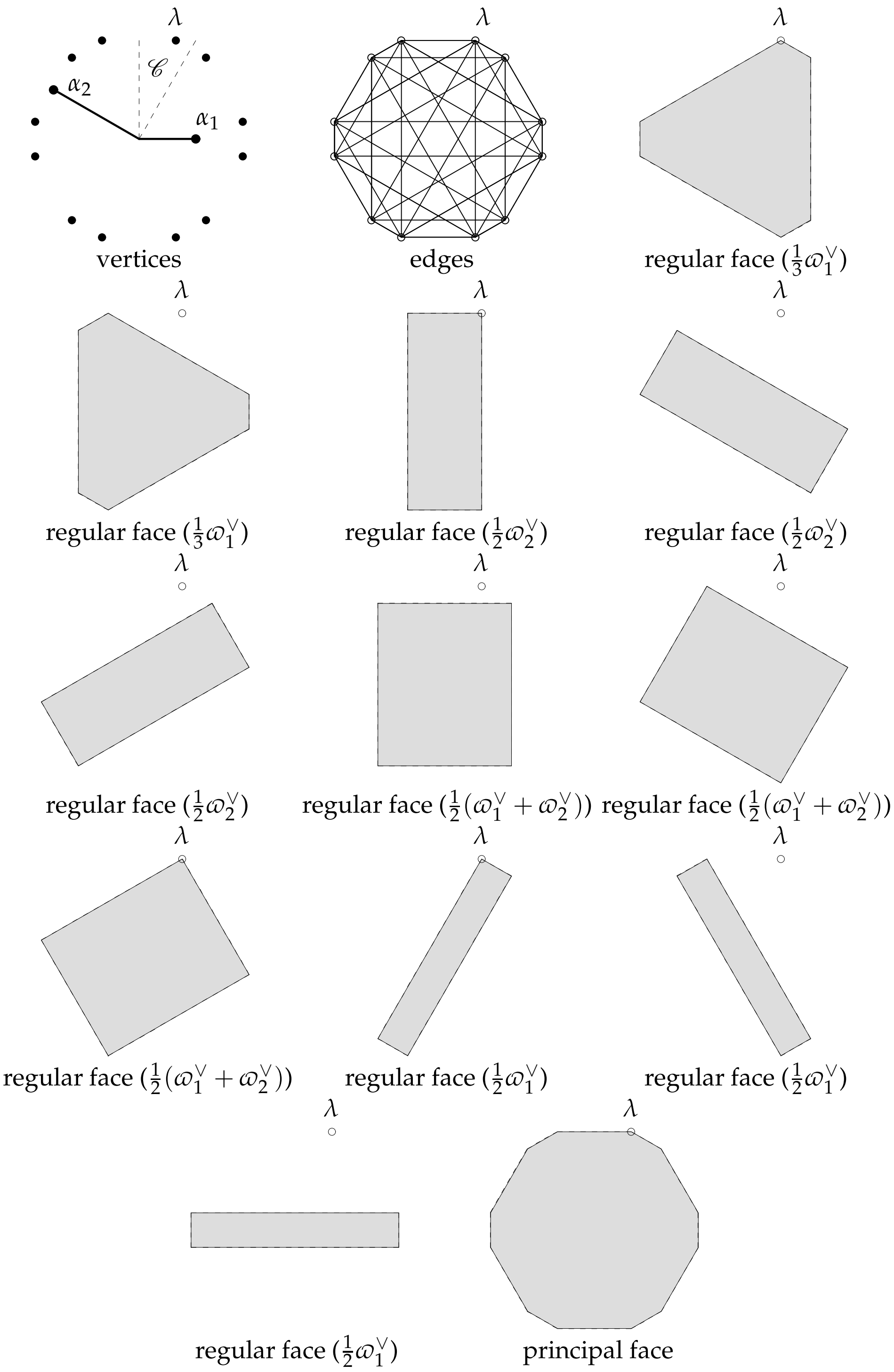

FIGURE 2. Generalized faces of Kostant polytope of type $\mathbf{G}_2$

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