On Singular Reduction of Hamiltonian Spaces

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A Jean-Marie Souriau

Le timbre de la trompette est noble et éclatant; il convient aux idées guerrières, aux cris de fureur et de vengeance, comme aux chants de triomphe, il se prête à l'expression de tous les sentiments énergiques, fiers et grandioses, à la plus part des accents tragiques. Il peut même figurer dans un morceau joyeux, pourvu que la joie y prenne un caractère d'emportement ou de grandeur pompeuse.

—H. Berlioz [4]

Introduction

Symplectic reduction of Hamiltonian spaces is a rich source of symplectic manifolds. It has been widely applied to the study of Hamiltonian systems with symmetries since the days of Jacobi (see [16] for a historical survey). Symplectic reduction also arises naturally in classical field theories, such as Yang-Mills theory, and in Guillemin's and Sternberg's work [8] on asymptotic multiplicity formulas for group representations. A general framework for symplectic reduction at regular values of a momentum map has been set up by Meyer [18] and by Marsden and Weinstein [17]. In many of the applications, however, one would like to carry out reduction at singular values of a momentum map. It is therefore of interest to devise a reduction scheme for singular levels of a momentum map and to study the singularities arising from it.

Over the past decade many results have been obtained in this direction. See [2] for a survey of and a comparison between various different approaches. Our point of view is that a reduced space of a Hamiltonian action of a compact Lie group is a *stratified*

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symplectic space. (Our results hold also if the group is noncompact, provided that the action is proper.) Roughly speaking, a stratified symplectic space is a topological space X with the following properties:

i. the space X decomposes into a locally finite disjoint union of symplectic manifolds;

ii. these manifolds fit together in a nice way, i.e., the decomposition is a *stratification*;

iii. there is given a subclass of the set of continuous functions on X, a set of 'smooth functions', which forms a Poisson algebra;

iv. the inclusion of each stratum is a Poisson map;

v. there is a unique open stratum in X. It is open and dense in X.

This paper is an overview of joint work of the first author with J. Arms and M. Gotay [1], and of the second author with E. Lerman [13], [21]. The exposition is informal; for most of the proofs we shall refer to the original articles. Instead we shall consider a few simple examples of the theory in detail. Further examples will be described in [14]. We would like to suggest the following two problems for future research:

A. Try to generalize the results of this paper to any of the various infinite-dimensional situations arising in mathematical physics, e.g. Yang-Mills theory (cf. [3]) and the Einstein equations (cf. [11]);

B. Try to generalize the multiplicity formulas of Guillemin and Sternberg [8] to singular reduced spaces, using the concept of a stratified symplectic space.

1 A Poisson Bracket on a Reduced Phase Space

Throughout this article (M, ω) will denote a symplectic manifold and G a compact Lie group acting on M in a Hamiltonian fashion, with momentum map $J : M \to \mathbf{g}^*$. We shall always assume that the map J is equivariant with respect to the given action on M and the coadjoint action on $\mathbf{g}^*$. In the sequel we shall make frequent use of the following equality, which follows easily from the definition of a momentum map:

$$\operatorname{im} dJ_p = \operatorname{ann} \mathbf{g}_p, \tag{1}$$

for all points p in M. Here $\operatorname{im} dJ_p$ denotes the image of the tangent map dJ_p, $\mathbf{g}_p$ denotes the infinitesimal stabilizer of the point p, and $\operatorname{ann} \mathbf{g}_p$ denotes the annihilator of $\mathbf{g}_p$ in $\mathbf{g}^*$.

For any point μ in $\mathbf{g}^*$, let $\mathcal{O}_\mu$ denote the coadjoint orbit through μ. If μ is a regular value of the momentum mapping, the pre-image $J^{-1}(\mathcal{O}_\mu)$ of the coadjoint orbit is a G-invariant submanifold of M, and it follows immediately from (1) that the action of G on $J^{-1}(\mathcal{O}_\mu)$ is locally free. Marsden and Weinstein defined the reduced space M_μ to be the quotient orbifold

$$M_\mu = J^{-1}(\mathcal{O}_\mu)/G,$$

and they showed that there is a unique symplectic form ω_μ on the space M_μ such that the pullback of ω_μ to $J^{-1}(\mathcal{O}_\mu)$ equals the restriction of the form ω to $J^{-1}(\mathcal{O}_\mu)$.

If μ is not a regular value of J, then the set $J^{-1}(\mathcal{O}_\mu)$ is not in general a manifold, and one sees from (1) that the action of G on $J^{-1}(\mathcal{O}_\mu)$ is no longer locally free. However, the definition $M_\mu = J^{-1}(\mathcal{O}_\mu)/G$ still makes sense topologically. We will see that the space M_μ, which is not in general a manifold, or even an orbifold, has many nice properties.

In the first place, there is a natural concept of 'smooth functions' on M_μ. A continuous function $f : M_\mu \to \mathbf{R}$ is called *smooth* if there exists a smooth G-invariant function $F : M \to \mathbf{R}$ such that

$$F|_{J^{-1}(\mathcal{O}_\mu)} = f \circ \pi,$$

where $\pi : J^{-1}(\mathcal{O}_\mu) \to M_\mu$ is the quotient map. We denote the algebra of all smooth functions on M_μ by $C^\infty(M_\mu)$.

If the group G is finite, its 'Lie algebra' is $\{0\}$ and the 'reduced space' is just the quotient M/G. In this case, the smooth functions on M/G are the G-invariant functions on M.

Secondly, we define the bracket of two smooth functions f and h on M_μ by

$$\{f, h\}_{M_\mu} = \{F, H\}_M,$$

where F and H are smooth G-invariant functions on M such that $F|_{J^{-1}(\mathcal{O}_\mu)} = f \circ \pi$, resp. $H|_{J^{-1}(\mathcal{O}_\mu)} = h \circ \pi$. One then checks that the bracket is well-defined, i.e., it does not depend on the choice of F and H, and so the algebra $C^\infty(M_\mu)$ becomes a Poisson algebra with this bracket; see [1] for a proof. Moreover, it turns out that if μ is a regular value of the momentum map, this Poisson structure coincides with the Poisson structure defined by the Marsden-Weinstein symplectic form ω_μ.

1.1. EXAMPLE. Let M be $\mathbf{C}^2$ with its standard symplectic form and let G be the circle $\{\, e^{i\theta} : \theta \in \mathbf{R} \,\}$, acting on M by $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2)$. This is the $(1, -1)$-*resonance*, the action generated by the Hamiltonian

$$J(z_1, z_2) = (|z_1|^2 - |z_2|^2)/2.$$

We want to compute the reduced space at the zero level. To this end we compute the algebra of G-invariant *real* polynomials on $\mathbf{C}^2$. It is easy to check that this algebra is generated by the following polynomials:

$$\begin{array}{rclrcl} \sigma_1 & = & |z_1|^2 & \sigma_2 & = & (z_1 z_2 + \overline{z_1 z_2})/2 \\ \sigma_3 & = & (z_1 z_2 - \overline{z_1 z_2})/2i & \sigma_4 & = & 2J = |z_1|^2 - |z_2|^2. \end{array}$$

(We choose this particular set of generators because it is well-suited to our further computations.) The only relation among the generators is: $\sigma_1(\sigma_1 - \sigma_4) = \sigma_2^2 + \sigma_3^2$. Define the *Hilbert map* $\sigma : M \to \mathbf{R}^4$ by

$$\sigma(m) = (\sigma_1(m), \sigma_2(m), \sigma_3(m), \sigma_4(m)).$$

The image $\sigma(J^{-1}(0))$ of the zero level set is contained in the hyperplane $\{ (x_1, x_2, x_3, x_4) : x_4 = 0 \}$. Let us identify this hyperplane with $\mathbf{R}^3$. Then $\sigma(J^{-1}(0))$ is the quadratic half-cone in $\mathbf{R}^3$ given by

$$x_1^2 = x_2^2 + x_3^2 \quad \text{and} \quad x_1 \geq 0. \tag{2}$$

Since the Hilbert map is invariant, its restriction to $J^{-1}(0)$ descends to a map $\tilde{\sigma} : M_0 \to \mathbf{R}^3$. It is not difficult to show that this map is a homeomorphism onto its image. This describes the reduced space as a topological space.

We claim that the algebra of smooth functions on M_0 is isomorphic to the algebra $C^\infty(\mathbf{R}^3)/\mathcal{I}$, where $\mathcal{I}$ is the ideal in $C^\infty(\mathbf{R}^3)$ of all functions vanishing on the set given by (2), and where the isomorphism is given by pulling back functions via $\tilde{\sigma}$. Indeed, if h is a smooth function on $\mathbf{R}^3$, the function H given by $H(m) = h(\sigma_1(m), \sigma_2(m), \sigma_3(m))$ is a smooth invariant function on $M = \mathbf{C}^2$, and therefore descends to a smooth function on M_0. This gives us the map

$$\tilde{\sigma}^* : C^\infty(\mathbf{R}^3) \to C^\infty(M_0). \tag{3}$$

By definition, its kernel is the ideal $\mathcal{I}$. To see that the map is surjective, pick an arbitrary smooth function f on M_0. Let F be an invariant smooth function on M such that $F|_{J^{-1}(0)}$ descends to f. By a theorem of Schwarz [20], a smooth function on $\mathbf{R}^n$ invariant under a linear action of a compact Lie group K is always a smooth function of the K-invariant *polynomials* on $\mathbf{R}^n$. In our case, this means that we can find a smooth function h defined on $\mathbf{R}^4$ such that $h \circ \sigma = F$. But then $\tilde{\sigma}^*(h|_{\mathbf{R}^3}) = f$. So $\tilde{\sigma}$ is surjective.

We can describe the Poisson bracket on M_0 in terms of the generators $\bar{x}_1, \bar{x}_3, \bar{x}_3$ of the algebra $C^\infty(\mathbf{R}^3)/\mathcal{I}$. (Here $\bar{f}$ denotes the equivalence class $f \bmod \mathcal{I}$ of a function f.) It is given by the following table:

$\{\cdot,\cdot\}$	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$
$\bar{x}_1$	0	$4\bar{x}_3$	$4\bar{x}_2$
$\bar{x}_2$	$-4\bar{x}_3$	0	$-4\bar{x}_1$
$\bar{x}_3$	$-4\bar{x}_2$	$4\bar{x}_1$	0

1.2. Remark. In the above table, replace the $\bar{x}_i$'s by the coordinates x_i on $\mathbf{R}^3$. It is easy to check that the resulting table is the structure matrix of the Lie-Poisson bracket on $\mathbf{sl}(2, \mathbf{R})^*$ (with respect to a suitable basis). This means that there exists a Poisson bracket on $\mathbf{R}^3$ such that the map (3) becomes a morphism of Poisson algebras and the ideal $\mathcal{I}$ a Poisson ideal.

Now let V be an arbitrary symplectic representation space for a compact Lie group, and let j be any embedding of the reduced space V_0 into a Euclidean space $\mathbf{R}^n$ constructed by means of invariant polynomials, analogous to the embedding of Example 1.1.

1.3. Conjecture. *There exists a Poisson bracket on $\mathbf{R}^n$ such that the embedding j of the reduced space V_0 into $\mathbf{R}^n$ becomes a Poisson map.*

A proof of this conjecture in the special case where the algebra of invariant polynomials has a basis of polynomials of degree ≤ 2 can be found in [13].

As illustrated by Example 1.1, there is a natural notion of an embedding of a reduced space into a Euclidean space. Likewise, there is a natural concept of diffeomorphisms between reduced spaces, and of a smooth action of a Lie group on a reduced space.

1.4. Example. Here is another way of computing the reduced space of Example 1.1. Consider the diagonal map $\Delta : \mathbf{C} \to M = \mathbf{C}^2$, assigning $w \mapsto (w, w)$. The image of Δ is contained in the zero level set $J^{-1}(0)$, and one readily checks that it intersects each orbit in $J^{-1}(0)$. In other words, the diagonal map induces a surjective map $\mathbf{C} \to M_0$. Under this map two points w, w' in $\mathbf{C}$ are mapped to the same orbit in the reduced space M_0 if and only if they are antipodes, $w' = -w$. So we get a bijection between orbit spaces,

$$\widetilde{\Delta} : \mathbf{C}/\mathbf{Z}_2 \to M_0$$

(where the nontrivial element in $\mathbf{Z}_2$ acts on $\mathbf{C}$ as the antipodal map). We claim that this map is an isomorphism in the sense that smooth functions on M_0 pull back to smooth functions on $\mathbf{C}/\mathbf{Z}_2$ and that the pullback map

$$\widetilde{\Delta}^* : C^\infty(M_0) \to C^\infty(\mathbf{C}/\mathbf{Z}_2)$$

is an isomorphism of Poisson algebras. In other words, the reduced space M_0 is isomorphic to the symplectic orbifold $\mathbf{C}/\mathbf{Z}_2$. That the map $\widetilde{\Delta}^*$ is well-defined follows from the fact that G-invariant functions on $M = \mathbf{C}^2$ restrict to $\mathbf{Z}_2$-invariant functions on the diagonal. That it is a morphism of Poisson algebras follows from the fact that the diagonal map Δ is symplectic. It is easy to see that $\widetilde{\Delta}^*$ is injective. Finally, we have to show that it is surjective. Consider the algebra of $\mathbf{Z}_2$-invariant real polynomials on $\mathbf{C} = \mathbf{R}^2$. It is generated by the three polynomials

$$\rho_1 = |w|^2 \qquad \rho_2 = (w^2 + \overline{w}^2)/2 \qquad \rho_3 = (w^2 - \overline{w}^2)/2i.$$

Each of these is the pullback of a G-invariant polynomial on M, namely:

$$\rho_1 = \sigma_1 \circ \Delta \qquad \rho_2 = \sigma_2 \circ \Delta \qquad \rho_3 = \sigma_3 \circ \Delta.$$

Now let h be an arbitrary $\mathbf{Z}_2$-invariant smooth function on $\mathbf{C}$. By Schwarz's theorem (loc. cit.) there exists a smooth function of three variables H such that

$$h(w) = H(\rho_1(w), \rho_2(w), \rho_3(w)).$$

Define the G-invariant smooth function F on $\mathbf{C}^2$ by $F = H \circ \sigma$. Then F descends to a smooth function f on M_0, and we have $\widetilde{\Delta}^*(f) = h$. So $\widetilde{\Delta}^*$ is surjective.

1.5. Example. Consider the symplectic manifold $M \times \mathcal{O}_\mu^-$, the symplectic product of M with the coadjoint orbit $\mathcal{O}_\mu$ through μ, endowed with the opposite of the Kirillov symplectic form. The diagonal action of G on $M \times \mathcal{O}_\mu^-$ is Hamiltonian with momentum map J_μ given by $J_\mu(m, \nu) = J(m) - \nu$. It is easy to check that the cartesian projection $\Pi : M \times \mathcal{O}_\mu^- \to M$ restricts to an equivariant bijection $J_\mu^{-1}(0) \cong J^{-1}(\mathcal{O}_\mu)$. As a result, Π descends to a bijection between reduced spaces,

$$\widetilde{\Pi} : (M \times \mathcal{O}_\mu^-)_0 \xrightarrow{\sim} M_\mu.$$

1.6. Proposition (the 'shifting trick'). *The map $\widetilde{\Pi}$ is an isomorphism of reduced spaces.*

Sketch of Proof. We want to show that $\widetilde{\Pi}$ induces an isomorphism of Poisson algebras,

$$\widetilde{\Pi}^* : C^\infty(M_\mu) \xrightarrow{\cong} C^\infty((M \times \mathcal{O}_\mu^-)_0).$$

It is not hard to show that smooth functions on $(M \times \mathcal{O}_\mu^-)_0$ pull back to smooth functions on M_μ and that $\widetilde{\Pi}^*$ is injective. Moreover, the map $\widetilde{\Pi}^*$ is a morphism of Poisson algebras, because the Cartesian projection $M \times \mathcal{O}_\mu^- \to M$ is a Poisson map.

The gist of the argument is showing that $\widetilde{\Pi}^*$ is surjective. Pick an arbitrary smooth function ϕ on $(M \times \mathcal{O}_\mu^-)_0$. Let Φ be a smooth G-invariant function on $M \times \mathcal{O}_\mu^-$ such that the restriction of Φ to $J_\mu^{-1}(0)$ equals the pullback of ϕ to $J_\mu^{-1}(0)$. Because the group G is compact, the coadjoint orbit $\mathcal{O}_\mu$ is a *closed* subset of $\mathbf{g}^*$. Therefore the product $M \times \mathcal{O}_\mu$ is included in $M \times \mathbf{g}^*$ as a closed submanifold, and we can find a smooth extension $\bar{\Phi}$ of Φ to the whole of $M \times \mathbf{g}^*$. We may furthermore assume that $\bar{\Phi}$ is G-invariant (if not, replace it by its average over G). Now let $j : M \to M \times \mathbf{g}^*$ be the natural map from M onto the graph of the momentum map J, $j(m) = (m, J(m))$, and define a function F on M by putting $F = \bar{\Phi} \circ j$. Then F is a smooth G-invariant function on M, and therefore induces a smooth function f on the reduced space M_μ. For all (m, ν) in $J_\mu^{-1}(0)$ one has

$$F \circ \Pi\,(m, \nu) = \bar{\Phi}(m, J(m)) = \bar{\Phi}(m, \nu) = \Phi(m, \nu).$$

This shows that $\widetilde{\Pi}^* f = \phi$. Thus $\widetilde{\Pi}^*$ is surjective. $\square$

As far as we know, this proposition was first proved by Guillemin and Sternberg [8] under the assumption of regularity. It allows us to restrict our attention from now on to reduction at the zero level.

1.7. Remark. The shifting trick breaks down for Hamiltonian actions of a noncompact group. The reason is that for a noncompact group G the coadjoint orbits are not necessarily closed in $\mathbf{g}^*$; cf. also [1, § 2, last two paragraphs]. It is not hard to show that the shifting trick works for reduction at a closed coadjoint orbit of a noncompact group G, provided the action of G on M is proper in the sense of [19]. All the other results in this paper go through for Hamiltonian actions of an arbitrary Lie group G, provided that the action is proper. (The only statement that requires a slight modification is the density theorem, Theorem 2.7; cf. the remark following that theorem.)

Many other concepts defined for smooth symplectic manifolds carry over to singular reduced phase spaces in a routine manner. For instance, one defines a *smooth curve* in M_μ as a continuous map $\gamma : [0, 1] \to M_\mu$ such that the pullback $f \circ \gamma$ of a smooth function f on M_μ is a smooth function on the interval $[0, 1]$. If h is a smooth function on M_μ, it makes no sense to talk about the Hamiltonian vector field of h on M_μ, since M_μ is not a manifold. One can however define the *Hamiltonian derivation* $ad\, h$ of h. It operates on a smooth function f on M_μ as follows:

$$ad\, h \cdot f = \{h, f\}_{M_\mu}. \tag{4}$$

Now an *integral curve* of the Hamiltonian h through a point m is defined as a smooth curve $\gamma(t)$ with $\gamma(0) = m$, such that for all functions $f \in C^\infty(M_\mu)$

$$\frac{d}{dt} f(\gamma(t)) = -\, ad\, h \cdot f\,(\gamma(t)). \tag{5}$$

One proves the existence and uniqueness of such integral curves by considering the flow of a lift of the Hamiltonian h to M; see [13].

If K is a Lie group acting smoothly on M_0, then for each element ξ of the Lie algebra $\mathbf{k}$ of K one has the associated *fundamental derivation* ξ_{M_0} on the algebra $C^\infty(M_0)$. It operates on a smooth function f as follows:

$$(\xi_{M_0} \cdot f)(x) = -\left.\frac{d}{dt} f(\exp t\xi \cdot x)\right|_{t=0},$$

for x in M_0. This is well-defined since $f(\exp t\xi \cdot x)$ depends smoothly on t. We call a K-action on M_0 *Hamiltonian* if it has a momentum map, i.e., a smooth $Ad(K)$-equivariant map $I : M_0 \to \mathbf{k}^*$ such that for each ξ in $\mathbf{k}$ one has

$$ad\, I^\xi = -\xi_{M_0}.$$

Here I^ξ is the ξ-th component of I, defined by $I^\xi(x) = \langle I(x), \xi\rangle$, and $ad\, I^\xi$ is the Hamiltonian derivation on $C^\infty(M_0)$ corresponding to the function I^ξ. Given a Hamiltonian K-action on M_0, one can reduce once more and obtain a space with a Poisson algebra.

An important special case of a Hamiltonian action on a reduced space arises in the following situation. Assume that the group G is the product of two Lie groups G_1 and G_2. Then the momentum map for the G-action on M splits into a product of two maps, $J = J_1 \times J_2$, where $J_1 : M \to \mathbf{g}_1^*$ and $J_2 : M \to \mathbf{g}_2^*$ are momentum maps for the actions of G_1 and G_2, respectively. The actions of G_1 and G_2 on M commute, and it follows from the G-equivariance of J that J_1 is invariant with respect to the action of G_2. Let X_1 be the reduced space of M with respect to the G_1-action. Then the action of G_2 on M descends to an action on X_1 and the map J_2 descends to a smooth map $\bar{J}_2 : M \to \mathbf{g}^*$. It is not difficult to show that this G_2-action is Hamiltonian with momentum map $\bar{J}_2$. Let X_{12} denote the reduced space of X_1 with respect to this action.

We might just as well carry these two reductions out in reverse order. That is, first reducing M with respect to the G_2-action we obtain a Hamiltonian G_1-space X_2 with momentum map $\bar{J}_1$, and reducing once again we obtain a space X_{21}. We claim that the answer does not depend on the order of the reductions.

1.8. Theorem (reduction in stages). *The reduced spaces X_{12} and X_{21} are isomorphic.*

Reduction in stages is a useful computational tool and it plays an important role in [13]. The proof rests on the observation that both spaces can be obtained by reducing the original manifold M with respect to the action of the product group $G = G_1 \times G_2$; see [13] for the details.

2 A Stratification of a Reduced Phase Space

For Poisson manifolds Weinstein [23] has introduced the notion of *symplectic leaves*. If P is a Poisson manifold and p a point in P, the symplectic leaf of P through p is the set of all points q in P such that q can be joined to p by means of a finite number of trajectories of Hamiltonian vector fields on P. The symplectic leaves of a Poisson manifold are always symplectic manifolds.

In the previous section we showed that the reduced space M_0 of a Hamiltonian G-space M comes equipped with a natural Poisson algebra, and that there is a suitable concept of Hamiltonian flows on M_0, despite the fact that this space may be singular. Because of this, it makes sense to talk about the symplectic leaves of M_0 (cf. also Gonçalves [7]). It will turn out that the leaves of M_0 are smooth symplectic manifolds, which fit together in a neat way.

There is an alternative definition of these symplectic pieces, which is better suited to the study of the local structure of M_0. For a closed subgroup H of G the set of points in M of *orbit type* (H) is defined as

$$M_{(H)} = \{\, m \text{ in } M : \text{ stabilizer of } m \text{ is conjugate to } H \,\}.$$

It is a familiar fact from the theory of Lie group actions that $M_{(H)}$ is a locally closed submanifold of M (see e.g. Bredon [5]), possibly consisting of connected components of various dimensions. It follows from (1) that the restriction of the momentum map J to $M_{(H)}$ is of constant rank equal to the codimension of H in G. As a consequence, the intersection $J^{-1}(0) \cap M_{(H)}$ is a submanifold of M. In fact, we can say much more.

2.1. THEOREM. *The quotient $(M_0)_{(H)} := (J^{-1}(0) \cap M_{(H)})/G$ is a manifold and there exists a unique symplectic form $\omega_{(H)}$ on $(M_0)_{(H)}$ whose pullback to $J^{-1}(0) \cap M_{(H)}$ equals the restriction of the symplectic form ω to $J^{-1}(0) \cap M_{(H)}$.*

We have thus decomposed the reduced space into a collection of symplectic manifolds,

$$M_0 = \coprod_{H<G} (M_0)_{(H)}.$$

The symplectic pieces $(M_0)_{(H)}$ are called the *strata* of M_0. It is not hard to check that the inclusion maps $(M_0)_{(H)} \to M_0$ are Poisson, i.e., smooth functions on M_0 restrict to smooth functions on $(M_0)_{(H)}$, and the restriction map $C^\infty(M_0) \to C^\infty((M_0)_{(H)})$ is a morphism of Poisson algebras. The restriction map is not surjective, unless $(M_0)_{(H)}$ is closed. It is, however, easy to see that any smooth function on $(M_0)_{(H)}$ with compact support can be extended to a smooth function on M_0. (See the argument preceding Theorem 2.5.) Therefore the image of the restriction map is dense in $C^\infty((M_0)_{(H)})$. One can also show that the Hamiltonian flow of a smooth function h on M_0 leaves the strata $(M_0)_{(H)}$ invariant. Because of this, a Hamiltonian derivation on the algebra $C^\infty(M_0)$, as defined by (4), can be regarded as a collection of Hamiltonian vector fields, one on each stratum of M_0, glued together in a smooth manner.

Sketch of Proof. Let p be a point in the zero level set of J and let H be the stabilizer of p. Let $\mathbf{m}$ denote the tangent space at p to the orbit $G \cdot p$ through p. It is easy to see that any G-orbit in $J^{-1}(0)$ is isotropic, i.e., $\mathbf{m} \subset \mathbf{m}^\omega$, where $\mathbf{m}^\omega$ denotes the skew orthogonal complement of $\mathbf{m}$. Therefore we can define a symplectic vector space $V = \mathbf{m}^\omega/\mathbf{m}$, called the *symplectic slice* at p to the G-action. It is a symplectic representation space for H. Marle [15] and, independently, Guillemin and Sternberg [9] have shown how to reconstruct the Hamiltonian space M in a neighbourhood of the orbit $G \cdot p$ from the data G, H and V. The tangent space to M at p can be written (noncanonically) as a direct sum, $T_pM = \mathbf{m} \oplus \mathbf{m}^* \oplus V$. The topological normal bundle of the orbit $G \cdot p$ is a vector bundle,

$$Y = G \times_H (\mathbf{m}^* \times V),$$

associated to the principal bundle $H \to G \to G/H$. The space Y can also be seen as a vector bundle with symplectic fibre V and symplectic base $G \times_H \mathbf{m}^* = T^*(G \cdot p)$. As a manifold, the space Y can be obtained from the Hamiltonian $G \times H$-space

$$T^*G \times V \tag{6}$$

by means of reduction with respect to the H-action. (Here G acts by left translations on T^*G, and H acts by right translations on T^*G and as the given representation on V.) This provides Y with a symplectic structure. It turns out that the left G-action on the space Y is Hamiltonian with momentum map J given by:

$$J[g, \mu, v] = (Ad^*g)(\mu + \Phi_V(v)).$$

Here $[g, \mu, v]$ denotes the class of a triple (g, μ, v) in $G \times \mathbf{m} \times V$, and $\Phi_V : V \to \mathbf{h}^*$ is the quadratic momentum map for the linear H-action on V, given by:

$$\langle \Phi_V(v), \eta \rangle = 1/2\, \omega_V(\eta \cdot v, v).$$

It now follows from the G-equivariant version of Weinstein's isotropic embedding theorem [22] that a neighbourhood of the orbit $G \cdot p$ in M is G-equivariantly symplectomorphic to a neighbourhood of the zero section in the vector bundle Y over G/H. This is the local normal form of Marle-Guillemin-Sternberg.

It is easy to compute the set $(M_0)_{(H)}$ inside the local model. It is equal to the subspace V_H of fixed points in the symplectic vector space V, which is again symplectic. This proves Theorem 2.1. $\square$

2.2. Remark. As a corollary of this proof we obtain a local model for the reduced space, namely the product space (6), successively reduced with respect to the H-action and the G-action. By Theorem 1.8, we may reverse the order of the two reductions. The reduced space of $T^*G \times V$ with respect to the G-action is equal to the H-space V; reducing once more we obtain the conical space $\Phi_V^{-1}(0)/H$. Consequently, any reduced space is in the neighbourhood of each point isomorphic to the reduced space of some *linear* Hamiltonian space. This observation plays an important role in the proof of the embedding theorem in [13], which says that every reduced space can be embedded in a Euclidean space in such a way that the image is a Whitney stratified space. Also, this enables one to show that the stratification of M_0 satisfies the condition of the frontier, i.e., the closure of a stratum is a union of (connected components of) strata.

2.3. EXAMPLE (the 'lemon'). Let M be the product $S^2 \times S^2$ of two copies of the unit sphere in $\mathbf{R}^3$, both equipped with the symplectic form given by restriction of the standard volume form on $\mathbf{R}^3$. Consider the diagonal action of the circle on M given by rotations round a fixed axis on each factor. This space turns up in the study of perturbed Keplerian systems with axial symmetry, cf. [6]. In that article the reduced space M_0 is computed by means of invariant polynomials. Eugene Lerman showed us the following alternative computation. The Hamiltonian space M can be identified with the product of two copies of $\mathbf{C}P^1$ with a circle action defined by:

$$e^{i\theta} \cdot ((z_1 : z_2), (z_3 : z_4)) = ((e^{i\theta} z_1 : e^{-i\theta} z_2), (e^{i\theta} z_3 : e^{-i\theta} z_4)).$$

A momentum map is given by:

$$J((z_1 : z_2), (z_3 : z_4)) = \frac{1}{2} \left(\frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2} + \frac{|z_3|^2 - |z_4|^2}{|z_3|^2 + |z_4|^2} \right).$$

The image of J is the interval $[-1, 1]$, and its critical values are -1, 0 and 1.

There are two orbit types in M. The fixed point set consists of four points:

$$\begin{array}{rclrcl} P_1 & = & ((1:0),(1:0)) & P_2 & = & ((0:1),(1:0)) \\ P_3 & = & ((1:0),(0:1)) & P_4 & = & ((0:1),(0:1)) \end{array}$$

For all other points the stabilizer is $\{ e^{i\theta} : \theta \equiv 0 \bmod \pi \} \cong \mathbf{Z}_2$. The points P_2 and P_3 are contained in the zero level set, and P_1 and P_4 are the points where J takes its maximum, respectively its minimum.

The reduced space M_{-1} is obviously a point. The Marle-Guillemin-Sternberg local model tells us that near P_4 the Hamiltonian space M can be identified with a neighbourhood of the origin in the tangent space to P_4, which is a complex plane $\mathbf{C}^2$ with its standard S^1-action, $e^{i\theta} \cdot (w_1, w_2) = (e^{i\theta} w_1, e^{i\theta} w_2)$, and momentum map

$$J(w_1, w_2) = 1/2\,(|w_1|^2 + |w_2|^2) - 1.$$

It follows that for small ε the inverse image $J^{-1}(-1+\varepsilon)$ is a three-sphere and the reduced space $M_{-1+\varepsilon}$ is a $\mathbf{C}P^1$. But then M_ξ has to be a $\mathbf{C}P^1$ for all ξ in $(-1, 0)$. Guillemin and Sternberg [10] have shown that the space M_ε for $\varepsilon > 0$ can be obtained from $M_{-\varepsilon}$ by a succession of blowing-ups and blowing-downs. Since $M_{-\varepsilon}$ has only one degree of freedom, there can be no blowing-ups and blowing-downs, so M_ε is again a $\mathbf{C}P^1$. It also follows from Guillemin's and Sternberg's analysis that the reduced space M_0 at the critical level 0 is *homeomorphic* to $\mathbf{C}P^1$. The stratification of M_0 consists of three pieces: two single points, corresponding to the fixed points P_2 and P_3, and one open piece. Using the local model, one can easily figure out what M_0 looks like near the singular points. The symplectic slices at P_2 and P_3 are both equal to the complex plane $\mathbf{C}^2$ with the following S^1-action: $e^{i\theta} \cdot (w_1, w_2) = (e^{i\theta} w_1, e^{-i\theta} w_2)$. This is the $(1, -1)$-resonance discussed in Example 1.1. As a consequence, near both singular points the stratified symplectic space M_0 is isomorphic to the cone in $\mathbf{R}^3$ given by (2), and M_0 can be pictured as two copies of this cone glued together along the base.

From the previous examples the reader may be tempted to infer that reduced spaces are always orbifolds. The following example shows that this is not the case.

2.4. EXAMPLE. The $(1,1,-1,-1)$-*resonance* is the circle action on $\mathbf{C}^4$ generated by the Hamiltonian

$$J(z_1,z_2,z_3,z_4) = 1/2\,(|z_1|^2+|z_2|^2-|z_3|^2-|z_4|^2).$$

We want to describe the topology of the reduced space at the zero level and show that it has a singularity that is worse than an orbifold singularity. The zero level set is given by:

$$|z_1|^2+|z_2|^2 = |z_3|^2+|z_4|^2.$$

Consider the unit seven-sphere S^7 in $\mathbf{C}^4$, given by $|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2 = 1$. Its intersection with the zero level set of J is an S^1-invariant submanifold of $\mathbf{C}^4$, namely the product of two three-spheres of radius $1/2$,

$$J^{-1}(0)\cap S^7 = S^3\times S^3 \subset \mathbf{C}^2\times\mathbf{C}^2.$$

The circle action on the first copy of S^3 is given by $e^{i\theta}\cdot(z_1,z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$, and on the second copy of S^3 it is given by $e^{i\theta}\cdot(z_3,z_4) = (e^{-i\theta}z_3, e^{-i\theta}z_4)$. The quotient of $S^3\times S^3$ by the S^1-action is denoted by $S^3\times_{S^1} S^3$. It is an S^3-bundle over the complex projective line $\mathbf{C}P^1$, associated to the Hopf fibration $S^3\to\mathbf{C}P^1$. Topologically, the reduced space $(\mathbf{C}^4)_0$ can now be written as a *cone*,

$$(\mathbf{C}^4)_0 \sim C(S^3\times_{S^1} S^3),$$

i.e., the product $(S^3\times_{S^1}S^3)\times[0,\infty)$ with the boundary $(S^3\times_{S^1}S^3)\times\{0\}$ collapsed to a point. It is obvious from this description that the reduced space cannot be an orbifold. For instance, an easy computation shows that the local homology in degree 3 at the singular point is nonzero,

$$H_3((\mathbf{C}^4)_0, (\mathbf{C}^4)_0 - \{\text{vertex}\}; \mathbf{Q}) = \mathbf{Q},$$

which is impossible for a six-dimensional orbifold. (All orbifolds are rational homology manifolds; see e.g. [5, Chapter III].)

One can desingularize the reduced space in the following way. Consider the S^1-equivariant map Ψ from $\mathbf{C}^4$ to itself defined by:

$$\Psi : (z_1,z_2,z_3,z_4) \mapsto ((|z_3|^2+|z_4|^2)^{1/2}z_1, (|z_3|^2+|z_4|^2)^{1/2}z_2, z_3, z_4).$$

It maps the subset

$$S^3\times\mathbf{C}^2 = \{\,(z_1,z_2,z_3,z_4) : |z_1|^2+|z_2|^2 = 1\,\}$$

of $\mathbf{C}^4$ onto $J^{-1}(0)$. Therefore it descends to a surjective map $\widetilde{\Psi} : S^3\times_{S^1}\mathbf{C}^2 \to (\mathbf{C}^4)_0$. The space $S^3\times_{S^1}\mathbf{C}^2$ is a $\mathbf{C}^2$-bundle over $\mathbf{C}P^1$. The map $\widetilde{\Psi}$ is a diffeomorphism everywhere, except on the zero section of this bundle, which is mapped to the vertex of $(\mathbf{C}^4)_0$. The reduced space $(\mathbf{C}^4)_0$ can therefore be regarded as a plane bundle over $\mathbf{C}P^1$ with the zero section collapsed to a point.

Why are the strata of a reduced space equal to its symplectic leaves, as claimed at the outset of this section? In fact, the strata may be disconnected, whereas the leaves are by definition always connected, so in general they are not the same. We claim, however, that the symplectic leaves of M_0 are precisely the connected components of the strata $(M_0)_{(H)}$. Indeed, let m be a point in $(M_0)_{(H)}$. It is not hard to show that the symplectic leaf of m has to be contained in $(M_0)_{(H)}$. It suffices therefore to prove that it is *open* in $(M_0)_{(H)}$. This amounts to showing that an arbitrary point in $(M_0)_{(H)}$ sufficiently close to p can be joined to p by means of an integral curve of a Hamiltonian that is defined *globally* on M_0. Of course, it is easy to find such a Hamiltonian, call it f, that lives on $(M_0)_{(H)}$ only. The problem is to extend f to a smooth function on the whole of M_0. We may assume that f has compact support on $(M_0)_{(H)}$. We can lift f to a compactly supported G-invariant function F on the locally closed submanifold $J^{-1}(0) \cap M_{(H)}$ of M. Because F has compact support, we can extend it to a smooth function $\check{F}$ defined on the whole of M, and we may assume that $\check{F}$ is still G-invariant. So $\check{F}$ descends to a smooth function $\check{f}$ on the reduced space, whose restriction to $(M_0)_{(H)}$ equals f, as desired. We have proved:

2.5. Theorem. *The symplectic leaves of the reduced space M_0 coincide with the connected components of its strata $(M_0)_{(H)}$, $H < G$.*

This implies that the stratification of a reduced space depends only on its Poisson algebra of smooth functions.

The manifold of orbit type (H) contains the *manifold of symmetry* H, defined by

$$M_H = \{\, m \text{ in } M : \text{stabilizer of } m \text{ is equal to } H \,\}.$$

This is a symplectic submanifold of M, which is invariant under the action of the normalizer $N_G(H)$ of H in G. Let $\mathbf{n}$ denote the Lie algebra of $N_G(H)$. The action of $N_G(H)$ on M_H is Hamiltonian; a momentum map is given by the restriction of J to M_H, followed by the canonical projection $\mathbf{g}^* \to \mathbf{n}^*$. Because H iself acts trivially on M_H, the action descends to an action of the quotient group $L = N_G(H)/H$, and the momentum map descends to a momentum map for the L-action, $J_H : M_H \to \mathbf{l}^*$. Since L acts freely on M_H, the origin in $\mathbf{l}^*$ is a regular value of J_H.

2.6. Theorem. *The stratum $(M_0)_{(H)}$ of the reduced space of type (H) is symplectomorphic to the Marsden-Weinstein reduced space $J_H^{-1}(0)/L$.*

Because of this theorem, the procedure for lifting a Hamiltonian flow on M_0 to M is completely analogous to the regular case. Lifting an integral curve of a Hamiltonian through a point in $(M_0)_{(H)}$ amounts to solving a differential equation on the group L. Theorem 2.6 also enables one to write down a version of Smale's criterion for relative equilibria from which the assumption of regularity has been removed. See [13] for a further discussion.

We conclude by quoting one more theorem from [13]. Its proof hinges on the fact proved by Kirwan [12] that the fibres of a proper momentum map are connected.

2.7. Theorem. *Assume that the momentum map $J : M \to \mathbf{g}^*$ is proper. Then there exists a unique open stratum in the reduced space M_0. It is connected and dense.*

2.8. Remark. If the momentum map is not proper, the reduced space is not necessarily connected. Similarly, if G is a noncompact Lie group acting properly on M, the reduced space M_0 is well-defined as a stratified symplectic space, but it needn't be connected. However, in both these cases it is still true that each component C of M_0 has a unique open stratum, which is connected and dense in C.

2.9. Remark. As an immediate corollary to Theorem 2.7 we see that the Poisson algebra $C^\infty(M_0)$ of the reduced space is nondegenerate, i.e., its centre consists of the (locally) constant functions only.

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