Stratified Symplectic Spaces and Reduction

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Abstract

Let (M, ω) be a Hamiltonian G-space with proper momentum map $J : M \to \mathbf{g}^*$. It is well-known that if zero is a regular value of J and G acts freely on the level set $J^{-1}(0)$, then the reduced space $M_0 := J^{-1}(0)/G$ is a symplectic manifold. We show that if the regularity assumptions are dropped the space M_0 is a union of symplectic manifolds, i.e., it is a *stratified symplectic space*. Arms et al., [2], proved that M_0 possesses a natural Poisson bracket. Using their result we study Hamiltonian dynamics on the reduced space. In particular we show that Hamiltonian flows are strata-preserving and give a recipe for a lift of a reduced Hamiltonian flow to the level set $J^{-1}(0)$. Finally we give a detailed description of the stratification of M_0 and prove the existence of a connected open dense stratum.

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Contents

Introduction

Let (M, ω) be a connected Hamiltonian G-space with Ad^*-equivariant moment map $J : M \to \mathbf{g}^*$. Let us assume that the Lie group G is compact. (Most of the results of the paper hold for proper actions of arbitrary Lie groups. The proofs, however, are technically more difficult.)

Recall the Marsden–Weinstein reduction procedure (cf. [25]). If zero is a regular value of the moment map then the zero level set $Z = J^{-1}(0)$ is a submanifold and the action of G on Z is locally free (i.e., all the stabilizers are discrete). Let us assume that the action is actually free. Then the orbit space $M_0 = Z/G$ is a manifold. The manifold Z is coisotropic and (this is the miracle of the reduction) the leaves of the null foliation of $\omega|_Z$ are the G-orbits. It follows that there exists a unique symplectic form ω_0 on M_0 such that

$$\pi^*\omega_0 = \omega|_Z,$$

where $\pi : Z \to M_0$ is the orbit map. The pair (M_0, ω_0) is the Marsden–Weinstein reduced space.

In general, for a regular value μ of J different from zero, Marsden and Weinstein defined the reduced space at μ to be the quotient

$$M_\mu = J^{-1}(\mathcal{O}_\mu)/G,$$

where $\mathcal{O}_\mu$ denotes the coadjoint orbit through μ. Consider the symplectic manifold $M \times \mathcal{O}_{-\mu}$, the symplectic product of M with the coadjoint orbit through $-\mu$. The diagonal action of G on $M \times \mathcal{O}_{-\mu}$ is Hamiltonian with a moment map J_μ sending $(m, f) \in M \times \mathcal{O}_{-\mu}$ to $J(m) + f$. Zero is a regular value of J_μ and it is easy to see that the Marsden–Weinstein reduced space at μ can be identified with

$$M_\mu = J_\mu^{-1}(0)/G.$$

This is the so-called 'shifting trick', which allows one to talk exclusively about reduction at zero.

If h is a G-invariant Hamiltonian on M, i.e., $h \in C^\infty(M)^G$, then the restriction $h|_Z$ descends to a smooth function h_0 on the reduced space M_0 so that

$$\pi^* h_0 = h|_Z.$$

Marsden and Weinstein gave a recipe for lifting the reduced flow, that is to say for computing the Hamiltonian flow of h on Z from the knowledge of the flow of h_0 on M_0. They showed that the lifting problem amounts to solving a differential equation on the group G.

If zero is a singular value of J then the level set Z is not a manifold. Moreover there are jumps in the dimension of the orbits of the points of Z. A lot of work has been done over the last ten years in the direction of finding a 'correct' reduction procedure for singular values of the moment map. See [3] for a description and comparison of several different approaches.

Our point of view is that the reduced space M_0 is a *stratified symplectic space*. Roughly speaking it means that:

i. M_0 is a union of symplectic manifolds (the *symplectic strata*);

ii. these manifolds fit together nicely;

iii. there exists a naturally defined subclass of the class of continuous functions on M_0, a set of 'smooth functions', whose members restrict to smooth functions on the strata;

iv. these smooth functions form a Poisson algebra and the bracket agrees with the brackets on the strata defined by the symplectic forms.

We will see that the notions of Hamiltonian dynamics, Liouville volume, group actions, moment maps, etc. all make sense in the setting of stratified symplectic spaces.

If the manifold M is Kähler and the group G acts holomorphically on M, we can show that all strata of the reduced space are Kähler manifolds. In this case the action of G extends to an action of the complexified group $G^{\mathbf{C}}$, and Kirwan [19] has identified the reduced space with a 'Kähler quotient',

$$M_0 = M^{\mathrm{ss}}//G^{\mathbf{C}},$$

generalizing results of Guillemin and Sternberg [15] and Kempf and Ness [18]. In many cases we can show that the symplectic stratification of M_0 is identical to the stratification by $G^{\mathbf{C}}$-orbit types. We believe that this is true in general. However, the minimal complex-analytic stratification of M_0 is in general coarser than the stratification by orbit types. These questions and the related problem of geometric quantization of singular spaces will be discussed elsewhere.

The paper is organized as follows. In Section 1 we review the definition of a stratified space and explain what one may mean by a smooth structure on a singular space. In Section 2 we prove the existence of a decomposition of the reduced space into a union of symplectic manifolds. Section 3 is a discussion of Hamiltonian dynamics on M_0. We review the definition of the Poisson bracket and describe a procedure for lifting a reduced flow. In Section 4 we discuss reduction in stages and momentum maps on stratified symplectic spaces. Section 5 is a study of a conical local model for M_0. In Section 6 we show that the reduced space can be embedded into a Euclidean space. This enables us to show that the symplectic decomposition is indeed a stratification. In Section 7 we prove a tubular neighbourhood theorem for a stratum. Finally, in the Appendix, Section 8, we explain briefly Sternberg's minimal coupling procedure.

A summary of this work appears in [8] (cf. also [32]). A number of applications will be discussed in [21].

Note on terminology. We use the terms momentum mapping, momentum map, moment mapping and moment map interchangeably.

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1 Stratified Spaces

As with CW and simplicial complexes, the main idea of a stratification is that of a partition of a nice topological space X into a disjoint union of manifolds.

1.1. DEFINITION (cf. [12, page 36]). Let X be a Hausdorff and paracompact topological space and let $\mathcal{I}$ be a partially ordered set with order relation denoted by $\leq$. An *$\mathcal{I}$-decomposition* of X is a locally finite collection of disjoint locally closed manifolds $S_i \subset X$ (one for each $i \in \mathcal{I}$) called *pieces* such that

i. $X = \bigsqcup_{i\in\mathcal{I}} S_i$;

ii. $S_i \cap \bar{S}_j \neq \emptyset \iff S_i \subset \bar{S}_j \iff i \leq j$.

Condition ii. is the *frontier condition.* If $S_i \subset \bar{S}_j$ we write $S_i \leq S_j$. If $S_i \leq S_j$ and $S_i \neq S_j$ we write $S_i < S_j$. We call the space X an *$\mathcal{I}$-decomposed space.*

Define the dimension of a decomposed space X to be $\dim X = \sup_{i\in\mathcal{I}} \dim S_i$. We will only consider finite-dimensional spaces. Note that we do not require the pieces to be connected.

1.2. EXAMPLE. Let M be a smooth manifold and G a compact Lie group acting on M by diffeomorphisms. For a subgroup H of G denote by $M_{(H)}$ the set of all points whose stabilizer is conjugate to H, the *stratum* of M of *orbit type* (H). Here the indexing set $\mathcal{I}$ is the set of all possible stabilizer subgroups modulo the conjugacy relation: $H \sim K \iff$ there exists $g \in G$ with $gHg^{-1} = K$. The ordering is by reverse subconjugacy: the class of H is 'bigger' than the class of K, $(H) > (K)$, if and only if there exists $g \in G$ with $H \subset gKg^{-1}$.

1.3. Remark. There is a problem in the above example which was swept under the rug: a stratum $M_{(H)}$ may have components of different dimensions. For instance consider the action of a circle on the complex projective plane $\mathbf{C}P^2$ given by $e^{i\theta} \cdot (z_0 : z_1 : z_2) = (e^{i\theta} z_0 : z_1 : z_2)$. The set fixed by the action consists of a point $(1 : 0 : 0)$ and a line at infinity $\{(0 : z_1 : z_2)\}$. The solution is either to allow the pieces of a decomposition to have components of different dimensions or to refine the decomposition in question. To keep the notation manageable we may sometimes refine the decomposition without saying so explicitly.

1.4. Example. We keep the notation of Example 1.2. The orbit space M/G is not a manifold, but it is a decomposed space. Its pieces are the manifolds $M_{(H)}/G$. For instance, if M is $\mathbf{R}^n$ with the standard action of $SO(n)$ then the orbit space is the closed half-line $\{x \in \mathbf{R} : x \geq 0\} = \{0\} \cup \{x > 0\}$.

1.5. Definition. Given a decomposition $\{S_i\}_{i \in \mathcal{I}}$ of a space X define the *depth* of a piece S in X to be the integer

$$\operatorname{depth}_X S = \sup\{n : \text{ there exist pieces } S = S^0 < S^1 < \cdots < S^n\}.$$

Note that the depth of a piece S is bounded by its codimension $\dim X - \dim S$ and so is finite.

1.6. Definition. We define the *depth* of X to be

$$\operatorname{depth} X = \sup_{i \in \mathcal{I}} \operatorname{depth}_X S_i.$$

Again we have $\operatorname{depth} X \leq \dim X$. For example, if X is a manifold and has only one piece, namely X itself, then $\operatorname{depth} X = 0$. If X is a *cone* over a manifold Y, $X = \mathring{C}Y$, i.e., if X is obtained by collapsing the boundary $Y \times \{0\}$ of the half-open cylinder $Y \times [0, \infty)$, then X decomposes into two pieces: $Y \times (0, \infty)$ and the vertex, which is the collapsed boundary, and so has depth equal to 1. In general given a space X with a decomposition $\{S_i\}$ the cone over X has a decomposition consisting of the pieces of the form $S_i \times (0, \infty)$ and of the vertex. Therefore

$$\operatorname{depth} \mathring{C}X = \operatorname{depth} X + 1.$$

A decomposition of a space is called a *stratification* if the pieces fit together in a particularly nice way. The definition is a recursion on the depth of the space.

1.7. Definition (cf. [11]). A space X is called a *stratified space* if the pieces of X, called *strata,* satisfy the following condition:

> Given a point x in a piece S there exist an open neighbourhood U of x in X, an open ball B around x in S, a stratified space L, called the *link* of x, and a homeomorphism
>
> $$\varphi : B \times \mathring{C}L \to U$$
>
> that preserves the decompositions.

1.8. Example. It is well-known that the orbit space M/G of Example 1.4 is a stratified space (see, for example, [6]). The proof is easy. The key ingredient is the existence of slices.

So far our definitions have been purely topological. However, sometimes when the strata of a space X are smooth manifolds, it makes sense to single out a subalgebra $C^\infty(X)$ of the algebra of continuous functions $C^0(X)$ having the property that for any $f \in C^\infty(X)$ the restriction to a stratum S is smooth, $f|_S \in C^\infty(S)$. We will call such an algebra $C^\infty(X)$ a *smooth structure* on X. Given two spaces X and Y with smooth structures $C^\infty(X)$ and $C^\infty(Y)$, a continuous map $\varphi : X \to Y$ is *smooth* if for any $f \in C^\infty(Y)$ the composition $f \circ \varphi$ is smooth, $f \circ \varphi \in C^\infty(X)$. For example, according to this definition, the inclusion of a stratum into the space is smooth. In the same vein one can talk about two stratified spaces being diffeomorphic, etc.

There are two basic examples of smooth structures on stratified spaces.

1.9. Example (quotient smooth structures). Let a compact Lie group G act on a manifold M. The smooth structure on the orbit space M/G is the smallest subalgebra of $C^0(M/G)$ making the orbit map $\pi : M \to M/G$ smooth, i.e.,

$$C^\infty(M/G) = \{f : M/G \to \mathbf{R} : \text{ the composition } f \circ \pi \text{ is smooth } \}.$$

The algebra $C^\infty(M/G)$ is isomorphic to the space $C^\infty(M)^G$ of G-invariant functions.

Note that the same stratified space can support a variety of smooth structures. For example, the orbit space for the standard action of the cyclic group $\mathbf{Z}_n$ on the complex line $\mathbf{C}$ is again $\mathbf{C}$ stratified as

$$\mathbf{C} = \{0\} \cup \mathbf{C}^\times,$$

i.e., as a cone over S^1. However, the algebras $C^\infty(\mathbf{C}/\mathbf{Z}_n) = C^\infty(\mathbf{C})^{\mathbf{Z}_n}$ are different for different n.

1.10. Example (Whitney smooth functions). If a stratified space X is a subspace of a smooth manifold M we define

$$C^\infty(X) = \{f : X \to \mathbf{R} : \text{ there exists } \bar{F} \in C^\infty(M) \text{ with } f = \bar{F}|_X\}.$$

(See Whitney [36].) The algebra $C^\infty(X)$ is isomorphic to the quotient of $C^\infty(M)$ by the ideal of functions that vanish on X.

1.11. Example. Arms, Cushman and Gotay [2] used a combination of the approaches of the examples 1.9 and 1.10 to define smooth structures in a Hamiltonian setting as follows.

Let M now be a Hamiltonian G-space with moment map $J : M \to \mathbf{g}^*$. We will see in the next few sections that the zero fibre $Z = J^{-1}(0)$ and its quotient, the reduced space $M_0 = Z/G$, are stratified spaces. Arms et al. (op. cit.) define a function $f_0 : M_0 \to \mathbf{R}$ to be *smooth* if there exists a function $F \in C^\infty(M)^G$ with $F|_Z = \pi^* f_0$. (Here again $\pi : Z \to M_0$ is the orbit map.) In other words $C^\infty(M_0)$ is isomorphic to $C^\infty(M)^G/I^G$, where I^G is the ideal of invariant functions vanishing on Z. They also show that the algebra $C^\infty(M_0)$ inherits a Poisson algebra structure from $C^\infty(M)$. We will see that this Poisson bracket is compatible with the symplectic forms on the strata of M_0.

This example leads us to a working definition of a stratified symplectic space.

1.12. Definition. A *stratified symplectic space* X is a stratified space with a smooth structure $C^\infty(X)$ such that:

i. each stratum S is a symplectic manifold;

ii. $C^\infty(X)$ is a Poisson algebra;

iii. the embeddings $S \hookrightarrow X$ are Poisson.

2 A Decomposition of a Reduced Phase Space

The main result of this section is

2.1. Theorem. *Let (M, ω) be a Hamiltonian G-space with moment map $J : M \to \mathbf{g}^*$. The intersection of the stratum $M_{(H)}$ of orbit type (H) with the zero level set Z of the moment map is a manifold and the orbit space*

$$(M_0)_{(H)} = (M_{(H)} \cap Z)/G$$

has a natural symplectic structure $(\omega_0)_{(H)}$ whose pullback to $Z_{(H)} := M_{(H)} \cap Z$ coincides with the restriction to $Z_{(H)}$ of the symplectic form on M. Consequently the stratification of M by orbit types induces a decomposition of the reduced space $M_0 = Z/G$ into a disjoint union of symplectic manifolds,

$$M_0 = \bigsqcup_{H<G} (M_0)_{(H)}. \tag{1}$$

The proof of Theorem 2.1 is an application of the local normal form for the moment map discovered independently by Marle [24] and Guillemin and Sternberg [16]. Their result is based on the fact that an orbit through a point in the zero level set of the moment map is embedded isotropically. A theorem of Weinstein says that given an isotropic embedding of a manifold X in a symplectic manifold Y, the symplectic form on a small neighbourhood of X is completely determined by the symplectic normal bundle $N(X) = (TX)^\omega/TX$ of the embedding. (The fibre $(T_pX)^\omega$ of $(TX)^\omega$ is the symplectic perpendicular of T_pX in T_pY.) Later in the paper we will need a generalization of the isotropic embedding theorem, and so it seems appropriate to prove this generalization now.

2.2. Theorem (Constant Rank Embedding). *Let B be a manifold furnished with a closed two-form τ of constant rank. Then there exists a one-to-one correspondence (modulo appropriate equivalences) between*

1. *symplectic vector bundles over B, and*
2. *embeddings i of B into higher dimensional symplectic manifolds (A, σ) such that $i^*\sigma = \tau$.*

2.3. Remark. Before proving the theorem, we would like to point out three special cases.

Case 1: The form τ on the manifold B is zero. Then the theorem gives a one-to-one correspondence between isotropic embeddings and symplectic vector bundles. This is the isotropic embedding theorem of Weinstein [34].

Case 2: The form τ has maximal rank, i.e., the manifold B is symplectic. Then this is the symplectic embedding theorem (cf. [22]).

Case 3: If N is the zero bundle over B then the theorem reduces to the coisotropic embedding theorem of Gotay [13].

Conversely, Theorem 2.2 can be regarded as a synthesis of the coisotropic and symplectic embedding theorems.

Proof of Theorem 2.2. Given an embedding $i : (B, \tau) \to (A, \sigma)$, we associate to it the *symplectic normal bundle* $N(i)$, whose fibre at the point b in B is defined by

$$N(i)_b = \frac{(T_bB)^\sigma}{T_bB \cap (T_bB)^\sigma}.$$

Here we have identified the manifold B with its image $i(B)$ in A. Since the pullback of the form σ to B equals the form τ and since τ has constant rank, $N(i)$ is a well-defined vector bundle over B. By construction, its fibres are symplectic.

Conversely, suppose we are given a manifold B with a two-form τ of constant rank and a symplectic vector bundle N over B. We are going to exhibit an embedding $i : B \to A$ of (B, τ) into a symplectic manifold (A, σ) such that the normal bundle $N(i)$ associated to the embedding is isomorphic to the bundle N, and such that $i^*\sigma = \tau$. Roughly speaking, we shall first embed (B, τ) coisotropically into a symplectic manifold, and then use the bundle N to embed this symplectic manifold into a higher dimensional one. The details are as follows.

The form τ determines a subbundle of the tangent bundle TB, namely the bundle $\mathcal{V}$ of vectors tangent to the null foliation, with typical fibre

$$\mathcal{V}_b = \{v \in T_bB : \tau(v, w) = 0 \text{ for all } w \in T_bB\}.$$

Let $\mathcal{V}^*$ be the dual bundle. There is a natural surjection from the cotangent bundle T^*B to $\mathcal{V}^*$,

$$T^*B \to \mathcal{V}^*,$$

and by choosing a section s to this surjection we can regard $\mathcal{V}^*$ as a subbundle of T^*B. (This amounts to choosing a splitting $TB = \mathcal{V} \oplus \mathcal{H}$.) By restricting the canonical symplectic form γ on T^*B to $\mathcal{V}^*$ and adding to this the pullback of the form τ to $\mathcal{V}^*$, we obtain a closed two-form μ on $\mathcal{V}^*$,

$$\mu = (\pi_{\mathcal{V}^*})^*\tau + s^*\gamma.$$

It is not difficult to show that this form is nondegenerate in a neighbourhood U of the zero section of the bundle $\mathcal{V}^*$. The zero section is by construction coisotropic in $\mathcal{V}^*$, and the form μ pulls back to the form τ on B via the zero section. Caution: although the total space of $\mathcal{V}^*$ is symplectic in the neighbourhood U of the zero section, $\mathcal{V}^*$ is not a symplectic vector bundle over B, since its fibres are not symplectic.

Pulling back the bundle N to $\mathcal{V}^*$ we get a symplectic vector bundle $N^\#$ over $\mathcal{V}^*$, which fits into a commutative diagram,

$$\begin{array}{ccc} N^\# & \longrightarrow & N \\ \downarrow & & \downarrow \pi_N \\ \mathcal{V}^* & \xrightarrow{\pi_{\mathcal{V}^*}} & B. \end{array}$$

By means of the minimal coupling procedure (see the Appendix) we can put a closed two-form on the total space of the vector bundle $N^\# \to \mathcal{V}^*$. If we identify $\mathcal{V}^*$ with the zero section of this bundle, the minimal coupling form is nondegenerate in a neighbourhood of the subset U of $\mathcal{V}^*$ (see the Appendix, Example 8.6).

Note that we can also view the space $N^\#$ as the Whitney sum $\mathcal{V}^* \oplus N$ of the bundles $\mathcal{V}^*$ and N over the base B.

To summarize, we have found a neighbourhood A of the zero section in the bundle $\mathcal{V}^* \oplus N$ with a symplectic form σ. It is easy to check that the pullback of σ via the zero section equals the form τ, and that the symplectic normal bundle to the zero section is just N.

It remains to prove the uniqueness part of the theorem. Suppose that $i' : B \to A'$ is another symplectic embedding of B with symplectic normal bundle N. The geometric normal bundle $((i')^*TA')/TB$ of B in A' can be identified with the direct sum bundle $\mathcal{V}^* \oplus N$, where $\mathcal{V}^*$ is defined as above. It follows from the tubular neighbourhood theorem that there exist an open neighbourhood U of $i(B)$ in A, an open neighbourhood U' of $i'(B)$ in A', and a diffeomorphism f from U onto U', such that $f \circ i = i'$. The Darboux–Moser–Weinstein Theorem (see [34] or [17, Theorem 22.1]) now implies that, after shrinking the neighbourhoods U and U' if necessary, we can deform the map f to a symplectomorphism ϕ such that $\phi \circ i = i'$. □

2.4. Remark. In complete analogy with the Darboux Theorem, there exists an equivariant version of the constant rank embedding theorem: in the presence of a compact Lie group L of automorphisms of B preserving the form τ there is a one-to-one correspondence between symplectic L-vector bundles over B and L-equivariant symplectic embeddings of B.

Let us now return to the local normal form recipe of Marle, Guillemin and Sternberg. Let p be a point in the zero level set Z of the momentum map $J : M \to \mathbf{g}^*$, H the stabilizer of p and V the symplectic vector space $(T_p(G \cdot p))^\omega / T_p(G \cdot p)$, a fibre of the symplectic normal bundle of the orbit in M. We will refer to V as a *symplectic slice* for the action. The symplectic normal bundle of the orbit is $G \times_H V$, a vector bundle associated to the principal fibration $H \to G \to G \cdot p$.

We claim that the total space Y of the associated bundle $G \times_H ((\mathbf{g}/\mathbf{h})^* \times V)$ can be given the structure of a symplectic manifold making the embedding $G/H \hookrightarrow Y$ (as the zero section) isotropic with the corresponding normal bundle being $G \times_H V$.

To simplify the computations, we fix an $Ad(G)$-invariant inner product on the Lie algebra $\mathbf{g}$ of G. Then $\mathbf{g}$ splits $Ad(H)$-invariantly as $\mathbf{h} \times \mathbf{m}$ where $\mathbf{m}$ is the orthocomplement of the Lie algebra $\mathbf{h}$ of H, and we also have the dual splitting $\mathbf{g}^* = \mathbf{h}^* \times \mathbf{m}^*$. Note that $\mathbf{m}^*$ is isomorphic to $(\mathbf{g}/\mathbf{h})^*$.

The cotangent bundle of G is trivial: the map $\mathcal{L} : G \times \mathbf{g}^* \to T^*G$, $(g, \eta) \mapsto (g, (dL_{g^{-1}})^T \eta)$ is an isomorphism (here $L_g : G \to G$ denotes left multiplication by $g \in G$).

Consider the action of G on G by right multiplication:

$$R(a)(g) = ga^{-1}.$$

This action lifts to a Hamiltonian action $R^* : G \to \mathit{Diff}(T^*G)$. With respect to the trivialization $\mathcal{L}$ the action is given by

$$R^*(a) : (g, \eta) \mapsto (ga^{-1}, Ad^*(a)\eta),$$

where Ad^* denotes the coadjoint representation, $Ad^*(g) = (Ad(g^{-1}))^T$. The corresponding moment map J_R is given by the formula

$$J_R(g, \eta) = -\eta.$$

The action of G on G by left multiplication also lifts to a Hamiltonian action on $G \times \mathbf{g}^*$ which is given by

$$L^*(a) : (g, \eta) \mapsto (ag, \eta).$$

The corresponding moment map J_L is given by

$$J_L(g, \eta) = Ad^*(g)\eta.$$

Now, the restriction to H of the action R^* is a Hamiltonian action and the corresponding moment map Φ_R is J_R followed by the orthogonal projection of $\mathbf{g}^*$ onto $\mathbf{h}^*$. Note that Φ_R is H-equivariant regardless of whether H is connected.

The linear symplectic action of H on the vector space V, $H \to Sp(V, \omega_V)$ is also Hamiltonian. The corresponding moment map Φ_V is given by the formula

$$\langle \xi, \Phi_V(v) \rangle = 1/2\, \omega_V(\xi_V \cdot v, v), \tag{2}$$

where ξ_V denotes the image of $\xi \in \mathbf{h}$ in the Lie algebra $\mathbf{sp}(V, \omega_V)$ and $v \in V$. Again, it is easy to check that Φ_V is H-equivariant.

Therefore the product action of H on $T^*G \times V$ is Hamiltonian with H-equivariant moment map $\Phi : G \times \mathbf{m}^* \times \mathbf{h}^* \times V \to \mathbf{h}^*$ being simply the sum $\Phi_R + \Phi_V$,

$$\Phi : (g, \mu, \eta, v) \mapsto \Phi_V(v) - \eta.$$

Zero is a regular value of Φ. We claim that the reduced space $\Phi^{-1}(0)/H$ can be identified with $Y = G \times_H (\mathbf{m}^* \times V)$. Indeed, the map

$$\begin{array}{rcl} G \times \mathbf{m}^* \times V & \to & \Phi^{-1}(0) \subset G \times \mathbf{m}^* \times \mathbf{h}^* \times V \\ (g, \mu, v) & \mapsto & (g, \mu, \Phi_V(v), v) \end{array}$$

is an H-equivariant diffeomorphism. This endows Y with a symplectic structure. We leave it to the reader to check that the embedding of G/H to into Y is isotropic and that the normal bundle is $G \times_H V$.

The equivariant version of the isotropic embedding theorem now implies that there exist a neighbourhood U_0 of the zero section of Y, a neighbourhood U in M of the orbit $G \cdot p$, and a G-equivariant symplectic diffeomorphism

$$\varphi : U_0 \to U.$$

Next we describe the Hamiltonian action of G on our model space Y. Recall that the actions L^* and R^* of G on T^*G commute. We regard L^* as an action on the product $T^*G \times V$ by letting G act trivially on V. Then L^* commutes with the product action of H and the moment map $J_L : G \times \mathbf{g}^* \times V \to \mathbf{g}^*, \quad (g, \eta, v) \mapsto Ad^*(g)\eta$ is H-invariant. Consequently the action L^* descends to an action on the H-reduced space $\Phi^{-1}(0)/H$ and the corresponding moment map $J : G \times_H (\mathbf{m}^* \times V) \to \mathbf{g}^*$ sends a point $[g, \mu, v]$ to $Ad^*(g)(\mu + \Phi_V(v))$. (Here $[g, \mu, v]$ denotes the image of $(g, \mu, v) \in G \times \mathbf{m}^* \times V$ under the projection onto $(G \times \mathbf{m}^* \times V)/H = G \times_H (\mathbf{m}^* \times V)$.) We have proved

2.5. Proposition (local normal form for the moment map). *Let H be the stabilizer of $p \in Z$ and V be the symplectic slice to the orbit $G \cdot p$. Then a neighbourhood of the orbit is equivariantly symplectomorphic to a neighbourhood of the zero section of $Y = G \times_H (\mathbf{m}^* \times V)$ with the G-moment map J given by the formula*

$$J([g, \mu, v]) = Ad^*(g)(\mu + \Phi_V(v)). \tag{3}$$

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We compute with the model Y. The intersection of the zero level set J with the fibre of the bundle $\mathbf{m}^* \times V \to Y \to G/H$ is simply the quadratic cone $\{0\} \times \Phi_V^{-1}(0)$. On the other hand the intersection of the stratum $Y_{(H)}$ of orbit type (H) with the fibre consists of the points in $\mathbf{m}^* \times V$ whose stabilizer is conjugate to H. Therefore

$$(\mathbf{m}^* \times V) \cap J^{-1}(0) \cap Y_{(H)} = V_H,$$

the linear subspace of V which is fixed by H, which is symplectic. Since the set $J^{-1}(0) \cap Y_{(H)}$ is G-invariant and the action of G on G/H is transitive we have

$$J^{-1}(0) \cap Y_{(H)} = G \cdot \left((\mathbf{m}^* \times V) \cap J^{-1}(0) \cap Y_{(H)}\right) = G \times_H V_H = G/H \times V_H.$$

We conclude that the orbit space $(Y_{(H)} \cap J^{-1}(0))/G$ is the symplectic manifold V_H. The first assertion of the theorem now follows; the reduced space M_0 decomposes into a union of symplectic manifolds $(M_0)_{(H)}$, $H < G$.

It remains to show that the symplectic pieces satisfy the frontier condition (possibly after being decomposed further into connected components). Suppose a piece $(M_0)_{(H)}$ intersects nontrivially the closure of a piece $(M_0)_{(K)}$,

$$(M_0)_{(H)} \cap \overline{(M_0)}_{(K)} \neq \emptyset.$$

We want to show that the closure of $(M_0)_{(K)}$ contains every connected component of $(M_0)_{(H)}$ that it intersects nontrivially. Again we compute with the local model Y for a neighbourhood a point $x \in (M_0)_{(H)} \cap \overline{(M_0)}_{(K)}$. It is easy to see from the model that the group K has to be subconjugate to H. Without loss of generality we may assume that K is actually a subgroup of H. Let W be the symplectic perpendicular of V_H in the symplectic slice V. The space W is symplectic and H-invariant. (Since K is a subgroup of H, W is K-invariant as well.) Denote the momentum map corresponding to the action of H on W by Φ_W. It is nothing more than the restriction of Φ_V to W. Note that Φ_V is constant along the directions of V_H. It follows that

$$\Phi_V^{-1}(0) = \Phi_W^{-1}(0) \times V_H. \tag{4}$$

Moreover,

$$Y_{(K)} = G \times_H (\mathbf{m}^*_{(K)} \times V_{(K)}),$$

so that

$$J^{-1}(0) \cap Y_{(K)} = G \times_H (\Phi_W^{-1}(0) \cap W_{(K)}) \times V_H \tag{5}$$

Since the map Φ_W is homogeneously quadratic, the set is $\Phi_W^{-1}(0) \cap W_{(K)}$ is invariant under multiplication by positive scalars. Combined with (5) this implies that the closure of the set $J^{-1}(0) \cap Y_{(K)}$ in Y contains the whole of the set $J^{-1}(0) \cap Y_{(H)} = (G/H) \times V_H$. Consequently the closure of the piece $(Y_0)_{(K)}$ in Y_0 contains $(Y_0)_{(H)}$. □

2.6. Remark. In the strict sense of the word, the piece $(M_0)_{(H)}$ is not always a manifold, since it may contain components of different dimensions. As an example, consider the S^1-action on $\mathbf{C}^3$ generated by the Hamiltonian

$$z = (z_1, z_2, z_3) \mapsto |z_3|^2,$$

and, similarly, the action on $\mathbf{C}^2$ generated by the Hamiltonian

$$w = (w_1, w_2) \mapsto |w_2|^2.$$

These actions descend to Hamiltonian actions on $\mathbf{C}P^2$ and $\mathbf{C}P^1$, respectively. The diagonal action on $\mathbf{C}P^2 \times \mathbf{C}P^1$ has the Hamiltonian

$$J : ((z_1 : z_2 : z_3), (w_1 : w_2)) \mapsto \frac{|z_3|^2}{\|z\|^2} + \frac{|w_2|^2}{\|w\|^2},$$

which has the critical levels 0, 1 and 2. The level set $J^{-1}(1)$ contains a two-dimensional component of the fixed point set, namely

$$\{((z_1 : z_2 : 0), (0 : 1))\},$$

and also the isolated fixed point $((0 : 0 : 1), (1 : 0))$.

3 Dynamics on a Reduced Phase Space

We now discuss Hamiltonian dynamics on the reduced space $M_0 = Z/G$. Recall that although M_0 is not a manifold we can still define a space of smooth functions $C^\infty(M_0)$ as in Example 1.11.

Note that for each piece $(M_0)_{(H)}$ the space $\{f|_{(M_0)_{(H)}} : f \in C^\infty(M_0)\}$ is dense in the space of smooth functions on the piece. This is because the pullback to Z of a function compactly supported on $(M_0)_{(H)}$ can be easily extended to a smooth G-invariant function on the whole of M.

The symplectic structures on the pieces allow us to define on $C^\infty(M_0)$ a Poisson bracket simply by using the Poisson brackets on the function spaces $C^\infty((M_0)_{(H)})$. However, it is not a priori clear that the bracket of two smooth functions on M_0 is again a smooth function.

3.1. Proposition. *The bracket of two smooth functions on M_0 is again a smooth function.*

Proof. Let f and g be two smooth functions on M_0 and let $\bar{f}$, $\bar{g}$ be two G-invariant functions on M with $\bar{f}|_Z = \pi^* f$, $\bar{g}|_Z = \pi^* g$. Showing that

$$\pi^*(\{f, g\}_{M_0}) = \{\bar{f}, \bar{g}\}_M\Big|_Z \tag{6}$$

will prove the proposition.

It is enough to establish (6) for a point in M_0. So let p be a point in $Z_{(H)}$, the intersection of the zero level set with the stratum of type (H), for some subgroup $H < G$. Let p_0 be its image under the orbit map $\pi : Z_{(H)} \to (M_0)_{(H)}$. By Theorem 2.1, the pullback by π of $(\omega_0)_{(H)}$, the symplectic form on the piece $(M_0)_{(H)}$, is equal to the restriction to $Z_{(H)}$ of the symplectic form on M,

$$\pi^*(\omega_0)_{(H)} = \omega|_{Z_{(H)}}.$$

It follows that if we take the Hamiltonian vector field $\Xi_{\bar{f}}$ of $\bar{f}$, which is tangent to $Z_{(H)}$, evaluate it at p and push it forward by π, we will then get the Hamiltonian vector field of f evaluated at p_0,

$$\pi_*(\Xi_{\bar{f}}(p)) = \Xi_f(p_0).$$

By definition, the Poisson bracket of f and g at the points of $(M_0)_{(H)}$ can be written in terms of their Hamiltonian vector fields as

$$\{f, g\} = (\omega_0)_{(H)}(\Xi_f, \Xi_g).$$

Therefore

$$\begin{aligned}
\pi^*\{f, g\}_{M_0} &= \pi^*[(\omega_0)_{(H)}(\Xi_f, \Xi_g)] \\
&= \pi^*[(\omega_0)_{(H)}(\pi_*\Xi_{\bar{f}}, \pi_*\Xi_{\bar{g}})] \\
&= \pi^*\left((\omega_0)_{(H)}\right)(\Xi_{\bar{f}}, \Xi_{\bar{g}}) \\
&= \omega(\Xi_{\bar{f}}, \Xi_{\bar{g}}) = \{\bar{f}, \bar{g}\}_M.
\end{aligned}$$

□

Having defined a bracket, we now come to the question of defining Hamiltonian flows on the reduced space. Given a Hamiltonian $h \in C^\infty(M_0)$, an *integral curve* of h through a point m_0 is a (generalized) smooth curve $\gamma(t)$ with $\gamma(0) = m_0$, such that for all functions $f \in C^\infty(M_0)$

$$\frac{d}{dt} f(\gamma(t)) = \{f, h\}_{M_0}(\gamma(t)). \tag{7}$$

(Hamilton's equation on the reduced space.) Since the reduced space M_0 is not a manifold, (7) cannot be reduced to a system of ordinary differential equations. For this reason we need to give an argument proving the existence and uniqueness of integral curves.

To show the existence, we pick a smooth G-invariant function $\bar{h}$ on M with $\bar{h}|_Z = \pi^* h$. The Hamiltonian flow $\bar{\Phi}_t$ of $\bar{h}$ is G-equivariant and leaves the level set Z invariant. It therefore descends to a smooth flow Φ_t on the reduced space M_0. Let m be a point in the zero level set with $\pi(m) = m_0$ and f a smooth function on M_0. Pick $\bar{f} \in C^\infty(M)$ with $\bar{f}|_Z = \pi^* f$. Then $f(\Phi_t(m_0)) = \bar{f}(\bar{\Phi}_t(m))$. The definition of the bracket on M_0 now implies that

$$\frac{d}{dt} f(\Phi_t(m_0)) = \frac{d}{dt} \bar{f}(\bar{\Phi}_t(m)) = \{\bar{f}, \bar{h}\}_M(\bar{\Phi}_t(m)) = \{f, h\}_{M_0}(\Phi_t(m_0)). \quad (8)$$

Next we prove the uniqueness of integral curves. Let $\gamma(t)$ be an integral curve of h starting at m_0. We would like to show that $\gamma(t) = \Phi_t(m_0)$, or, equivalently, that

$$\Phi_{-t}(\gamma(t)) = m_0$$

for all t.

Using integration over G one can easily show the existence of a G-invariant partition of unity on M subordinate to a cover of G-invariant open sets. This implies that smooth functions on the reduced space separate points, i.e., if $f(x) = f(y)$ for all $f \in C^\infty(M_0)$, then $x = y$. Therefore it suffices to show that for all t and for all $f \in C^\infty(M_0)$

$$f(\Phi_{-t}(\gamma(t))) = f(m_0).$$

Put $f_t = f \circ \Phi_{-t}$. Equation (8) together with the fact that h is preserved by its flow Φ_t implies that

$$\frac{df_t}{dt} = \{h, f_t\}_{M_0}. \quad (9)$$

Combining (7) and (9) we get

$$\frac{d}{dt} f(\Phi_{-t}(\gamma(t))) = \{h, f\}_{M_0}(\gamma(t)) + \{f, h\}_{M_0}(\gamma(t)) = 0.$$

This proves that an integral curve of h through m_0 is unique.

Because of G-equivariance, the lift $\bar{\Phi}_t$ of the flow Φ_t preserves the orbit type stratification of M. As an immediate consequence, we have the following lemma.

3.2. Lemma. *The Hamiltonian flows of the functions in $C^\infty(M_0)$ preserve the symplectic pieces of M_0. The restriction of the Hamiltonian flow of a function $h \in C^\infty(M_0)$ to a piece $(M_0)_{(H)}$ equals the Hamiltonian flow of the function $h|_{(M_0)_{(H)}}$.*

Using the definition of the bracket on $C^\infty(M_0)$ one can easily show that the Hamiltonian flows sweep out the symplectic pieces (more precisely their connected components). Consequently the smooth symplectic manifolds defined by Gonçalves [10] are exactly the connected components of the strata $(M_0)_{(H)}$. It also follows from this observation that

3.3. Proposition. *The decomposition of the reduced space is determined by the Poisson algebra of smooth functions.*

Note also that since the pieces are smooth manifolds the integral curves are smooth in the usual sense.

We will now consider a finer decomposition of the manifold M. For a subgroup H of G define

$$M_H = \{m \in M : \text{ stabilizer of } m \text{ is exactly } H\},$$

the *manifold of symmetry* H. This manifold is symplectic (see for example [17, Proposition 27.5]), and the symplectic form ω_H is simply the restriction of the symplectic form on M, $\omega_H = \omega|_{M_H}$. Since the Hamiltonian vector field of an invariant function $\bar{h}$ is tangent to M_H, it coincides with the Hamiltonian vector field (with respect to the form ω_H) of the restriction $\bar{h}|_{M_H}$. In other words, the decomposition of M into symmetry components,

$$M = \bigsqcup_{H<G} M_H,$$

is also a decomposition of the Hamiltonian system $(M, \bar{h})$:

$$(M, \bar{h}) = \bigsqcup_{H<G} (M_H, \bar{h}|_{M_H}).$$

The rest of the section is devoted to showing that the intersection of the zero level set Z with the manifold of symmetry H is a coisotropic manifold fibring over the reduced piece $(M_0)_{(H)}$, and to explaining the connection between the fibration $Z \cap M_H \to (M_0)_{(H)}$ and the regular Marsden–Weinstein reduction.

The action of G does not preserve the manifold M_H; in fact, under the action M_H sweeps out $M_{(H)}$. To see this, observe that if a point p lies in $M_{(H)}$ then the stabilizer of p is $g^{-1}Hg$ for some $g \in G$, so $g^{-1}p$ is in M_H.

This argument shows that the composition of the inclusion $M_H \hookrightarrow M_{(H)}$ with the orbit map $M_{(H)} \to M_{(H)}/G$ is surjective. Moreover, $M_H \to M_{(H)}/G$ is a principal fibration with the structure group $L = N_G(H)/H$, where $N_G(H)$ denotes the normalizer of H in G.

We can now reinterpret the embedding $M_H \hookrightarrow M_{(H)}$ as a map of fibre bundles. Both manifolds M_H and $M_{(H)}$ fibre over $M_{(H)}/G$ with typical fibres $N_G(H)/H$ and G/H respectively; they are both associated to the principal fibration $L \to M_H \to M_{(H)}/G$. The natural embedding of the fibres $N_G(H)/H \hookrightarrow G/H$ induces an embedding of the fibre bundles:

$$M_H = M_H \times_L (N_G(H)/H) \to M_H \times_L (G/H) = M_{(H)}.$$

The piece $(M_0)_{(H)}$ of orbit type H is a submanifold of the base $M_{(H)}/G$ and the fibration $M_{(H)} \cap Z \to (M_0)_{(H)}$ is a restriction of the fibration $M_{(H)} \to M_{(H)}/G$ to the submanifold $(M_0)_{(H)}$ (cf. Theorem 2.1). We claim that the intersection of Z with M_H is a restriction to $(M_0)_{(H)}$ of the fibration $M_H \to M_{(H)}/G$. Indeed,

$$M_H|_{(M_0)_{(H)}} = M_H \cap \left(M_{(H)}|_{(M_0)_{(H)}}\right) = M_H \cap (M_{(H)} \cap Z) = M_H \cap Z.$$

The preceding discussion can now be summarized in a lemma.

3.4. Lemma. *The manifold M_H of points with symmetry H intersects the zero level set Z of the moment map $J : M \to \mathbf{g}^*$ in a manifold which fibres over the piece $(M_0)_{(H)}$ of orbit type H with typical fibre $N_G(H)/H$.*

We are now going to relate the fibration $M_H \cap Z \to (M_0)_{(H)}$ to the regular Marsden–Weinstein reduction procedure.

Let M'_H denote the union of the components of M_H which intersect Z nontrivially. The action of the group $N_G(H)$ on M is Hamiltonian and, as a result, the action of $N_G(H)$ on M'_H is Hamiltonian as well (although it is not effective). It follows that the action of $L = N_G(H)/H$ on M'_H is Hamiltonian.

Indeed, let us show that the restriction J' of the moment map J to M'_H can be interpreted as a moment map for the action of L on M'_H. For any $a \in H$ and $p \in M'_H$ we have

$$J(p) = J(a \cdot p) = Ad^*(a)J(p).$$

So the image of M'_H under J' is contained in $(\mathbf{g}^*)^{\mathbf{h}}$, the subspace of $\mathbf{g}^*$ infinitesimally fixed by H. On the other hand, since H fixes the points of M'_H, for any $\xi \in \mathbf{h}$ the function $\langle \xi, J \rangle$ is locally constant on M'_H. It follows from the fact that $Z \cap M'_H \neq \emptyset$ that $\langle \xi, J \rangle$ is actually zero on M'_H, that is, the image $J'(M'_H)$ is contained in $\operatorname{ann}_{\mathbf{g}^*} \mathbf{h}$, the annihilator of $\mathbf{h}$ in $\mathbf{g}^*$. The vector space $\mathbf{l}^*$, the dual of the Lie algebra of L, can be identified with $\operatorname{ann}_{\mathbf{n}^*} \mathbf{h}$, the annihilator of $\mathbf{h}$ in $\mathbf{n}^*$. It is not hard to show that the natural map

$$(\mathbf{g}^*)^{\mathbf{h}} \cap \operatorname{ann}_{\mathbf{g}^*} \mathbf{h} \to \mathbf{l}^*,$$

given by restriction of functionals, is an isomorphism. This allows us to identify the vector spaces $(\mathbf{g}^*)^{\mathbf{h}} \cap \operatorname{ann}_{\mathbf{g}^*} \mathbf{h}$ and $\mathbf{l}^*$.

3.5. Theorem. *Zero is a regular value of the moment map $J' : M'_H \to \mathbf{l}^*$. Consequently the piece $(M_0)_{(H)}$ of the reduced space M_0 of type (H) is the regular Marsden–Weinstein reduced space $(J')^{-1}(0)/L$.*

Proof. Since L acts freely on M'_H, zero is a regular value of J'. We have shown that $(J')^{-1}(0) = Z \cap M_H$. On the other hand, by Lemma 3.4, $(M_H \cap Z)/L = (M_0)_{(H)}$. It remains to show that the symplectic forms on $(J')^{-1}(0)/L$ and $(M_0)_{(H)}$ coincide. This is easy and is left to the reader. □

This theorem provides us with a simple recipe for lifting integral curves of a reduced Hamiltonian flow on the reduced space M_0 to the level set Z. Namely, let $\bar{h}$ be an invariant smooth function on the manifold M, and let h be the smooth function on the reduced space induced by $\bar{h}$. Let $\bar{\Phi}_t$, resp. Φ_t, denote the Hamiltonian flow of $\bar{h}$, resp. h. If $\gamma(t)$ is an integral curve of the function h, then it lies inside some symplectic piece $(M_0)_{(H)}$, and the classical recipe for lifting a reduced flow (see e.g. [1]) can be used to lift $\gamma(t)$ to an integral curve of the Hamiltonian $\bar{h}$, lying in the manifold M_H.

As another application of Theorem 3.5 we give a generalization of Smale's criterion for relative equilibria of Hamiltonian systems.

3.6. Definition. A point $p \in Z$ is called a *relative equilibrium (with respect to the G-action)* of the Hamiltonian vector field $\Xi_{\bar{h}}$ on M if the trajectory $\bar{\Phi}_t(p)$ through p is contained in the orbit $G \cdot p$.

In terms of reduction, relative equilibria can be characterized as follows.

3.7. Lemma. *Let p be a point in $Z \cap M_H$. Then the following statements are equivalent.*

1. *The point p is a relative equilibrium of the Hamiltonian vector field $\Xi_{\bar{h}}$.*
2. *The image p_0 of p under the orbit map $\pi : Z \to M_0$ is fixed under the flow Φ_t of the reduced Hamiltonian h on M_0.*
3. *The point p_0 is an equilibrium point of the Hamiltonian vector field Ξ_h on the symplectic piece $(M_0)_{(H)}$.*
4. *There is a one-parameter subgroup $\{g_t\}$ of G such that*

$$\bar{\Phi}_t(p) = g_t \cdot p.$$

Proof. The implications $1 \Rightarrow 2$ and $4 \Rightarrow 1$ are immediate from Definition 3.6, and $2 \Leftrightarrow 3$ is obvious from Lemma 3.2.

We use Theorem 3.5 to show $3 \Rightarrow 4$. Suppose that $p_0 \in (M_0)_H$ is an equilibrium point for Ξ_h, i.e., $\Phi_t(p_0) = p_0$ for all time t. Since the projection $Z \cap M_H \to (M_0)_{(H)}$ is a principal $N_G(H)/H$-bundle, there are unique elements $\breve{g}_t \in N_G(H)/H$ such that $\bar{\Phi}_t(p) = \breve{g}_t \cdot p$. It is easy to see that there exists a one-parameter subgroup $\{g_t\}$ of $N_G(H) < G$ which projects down to $\{\breve{g}_t\}$ under the map $N_G(H) \to N_G(H)/H$ and satisfies $\bar{\Phi}_t(p) = g_t \cdot p$. $\square$

In particular, if the symplectic piece of p_0 consists of one single point, the point p is automatically a relative equilibrium.

If p is a regular point of the momentum map J, it is a relative equilibrium if and only if it is a critical point of the energy-momentum map $J \times \bar{h} : M \to \mathbf{g}^* \times \mathbf{R}$. This is Smale's criterion for relative equilibria; see Marsden and Weinstein [25] for a proof and references. Using Theorem 3.5 we can give a more general criterion from which the regularity assumption has been removed.

3.8. Theorem. *A point in* $Z \cap M_H$ *is a relative equilibrium of the Hamiltonian vector field of the* G*-invariant function* $\bar{h}$ *if and only if it is a critical point of the map* $J' \times (\bar{h}|_{M_H}) : M_H \to \mathbf{l}^*$.

Proof. By Lemma 3.7, $p \in Z \cap M_H$ is a relative equilibrium of $\Xi_{\bar{h}}$ if and only if p_0 is a critical point of the map $h|_{(M_0)_{(H)}}$. By Theorem 3.5, this is equivalent to p being a critical point of $\bar{h}|_{Z \cap M_H}$. This in turn is equivalent to p being a critical point of the product map $J' \times (\bar{h}|_{M_H})$. □

3.9. Remark. Suppose that the manifold M is compact. Then the pieces of the reduced space all have finite Liouville volume. This can be seen as follows. The *set* M^H *of points fixed by* H is defined by

$$M^H = \{m \in M : H \text{ is a subgroup of the stabilizer of } m\}.$$

Note that $M_H = M^H \cap M_{(H)}$. Also, the fixed point set M^H is a closed symplectic submanifold of M, and the manifold M_H of symmetry H is open in M^H. Since the space M was assumed to be compact, the Liouville volume of M^H has to be finite. This implies that the volume of M_H is finite, and hence, the volume of the piece $(M_0)_{(H)} = (Z \cap M_H)/N_G(H)$ is also finite.

4 Reduction in Stages

In this section we show that the reduction procedure of Section 2 can be carried out in stages in complete analogy with the regular Marsden–Weinstein reduction procedure. The results of this section will be used later on to show that the decomposition of M_0 is actually a stratification in the sense of Definition 1.7.

Let us start by recalling the reduction-in-stages procedure under all the assumptions of regularity. The set-up is as follows. Let G_1 and G_2 be two groups acting on (M, ω) in a Hamiltonian fashion with corresponding equivariant moment maps $J_1 : M \to \mathbf{g}_1^*$ and $J_2 : M \to \mathbf{g}_2^*$. Assume that the actions commute, i.e., that we have an action of the product group $G_1 \times G_2$. The corresponding moment map $J : M \to \mathbf{g}_1^* \times \mathbf{g}_2^*$ is a product: $J(m) = (J_1(m), J_2(m))$. The equivariance of J is equivalent to J_1 being G_2-invariant and J_2 being G_1-invariant. So after averaging J_1 over the group G_2 and J_2 over G_1 we may assume that J is equivariant.

If zero is a regular value of J_1 and the action of G_1 on $J_1^{-1}(0)$ is free then

$$X_1 = J_1^{-1}(0)/G_1$$

is a symplectic manifold. The G_2-invariance of J_1 implies that the action of G_2 preserves the level set $J_1^{-1}(0)$. Since the actions of G_1 and G_2 commute, we get an action of G_2 on X_1. This action is Hamiltonian. To compute the corresponding moment map $J_2' : X_1 \to \mathbf{g}_2^*$ note that the restriction of J_2 to the level set $J_1^{-1}(0)$ is G_2-invariant and so descends to a smooth map from X_1 to $\mathbf{g}_2^*$. It is easy to check that this is a moment map for the action of G_2 on X_1.

If zero is a regular value of J_2' and G_2 acts freely on the zero level set in question then we can reduce one more time and obtain the space

$$X_{12} = (J_2')^{-1}(0)/G_2.$$

On the other hand, the reduction may be carried out in reverse order. That is, first one obtains a Hamiltonian G_1-space $X_2 = J_2^{-1}(0)/G_2$ with G_1-moment map $J_1' : X_2 \to \mathbf{g}_1^*$ and then the space

$$X_{21} = (J_1')^{-1}(0)/G_1.$$

The reduction-in-stages theorem asserts that the two spaces are isomorphic as symplectic manifolds. The proof is essentially an observation that either space can be obtained by reducing M at $(0,0) \in \mathbf{g}_1^* \times \mathbf{g}_2^*$ with respect to the action of the product $G_1 \times G_2$:

$$X_{12} = (J_1 \times J_2)^{-1}(0,0)/G_1 \times G_2 = X_{21}.$$

We claim that the assumptions of regularity can be removed. Let us look at what is involved in the proof of this claim. The space $X_1 = J_1^{-1}(0)/G_1$ is no longer neccessarily a manifold. However, a simple check of definitions shows that the induced action of G_2 on the space is smooth, that is, it preserves $C^\infty(X_1)$. Moreover, the action is decomposition-preserving. The reason is fairly simple: an orbit type stratum $M_{(H)}$, $H < G_1$, is preserved by the action of G_2. Since G_2 preserves the Poisson bracket on M, it preserves the reduced bracket on X_1.

Alternatively, Theorem 3.5 implies that the action of G_2 on each piece of X_1 is symplectic. In fact, the action of G_2 on a piece $(X_1)_{(H)}$ is Hamiltonian.

The corresponding moment map is computed as in the regular case: the map $J_2|_{M_H}$ is a moment map for the action of G_2 on M_H, the manifold of symmetry H; it is invariant under the action of $N_G(H)/H$ and so descends to a moment map on the piece $(X_1)_{(H)}$. These moment maps on the pieces fit together to form a map $J_2' : X_1 \to \mathbf{g}_2^*$. This map is also defined by the fact that its pullback to the zero level set $J_1^{-1}(0)$ equals the restriction of J_2,

$$J_2|_{J_1^{-1}(0)} = \pi_1^*(J_2'),$$

where $\pi_1 : J_1^{-1}(0) \to X_1$ is the orbit map. Therefore the map J_2' is smooth and it makes sense to *define* this map to be the moment map corresponding to the action of G_2 on the space X_1. We would like to repeat that the restriction of J_2' to each piece is a momentum map in the usual sense.

Proceeding by analogy we define the reduced space at zero corresponding to the action of G_2 on X_1 to be

$$X_{12} = (J_2')^{-1}(0)/G_2.$$

We now describe the smooth structure (i.e., the space of smooth functions) and the Poisson bracket on X_{12}. A continuous function $f : X_{12} \to \mathbf{R}$ is smooth if there exists a function $f' \in C^\infty(X_1)^{G_2}$ such that

$$f'|_{(J_2')^{-1}(0)} = \pi_{12}^* f,$$

where $\pi_{12} : (J_2')^{-1}(0) \to X_{12}$ is the orbit map. Similarly, for any two functions $f, g \in C^\infty(X_{12})$ we define their Poisson bracket at $x \in X_{12}$ by

$$\{f, g\}_{X_{12}}(x) = \{f', g'\}_{X_1}(x'),$$

where f' and g' are the corresponding functions in $C^\infty(X_1)^{G_2}$ and $x' \in (J_2')^{-1}(0)$ satisfies $\pi_{12}(x') = x$.

One can check, by adapting the arguments given earlier, that the bracket in $C^\infty(X_{12})$ is well-defined and that X_{12} decomposes into a union of symplectic spaces. For example, the decomposition of X_{12} can be obtained by first stratifying the pieces of X_1 by the G_2-orbit types and then looking at the intersections of these strata with the level set $(J_2')^{-1}(0)$. We will shortly see that such direct computations can be avoided.

Indeed, as a topological space X_{12} can also be obtained by reducing the M at the origin with respect to the action of the product $G_1 \times G_2$. That is, X_{12} is homeomorphic to

$$M_0 = (J_1 \times J_2)^{-1}(0,0)/G_1 \times G_2.$$

The reader may wish to keep in mind the diagram below.

$$\begin{array}{ccccc} J_1^{-1}(0) & \longrightarrow & M & \longleftarrow & (J_1 \times J_2)^{-1}(0,0) \\ \downarrow \pi_1 & & & & \Big\downarrow \pi \\ X_1 & \longleftarrow & (J_2')^{-1}(0) & & \\ & & \downarrow \pi_{12} & & \\ & & X_{12} & \longleftrightarrow & M_0 \end{array}$$

We want to show that $C^\infty(M_0)$ and $C^\infty(X_{12})$ are isomorphic as Poisson algebras. This will prove that reduction in stages holds without any assumptions of regularity and, as a byproduct, it will also show that X_{12} can be decomposed as a union of symplectic manifolds.

By definition, a function $f : M_0 \to \mathbf{R}$ is smooth if there exists a function $\bar{f} \in C^\infty(M)^{G_1 \times G_2}$ with

$$\bar{f}|_{(J_1 \times J_2)^{-1}(0,0)} = \pi^* f,$$

where $\pi : (J_1 \times J_2)^{-1}(0,0) \to M_0$ is the orbit map.

The restriction $\bar{f}|_{J_1^{-1}(0)}$ is G_1-invariant and so there exists a function $f' \in C^\infty(X_1)$ with $\pi_1^* f' = \bar{f}|_{J_1^{-1}(0)}$. Clearly f' is G_2-invariant and further satisfies

$$f'|_{(J_2')^{-1}(0)} = \pi_{12}^* f.$$

Therefore f is also a smooth function on X_{12}. Consequently we have an inclusion

$$C^\infty(M_0) \hookrightarrow C^\infty(X_{12}).$$

We need to show that the reverse inclusion also holds. So let $f : X_{12} \to \mathbf{R}$ be smooth. Then there exists a smooth G_2-invariant function $f' : X_1 \to \mathbf{R}$ such that

$$f'|_{(J_2')^{-1}(0)} = \pi_{12}^* f.$$

In turn there exists a smooth function $\tilde{f} \in C^\infty(M)^{G_1}$ with $\tilde{f}|_{J_1^{-1}(0)} = \pi_1^* f'$. Unfortunately $\tilde{f}$ need not be G_2-invariant. Let $\bar{f}$ be the average of $\tilde{f}$ with respect to the action of G_2,

$$\bar{f}(p) = \int_{G_2} \tilde{f}(g \cdot p)\, dg.$$

Then $\bar{f} \in C^\infty(M)^{G_1 \times G_2}$. Since the projection π_1 is G_2-equivariant, the restriction of $\bar{f}$ to the set $J_1^{-1}(0)$ equals the pullback by π_1 of the G_2-average of f'. But f' is G_2-invariant, so

$$\bar{f}|_{J_1^{-1}(0)} = \pi_1^* f'.$$

It follows that

$$\bar{f}|_{(J_1 \times J_2)^{-1}(0,0)} = \pi^* f,$$

i.e., that $f \in C^\infty(M_0)$. The same argument shows that the brackets on $C^\infty(M_0)$ and $C^\infty(X_{12})$ coincide. To summarize, we have proved

4.1. Theorem. *The reduction procedure described in Section 2 can be carried out in stages.*

4.2. Remark. The discussion above shows that there is a natural way of defining Hamiltonian group actions and momentum maps on stratified symplectic spaces. The reduced spaces would again, presumably, be stratified symplectic spaces.

The rest of the section is concerned with generalizing the theorem of reduction in stages to extensions of groups. These results will not be used in the rest of the of the paper. Let

$$1 \to A \to B \to B/A \to 1$$

be an exact sequence of compact Lie groups with the group B acting in a Hamiltonian fashion on a symplectic manifold (M, ω). Let $J_B : M \to \mathbf{b}^*$ denote a corresponding moment map. Then the action of A is also Hamiltonian and the moment map J_A is given by

$$J_A = i^* \circ J_B,$$

where $i^* : \mathbf{b}^* \to \mathbf{a}^*$ is the transpose of the inclusion map $i : \mathbf{a} \to \mathbf{b}$. We would like to show that the reduction of M at zero with respect to the action of B can be carried out in stages. That is to say, one can first reduce M at zero with respect to the action of A, the resulting space is a Hamiltonian B/A space and further reduction at zero with respect to the action of B/A gives one $J_B^{-1}(0)/B$.

4.3. Remark. We would like to remind the reader that if a discrete group Γ acts symplectically on a manifold (M, ω), then we consider such an action to be Hamiltonian with momentum map being identically zero. We take the reduced space to be the quotient M/Γ. Clearly it is a stratified symplectic space.

It is easy to show that the set $Z_A = J_A^{-1}(0)$ is B-invariant and that the image of Z_A under the moment map J_B is contained in $\mathrm{ann}_{\mathbf{b}^*} \mathbf{a}$, the annihilator of $\mathbf{a}$ in $\mathbf{b}^*$, which is isomorphic to $(\mathbf{b}/\mathbf{a})^*$. Also, by an averaging argument similar to the one given in the proof of Theorem 4.1, for any function $f \in C^\infty(Z_A/A)^{B/A}$ there exists a function $\hat{f} \in C^\infty(M)^B$ with

$$\hat{f}|_{Z_A} = \pi_A^* f,$$

where $\pi_A : Z_A \to Z_A/A$ is the orbit map.

The difficulty lies in showing the induced action of B/A on Z_A/A is Hamiltonian in a reasonable sense of the word and that the moment map for this action is compatible with J_B. In other words, one would like to show that $J_B|_{Z_A}$ descends to a map from the quotient to $(\mathbf{b}/\mathbf{a})^*$. Since J_B is equivariant, a sufficient condition for this to be possible is that the action of A on $\mathrm{ann}_{\mathbf{b}^*} \mathbf{a}$ be trivial.

4.4. Theorem. *Let M be a symplectic manifold and B a compact Lie group acting on M in a Hamiltonian fashion with corresponding moment map $J_B : M \to \mathbf{b}^*$. Let $1 \to A \to B \to C \to 1$ be an exact sequence of Lie groups. Suppose that either*

1. *A is the connected component of the identity of B, or*
2. *B is connected.*

Then one can reduce M first with respect to the action of A and obtain a Hamiltonian C-space. The result of the C-reduction of this space is isomorphic to the reduction of M with respect to the action of B.

Proof. If A is the connected component of the identity of B, then $(\mathbf{b}/\mathbf{a})^* = 0$. So by the discussion preceding the statement of the theorem the reduction can be carried out in stages.

Consider now the case where B is connected. Again it suffices to show that the action of A on $\mathrm{ann}_{\mathbf{b}^*}\,\mathbf{a}$ is trivial. Since A is compact, $\mathbf{b}$ carries an $Ad(A)$-invariant inner product, which allows us to identify $\mathrm{ann}_{\mathbf{b}^*}\,\mathbf{a}$ with

$$\mathbf{a}^\perp = \{x \in \mathbf{b} : x \perp \mathbf{a}\}.$$

Since $\mathbf{a}^\perp$ is A-invariant, $[\mathbf{a}, \mathbf{a}^\perp]$ is contained in $\mathbf{a}^\perp$. On the other hand, since A is normal in B, $\mathbf{a}$ is an ideal in $\mathbf{b}$ and so $[\mathbf{a}, \mathbf{a}^\perp]$ is contained in $\mathbf{a}$. It follows that $[\mathbf{a}, \mathbf{a}^\perp] = 0$. Thus the identity component A_0 of A acts trivially on $\mathbf{a}^\perp$.

Consider the map $\phi : B \times A_0 \to B$, $\phi(b, a) = bab^{-1}$. Since, by assumption, A is normal in B and $B \times A_0$ is connected, the image is contained in a connected component of A. Since $\phi(1, 1) = 1$, the image is actually A_0. Therefore A_0 is normal in B and B/A_0 is a group. Now the group A/A_0, which is finite, is normal in the group B/A_0, which is connected, and an argument similar to the preceding one shows that it is *central.* It follows that the adjoint action of A/A_0 on $Lie(B/A_0) \simeq \mathbf{a}^\perp$ is trivial. We conclude that the adjoint action of A on $\mathbf{a}^\perp$ is trivial. $\square$

5 The Local Structure of the Decomposition

Reduction in stages allows us to construct a simple model for a neighbourhood of a point in a reduced space. Let (M, ω) be a Hamiltonian G-space with momentum map $J : M \to \mathbf{g}^*$, and let $Z = J^{-1}(0)$ be the zero level set.

5.1. Theorem. *Let x be a point in the reduced space $M_0 = Z/G$ and p a point in the zero level set mapping to x under the orbit map $Z \to M_0$. Let H be the stabilizer of p, $V = T_p(G \cdot p)^\omega / T_p(G \cdot p)$ the fibre at p of the symplectic normal bundle of the orbit through p and ω_V the symplectic form on V. Let $\Phi_V : V \to \mathbf{h}^*$ be the momentum map corresponding to the linear action of H. Let $\bar{0}$ denote the image of the origin in the reduced space $\Phi_V^{-1}(0)/H$. Then a neighbourhood U_1 of x in M_0 is isomorphic to a neighbourhood U_2 of $\bar{0}$ in $\Phi_V^{-1}(0)/H$. More precisely, there exists a homeomorphism $\varphi : U_1 \to U_2$ that induces an isomorphism*

$$\varphi^* : C^\infty(U_2) \to C^\infty(U_1)$$

of Poisson algebras. In particular, φ carries the symplectic pieces of U_1 onto symplectic pieces of U_2.

PROOF. Recall that a neighbourhood of p in M is modelled by a space Y obtained by reducing the product $T^*G \times V$ at zero with respect to the action of H (Proposition 2.5), and that if the modelled space Y is reduced further with respect to the action of G the result is a model Y_0 for a neighbourhood of x in M_0. By Theorem 4.1 the order in which these two reductions are carried out can be reversed.

Let us reverse the order of the two reductions. The reduction of $T^*G \times V$ with respect to the action of G gives a symplectic vector space (V, ω_V) with the group H acting linearly. The second reduction gives $\Phi_V^{-1}(0)/H$. Therefore we may identify the model space Y_0 with $\Phi_V^{-1}(0)/H$ and the theorem is proved. □

Let us now describe the topology and the symplectic structure of the space $\Phi_V^{-1}(0)/H$ in more detail. Let V_H be the space of H-fixed vectors in V and W the symplectic perpendicular of V_H, as in the proof of Theorem 2.1. It follows from (4) that

$$\Phi_V^{-1}(0)/H = \Phi_W^{-1}(0)/H \times V_H, \tag{10}$$

where Φ_W is the corresponding H-momentum map on W.

Without loss of generality we may assume that W carries an H-invariant complex structure compatible with the symplectic form $\omega_W = \omega_V|_W$. This allows us to identify (W, ω_W) with $(\mathbf{C}^n, \Omega)$, $n = 1/2 \dim W$, with coordinates $z_1, \ldots, z_n$ and symplectic form

$$\Omega = \sqrt{-1} \sum_j dz_j \wedge d\bar{z}_j.$$

We may assume that H is a subgroup of $U(n)$ (if not, replace H by its image under the homomorphism $H \to U(n)$). Let $T \cong S^1$ denote the central circle subgroup of $U(n)$. The action of $e^{i\theta} \in T$ is given by $e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n)$. The momentum map Φ for the action of T on $\mathbf{C}^n$ is simply

$$\Phi(z) = \|z\|^2.$$

Consequently, if T is a subgroup of H, the zero level set of Φ_W consists of the origin only, and the reduced space $\Phi_V^{-1}(0)/H$ is the symplectic vector space V_H.

Now assume that T is not a subgroup of H. Then the Lie algebra $\mathbf{h}$ of H intersects the Lie algebra of T trivially. Since the momentum map Φ_W is homogeneously quadratic, the level set $\Phi_W^{-1}(0)$ is invariant under the multiplication by positive scalars. It follows that, for a subgroup K of H, the manifold $W_{(K)} \cap \Phi_W^{-1}(0)$ intersects the unit sphere $S^{2n-1} = \{z \in \mathbf{C}^n : \|z\|^2 = 1\}$ transversely. Thus the set

$$L = \Phi_W^{-1}(0) \cap S^{2n-1}$$

decomposes into a disjoint union of smooth manifolds,

$$L = \bigsqcup_{K<H} L_{(K)},$$

where

$$L_{(K)} = \Phi_W^{-1}(0) \cap S^{2n-1} \cap W_{(K)}.$$

Also, the map

$$\rho : L \times (0, \infty) \to \Phi_W^{-1}(0) \setminus \{0\}, \quad \rho(z,t) = \sqrt{t}\, z$$

is a decomposition-preserving H-equivariant homeomorphism. Consequently, the reduced space $(\Phi_W^{-1}(0) \setminus \{0\})/H$ is homeomorphic to $(L/H) \times (0, \infty)$. This homeomorphism extends to a homeomorphism

$$\mathring{C}(L/H) \xrightarrow{\sim} \Phi^{-1}(0)/H, \tag{11}$$

which preserves the decompositions and which is a diffeomorphism on each piece. From Theorem 5.1 and (11) we can now read off that there exist an open neighbourhood U of the point x in M_0, an open ball B in V_H and a decomposition-preserving homeomorphism

$$U \xrightarrow{\sim} \mathring{C}(L/H) \times B. \tag{12}$$

5.2. Remark. The space L/H is a *link* of the point x (cf. Definition 1.7). This will be proved in the next section.

The fact that the space $(\Phi_W^{-1}(0) \setminus \{0\})/H$ is conical is also reflected in its symplectic structure. Indeed, let us start by writing $W \setminus \{0\}$ as a cylinder over S^{2n-1} and by rewriting the symplectic form in these new coordinates.

Note first that the circle group T in the centre of $U(n)$ acts freely on the sphere S^{2n-1}, that the quotient S^{2n-1}/T is the projective space $\mathbf{C}P^{n-1}$ and that the principal fibration $T \to S^{2n-1} \to \mathbf{C}P^{n-1}$ is the Hopf fibration. The form

$$A = -1/2\sqrt{-1}\sum_j (\bar{z}_j\, dz_j - z_j\, d\bar{z}_j)|_{S^{2n-1}}$$

is a $U(n)$-invariant connection one-form on this fibration. It is easy to check that $\rho^*\Omega = d(tA)$, where $\rho : S^{2n-1} \times (0,\infty) \to \mathbf{C}^n \setminus \{0\}$ is the map defined by $\rho(z,t) = \sqrt{t}\, z$. Thus, if the group H is trivial, then $L = S^{2n-1}$ and

$$(\Phi_W^{-1}(0) \setminus \{0\})/H = S^{2n-1} \times (0,\infty),$$

while the reduced form is the exact form $d(tA)$. We are going to show that each piece of the reduced space $(\Phi_W^{-1}(0) \setminus \{0\})/H$ carries a connection one-form, A_{reduced}, and that the reduced symplectic structure can be written as

$$\omega_{\text{reduced}} = d(tA_{\text{reduced}}).$$

5.3. Theorem. *Let K be a closed subgroup of H.*

i. The connection one-form A descends to a one-form $A_{(K)}$ on $L_{(K)}/H$ and the reduced symplectic form on

$$(W_{(K)} \cap \Phi_W^{-1}(0))/H = L_{(K)}/H \times (0,\infty)$$

is the exact form $d(tA_{(K)})$.

ii. The induced action of T on the manifold $L_{(K)}/H$ is fixed point free.

iii. If the T-action on $L_{(K)}/H$ is free, $A_{(K)}$ is a connection one-form on the principal T-bundle $L_{(K)}/H \to L_{(K)}/(H \times T)$.

Proof. For $\xi \in \mathbf{h}$ and $z \in \mathbf{C}^n$ let $\xi \cdot z$ denote the value at z of the vector field corresponding to the infinitesimal action of ξ. Since the form tA is H-invariant, one easily sees that the momentum map Φ_W in the new coordinates is given by

$$\langle \xi, \Phi_W(z,t)\rangle = -tA(\xi \cdot z).$$

Therefore the intersecion $\Phi_W^{-1}(0) \cap S^{2n-1}$ is the set

$$L = \{z \in S^{2n-1} : A(\xi \cdot z) = 0 \ \text{ for all } \ \xi \in \mathbf{h}\}. \tag{13}$$

It follows that the restriction of the one-form A to the manifold $L_{(K)}$ vanishes along the directions of the H-orbits. Since A is H-invariant, it descends to a one-form $A_{(K)}$ on $L_{(K)}/H$. It is easy to see that the reduced symplectic form on $L_{(K)}/H \times (0, \infty)$ is simply $d(tA_{(K)})$. This proves the first statement.

To prove that the action of T on $L_{(K)}/H$ is fixed point free, it suffices to show that for each point $z \in L_{(K)}$ the T-orbit through z intersects the H-orbit discretely or, equivalently, that the corresponding tangent spaces intersect trivially,

$$T_z(T \cdot z) \cap T_z(H \cdot z) = 0. \tag{14}$$

But condition (13) says that the H-orbit is tangent to the horizontal distribution on S^{2n-1} defined by A. On the other hand, the T-orbit is tangent to the vertical distribution. This implies (14), which proves the second statement.

If the T-action on $L_{(K)}/H$ is actually free, it follows from the preceding paragraph that $A_{(K)}$ is a connection one-form for the principal T-fibration $L_{(K)}/H \to L_{(K)}/(H \times T)$. This proves the last statement. □

5.4. Remark. If the T-action on $L_{(K)}$ is not free, the map

$$L_{(K)}/H \to L_{(K)}/(H \times T)$$

is a *V-bundle,* the quotient $L_{(K)}/(H \times T)$ is a *symplectic V-manifold*, also called a *symplectic orbifold,* and the form $A_{(K)}$ is a connection one-form in a generalized sense. (See Satake [29], Baily [4] and Schwarz [30].) Heuristically, the collection of V-bundles $\{L_{(K)}/H \to L_{(K)}/(H \times T)\}_{K<H}$ may be considered as a single fibration of the stratified spaces

$$L/H \to L/(H \times T)$$

and the collection of one-forms $\{A_{(K)}\}$ as a kind of connection structure for this fibration.

Note that the set $L = \Phi_W^{-1}(0) \cap S^{2n-1}$ is exactly the zero level set of the map $F : W \to \mathbf{h}^* \times \mathbf{R}$ given by

$$F(w) = (\Phi_W(w), \|w\|^2 - 1), \tag{15}$$

which is a momentum map for the action of $H \times T$ on W.

5.5. Definition. The stratified symplectic space $\Sigma = L/(H \times T)$ obtained by reducing the vector space W at the zero level with respect to the $H \times T$-action will be called a *symplectic link* of the point $\bar{0} \in \Phi_V^{-1}(0)/H$ (and of the corresponding point in the reduced space M_0.)

By the reduction-in-stages theorem, Theorem 4.1, the symplectic link Σ can also be seen as the reduction at $0 \in \mathbf{h}^*$ of the smooth Hamiltonian H-space $\mathbf{C}P^{n-1}$ or, alternatively, as the reduction at $0 \in \mathbf{R}$ of the singular Hamiltonian S^1-space $\Phi_W^{-1}(0)/H$.

5.6. Remark. The decomposition of Σ considered as a reduced symplectic space is *finer* than the decomposition $\Sigma = \bigsqcup_{K<H} L_{(K)}/(H \times T)$ since, in general, the pieces $L_{(K)}/(H \times T)$ are V-manifolds and need to be decomposed further. This can be seen in the following example.

5.7. Example. Let W be $\mathbf{C}^3$ and let H be the circle $\{e^{i\phi} : \phi \in \mathbf{R}\}$ acting by
$$e^{i\phi} \cdot (z_1, z_2, z_3) = (e^{i\phi} z_1, e^{-i\phi} z_2, e^{-2i\phi} z_3).$$
(The $(1, -1, -2)$ resonance.) In this case the set L is given by
$$\left\{ \begin{array}{rcl} |z_1|^2 - |z_2|^2 - 2|z_3|^2 & = & 0 \\ |z_1|^2 + |z_2|^2 + |z_3|^2 & = & 1 \end{array} \right. .$$
The set L is smooth and the H-action on L is free. The $H \times T$-action on L, however, is not free. There are three $H \times T$-strata on L, namely
$$\{z : |z_1|^2 = |z_2|^2 = 1/2;\ z_3 = 0\},$$
where the stabilizer is isomorphic to $\mathbf{Z}_2$,
$$\{z : |z_1|^2 = 2/3;\ |z_3|^2 = 1/3;\ z_2 = 0\},$$
where the stabilizer is isomorphic to $\mathbf{Z}_3$, and the open stratum, where the action is free. Thus L/H is a manifold, Σ is an orbifold with two isolated singular points, and the T-quotient map $L/H \to \Sigma$ is not a circle bundle.

5.8. Remark. Kirwan has proved in [20, Remark 3.1] and [19, Remark 9.1] that the zero level set of a proper momentum map is connected. Obviously, the momentum map F in (15) is proper. Consequently the symplectic link of each point in M_0 is connected.

Using Kirwan's result, we show that the decomposition of the reduced space M_0 has the following remarkable property.

5.9. THEOREM. *Assume that the momentum map $J : M \to \mathbf{g}^*$ is proper. Then there is a unique piece $(M_0)_{(H)}$ which is open in the reduced space. It is connected and dense.*

PROOF. Let us first refine the decomposition of M_0 so that all the pieces are connected. Similarly, without saying so explicitly, given the symplectic link of a point of M_0, its decomposition into manifolds will be refined so that all the pieces are connected.

Let $\widehat{M}_0$ denote the union of all the open pieces of M_0. We will prove by induction on the dimension of M_0 that $\widehat{M}_0$ is dense in M_0. Let x be a point in M_0. Let

$$U \simeq \mathring{C}(L/H) \times B$$

be a model neighbourhood of x as in (12), and let $\pi : L/H \to \Sigma_x$ be the T-quotient map from the link to the symplectic link of x. We see from this model that, if the symplectic link of x is empty, x lies in $\widehat{M}_0$ (and hence in its closure). If Σ_x is nonempty, we may assume by induction on the dimension of M_0 that $\widehat{\Sigma}_x$, defined analogously to $\widehat{M}_0$, consists of a single open dense piece of Σ_x. The map $\pi : L/H \to \Sigma_x$ is continuous and furthermore has the property that a preimage of a piece of Σ is a submanifold of a piece of L/H. Consequently $\pi^{-1}(\widehat{\Sigma}_x)$ is an open subset of some piece $L_{(K)}/H$ of L/H. Therefore $L_{(K)}/H$ is open in L/H and so the intersection of the neighbourhood U of x with $\widehat{M}_0$ is of the form

$$\widehat{M}_0 \cap U \simeq B \times (L_{(K)}/H) \times (0, r).$$

with the point x being identified with the vertex of the cone on $L_{(K)}/H$. So x lies in the closure of $\widehat{M}_0$. Since x is arbitrary, this proves that $\widehat{M}_0$ is dense.

We now prove that $\widehat{M}_0$ is connected. Suppose not. Then $\widehat{M}_0$ is a disjoint union of two nonempty open sets U_1 and U_2. Then, since $\widehat{M}_0$ is dense, the union of the closures of U_1 and of U_2 is the whole of M_0, i.e. $\bar{U}_1 \cup \bar{U}_2 = M_0$. The space M_0 is connected by Kirwan's theorem (op. cit.), so the intersection $\bar{U}_1 \cap \bar{U}_2$ is nonempty. Let x be a point in the intersection. This point cannot lie in $\widehat{M}_0$, so the symplectic link Σ_x is nonempty. Let C_1 and C_2 be two (connected) pieces of M_0 with $C_1 \subset U_1$ and $C_2 \subset U_2$. To these two pieces there correspond two disjoint open pieces B_1 and B_2 of the link L/H. The quotient map π is open, so $\pi(B_1)$ and $\pi(B_2)$ are two disjoint open subsets of Σ_x. In fact, $\pi(B_1)$ is a symplectic V-manifold and so is a union of symplectic manifolds. Each of these manifolds is a piece in the decomposition of Σ_x. Therefore $\pi(B_1)$ contains an open piece of Σ_x. Similarly, $\pi(B_2)$ contains an open piece of Σ_x. But this contradicts the inductive assumption that $\widehat{\Sigma}_x$ is connected. We conclude that $\widehat{M}_0$ is connected. Thus we have proved that there exists a unique piece $(M_0)_{(H)}$ of the reduced space such that one of its connected components is open and dense. Clearly this implies that $(M_0)_{(H)}$ can only have one component. $\square$

5.10. Remark. If the momentum map is not proper, the reduced space is not necessarily connected. However, in this case it is obvious from the proof of Theorem 5.9 that each component C of M_0 has a unique open stratum, which is connected and dense in C.

As an immediate corollary to Theorem 5.9 and the preceding remark we have the following basic property of the Poisson algebra $C^\infty(M_0)$, first shown by Arms et al. [2].

5.11. Corollary. *The Poisson algebra of the reduced space is nondegenerate, i.e., its centre consists of the locally constant functions only.*

6 A Whitney Embedding of a Reduced Phase Space

The object of this section is to show that the symplectic decomposition of a reduced phase space is a stratification in the sense of Definition 1.7. We will actually prove a slightly stronger result, namely that a reduced space can be embedded into a Euclidean space as a *Whitney stratified set.* The idea that reduced spaces should be embeddable into Euclidean space is due to Richard Cushman [7] (cf. also [2]).

First we shall discuss embeddings and Whitney's condition B and prove an embedding result for reduced spaces. Then we shall indicate how this implies that the decomposition of M_0 into symplectic pieces is a stratification. For background material on Whitney stratifications and semialgebraic sets see Whitney [37], Łojasiewicz [23], Mather [26], Gibson [9] and Goresky and MacPherson [12]. For embedding results concerning orbit spaces see Schwarz [31], Mather [28] and Bierstone [5].

6.1. Definition. Let X be a space with a smooth structure $C^\infty(X)$. For each point $x \in X$ the *ideal* $\mathbf{m}_x$ of x is the set of all smooth functions vanishing at x. The *(Zariski) tangent space* T_xX of X at x is the vector space $(\mathbf{m}_x/\mathbf{m}_x^2)^*$. If Y is another space with a smooth structure and f a smooth map from X to Y, then the *derivative* $df_x : T_xX \to T_{f(x)}Y$ of f at a point x is the dual of the map $f^* : \mathbf{m}_{f(x)}/\mathbf{m}_{f(x)}^2 \to \mathbf{m}_x/\mathbf{m}_x^2$.

If there exist smooth partitions of unity with respect to an arbitrary open cover of the space X, then the ideal of a point is a maximal ideal in $C^\infty(X)$.

6.2. Example. If V is a finite-dimensional representation space of a compact Lie group H, then by a result of Hilbert the algebra of invariant polynomials $P(V)^H$ is finitely generated. Let $\bar{0}$ denote the image of the origin in the orbit space V/H. The Zariski tangent space of the orbit space V/H at the point $\bar{0}$ is finite-dimensional; its dimension is equal to the number of elements in a minimal set of homogeneous generators of $P(V)^H$. (For a proof of this fact, see e.g. Mather [28].)

6.3. Definition. Let X be a space with a smooth structure $C^\infty(X)$ and let Y be a manifold. A *proper embedding* of X into Y is a smooth, proper and injective map f such that the transpose $f^* : C^\infty(Y) \to C^\infty(X)$ is surjective.

If $f : X \to Y$ is a proper embedding, then for each $x \in X$ the derivative df_x is injective. This follows from the fact that $f^* : \mathbf{m}_{f(x)}/\mathbf{m}_{f(x)}^2 \to \mathbf{m}_x/\mathbf{m}_x^2$ is surjective. If X is equipped with a decomposition into smooth manifolds, one can show, under some mild additional assumptions on the smooth structure on X, that for each piece S in X the restriction of f is an embedding of S into Y. For example, it suffices to assume that the restriction map $C^\infty(X) \to C^\infty(S)$ has dense image.

6.4. Example. We continue Example 6.2. Let us choose a finite set of homogeneous generators $(\sigma_1, \dots, \sigma_k)$ of the algebra of invariant polynomials on V. We define the corresponding *Hilbert map* $\sigma : V \to \mathbf{R}^k$ by

$$\sigma(v) = (\sigma_1(v), \dots, \sigma_k(v)).$$

The Hilbert map descends to an injective map

$$\tilde{\sigma} : V/H \to \mathbf{R}^k.$$

A fundamental result of Schwarz [31] states that this map is a *proper embedding* of the orbit space V/H. Consequently, its derivative at the orbit $\bar{0} \in V/H$ is injective. If the set of generators $(\sigma_1, \ldots, \sigma_k)$ is minimal, the tangent space to V/H is k-dimensional, so $d\tilde{\sigma}_{\bar{0}}$ is even an isomorphism. Since the Hilbert map is polynomial, its image is by the Tarski–Seidenberg Theorem a semialgebraic subset of $\mathbf{R}^k$. Bierstone [5] has proved that the images of the strata of V/H are also semialgebraic in $\mathbf{R}^k$.

6.5. Example. Suppose that the representation space V of the previous example is actually symplectic. The reduced space $\Phi_V^{-1}(0)/H$ can be regarded in a natural way as a subset of the whole orbit space V/H. We claim that the restriction of the map $\tilde{\sigma}$ is a proper embedding of $\Phi_V^{-1}(0)/H$ into $\mathbf{R}^k$. Indeed, it is proper and injective, and by the previous example and the definition of $C^\infty(\Phi_V^{-1}(0)/H)$, the transpose map

$$C^\infty(\mathbf{R}^k) \to C^\infty(\Phi_V^{-1}(0)/H)$$

is surjective. We can say a little more about this embedding. Choose a finite set of generators $(\rho_1, \ldots, \rho_l)$ of the algebra $P(\mathbf{h}^*)^H$ of $Ad^*(H)$-invariant polynomials on $\mathbf{h}^*$, and by the corresponding Hilbert map ρ embed the quotient $\mathbf{h}^*/H$ into $\mathbf{R}^l$. The momentum map $\Phi_V : V \to \mathbf{h}^*$, being quadratic and H-equivariant, induces a map $\Phi_V^* : P(\mathbf{h}^*)^H \to P(V)^H$. It follows that we can find l polynomials ϕ_i in k variables such that

$$\Phi_V^*(\rho_i) = \phi_i \circ \sigma.$$

Define the polynomial map $\tilde{\Phi}_V : \mathbf{R}^k \to \mathbf{R}^l$ by

$$\tilde{\Phi}_V = (\phi_1, \ldots, \phi_l).$$

Then the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\Phi_V} & \mathbf{h}^* \\ \sigma \downarrow & & \downarrow \rho \\ \mathbf{R}^k & \xrightarrow{\tilde{\Phi}_V} & \mathbf{R}^l. \end{array}$$

It follows that

$$\tilde{\sigma}(\Phi_V^{-1}(0)/H) = \sigma(\Phi_V^{-1}(0)) = \tilde{\Phi}_V^{-1}(0) \cap \sigma(V).$$

The fact that the images of the strata in the orbit space V/H are semialgebraic and the fact that $\tilde{\Phi}_V$ is a polynomial map now imply that the images of the symplectic pieces in $\Phi_V^{-1}(0)/H$ are semialgebraic.

6.6. Example. Let G be a compact Lie group. As pointed out by Schwarz [31], any G-manifold M of finite type has the property that its orbit space can be properly embedded. In effect, by the Mostow–Palais Theorem (see e.g. [6]) there exists a proper G-equivariant embedding i of M into some representation space $\mathbf{R}^N$ for G. The map $i^* : C^\infty(\mathbf{R}^N) \to C^\infty(M)^G$ is surjective (use invariant integration to see this). The equivariant map i descends to a proper injective map $M/G \to \mathbf{R}^N/G$. Embed the quotient $\mathbf{R}^N/G$ into some $\mathbf{R}^n$ as in Example 6.4. Then the composition

$$M/G \to \mathbf{R}^N/G \to \mathbf{R}^n$$

is an embedding of M/G into $\mathbf{R}^n$.

Now suppose M is a Hamiltonian G-space. It is then an easy matter to construct an embedding of the reduced space M_0 into $\mathbf{R}^n$. First embed M/G into a Euclidean space $\mathbf{R}^n$ via a map i; the restriction of i to M_0 is then an embedding of M_0. Example 6.5 is a special case of this construction, where the image of the embedding is semialgebraic.

6.7. Theorem. *Let $i : M_0 \to \mathbf{R}^n$ be the embedding constructed in Example 6.6. Then the images of the symplectic pieces constitute a Whitney stratification of the subset $i(M_0)$ of $\mathbf{R}^n$.*

Let us first explain what we mean by this statement.

6.8. Definition. Let X be a subspace of $\mathbf{R}^n$. A decomposition of X is called a *Whitney stratification* if the pieces of X are smooth submanifolds of $\mathbf{R}^n$ and if for each pair of pieces P, Q with $P \le Q$ the following condition of Whitney holds:

Whitney's Condition B: Let p be an arbitrary point in P and let $\{p_i\}$ and $\{q_i\}$ be sequences in P, resp. Q, both converging to p. Assume that the lines l_i joining p_i and q_i converge (in the projective space $\mathbf{R}P^{n-1}$) to a line l, and that the tangent planes $T_{q_i}Q$ converge (in the Grassmannian of $(\dim Q)$-planes in $\mathbf{R}^n$) to a plane τ. Then l is contained in τ.

6.9. REMARK. If X is a Whitney stratified subset of $\mathbf{R}^n$ and O an open subset of $\mathbf{R}^m$, then $X \times O \subset \mathbf{R}^n \times \mathbf{R}^m$ is Whitney stratified by the products of O with strata in X.

6.10. REMARK. If Whitney's condition B holds for a pair of submanifolds $P, Q \subset \mathbf{R}^r$, and f is an embedding $\mathbf{R}^r \to \mathbf{R}^s$, then Whitney's condition B holds for the pair $f(P), f(Q) \subset \mathbf{R}^s$.

PROOF OF THEOREM 6.7. Let P, Q be an arbitrary pair of pieces in $i(M_0)$ with $P \leq Q$. We have to check the Whitney condition for P and Q. First we treat a very special case and then indicate how this implies the general case. Assume that we are in the situation of Example 6.5. We use the notation of that example. Suppose that the set of generators $(\sigma_1, \ldots, \sigma_k)$ is minimal. Assume that P is $\tilde{\sigma}(V_H)$, the image of the piece containing the point $\bar{0} \in \Phi_V^{-1}(0)/H$, and that Q is any other piece in $\tilde{\sigma}(\Phi_V^{-1}(0)/H)$. (As before, $\bar{0}$ denotes the image of the orgin in $\Phi_V^{-1}(0)/H$.) If $V_H = 0$, then P is a point, and the Whitney condition for P and Q is then a real algebraic version of a property first proved by Whitney [37] in the complex case; namely that a smooth semialgebraic variety is B-regular over any point in its closure. See Proposition 3 of Łojasiewicz [23, page 103] and also Gibson [9] for a discussion and further references.

If $V_H \neq 0$, then we know from formula (10) that

$$\Phi_V^{-1}(0)/H = \Phi_W^{-1}(0)/H \times V_H,$$

where W is the symplectic perpendicular of V_H. The minimality of the set of generators $(\sigma_1, \ldots, \sigma_k)$ implies that $\tilde{\sigma}(V_H)$ is a linear subspace of $\mathbf{R}^k$, and that

$$\tilde{\sigma}(\Phi_V^{-1}(0)/H) = \tilde{\sigma}(\Phi_W^{-1}(0)/H) \times \tilde{\sigma}(V_H).$$

The Whitney condition for P and Q now follows from the preceding case and Remark 6.9.

We are now ready to tackle the general case. Let x be an arbitrary point in $P \subset i(M_0)$. Pick a point p in M lying over the point in M_0 corresponding to x. Let H be the stabilizer and V the symplectic slice of p. Choose a minimal set of homogeneous generators $(\sigma_1, \ldots, \sigma_k)$ and embed V/H into $\mathbf{R}^k$ as before. We are going to show that there exists an embedding $\tilde{F}$ of a neighbourhood O of the origin in $\mathbf{R}^k$ into $\mathbf{R}^n$ with the following properties:

i. $\tilde{F}(0) = x$;

ii. $(\tilde{F} \circ \tilde{\sigma})(\Phi_V^{-1}(0)/H)$ is an open neighbourhood of x inside $i(M_0)$;

iii. the restriction of $\tilde{F}$ to $\tilde{\sigma}(\Phi_V^{-1}(0)/H)$ carries symplectic pieces in $\Phi_V^{-1}(0)/H$ to symplectic pieces in $i(M_0)$.

By Remark 6.10 the theorem will then follow from the special case treated above.

The local model in Proposition 2.5 provides us with an H-equivariant embedding $F : U \to M$, sending the origin in V to p, such that

$$J \circ F = \Phi_V | U. \tag{16}$$

Here[216z U is a ball of radius ε about the origin in the vector space V with respect to a norm $\|\cdot\|$ given by some H-invariant inner product. After taking quotients we get a map

$$\bar{F} : U/H \to M/G.$$

The fact that dF_0 is injective implies that $d\bar{F}_{\bar{0}}$ is injective. Hence the composition $i \circ \bar{F} : U/H \to \mathbf{R}^n$ has injective differential at $\bar{0} \in U/H$.

Now consider the restriction of the map $\tilde{\sigma} : V/H \to \mathbf{R}^k$ to U/H. The restriction map $C^\infty(V/H) \to C^\infty(U/H)$ is not surjective, because U is not closed in V. However, its image is certainly dense in $C^\infty(V/H)$. We claim there exist a neighbourhood O of $\tilde{\sigma}(U/H) \subset \mathbf{R}^k$ such that $\tilde{\sigma} : U/H \to O$ is a proper embedding, and a smooth map $\tilde{F} : O \to \mathbf{R}^n$ fitting into a commutative diagram:

$$\begin{array}{ccccc} V/H & \hookleftarrow & U/H & \xrightarrow{\bar{F}} & M/G \\ {\scriptstyle\tilde{\sigma}} \downarrow & & {\scriptstyle\tilde{\sigma}} \downarrow & & \downarrow {\scriptstyle i} \\ \mathbf{R}^k & \hookleftarrow & O & \xrightarrow{\tilde{F}} & \mathbf{R}^n. \end{array}$$

It will then automatically follow that $d\tilde{F}_0$ is injective, since $d\tilde{\sigma}_{\bar{0}}$ is an isomorphism (by the minimality of the basis of $P(V)^H$) and $d(i \circ \bar{F})_{\bar{0}}$ is injective. Moreover, by (16), the map $\tilde{F}$ will carry pieces in $\Phi_V^{-1}(0)/H$ to pieces in $i(M_0)$, and this will complete the proof.

We construct the neighbourhood O and the map $\tilde{F}$ as follows. Because of the invariance of the inner product on V there exists a polynomial g in k variables with $\|v\|^2 = g(\sigma(v))$ for all $v \in V$. Define

$$O = \{x \in \mathbf{R}^k : g(x) < \varepsilon\}.$$

Then σ maps U into O, and $\tilde{\sigma} : U/H \to O$ is a *proper* map. It now follows from Mather [28, Theorems 2 and 3] that $\tilde{\sigma}^* : C^\infty(O) \to C^\infty(U/H)$ is *surjective*. Let ξ_l be the l-th coordinate function on $\mathbf{R}^n$. The functions $i \circ \bar{F} \circ (\xi_l - x_l)$ are smooth on U/H, and can therefore be lifted to functions $f_l : O \to \mathbf{R}^n$. Put $\tilde{F} = (f_1, \dots, f_n)$. Then $\tilde{F}(0) = x$ and $\tilde{F} \circ \tilde{\sigma} = i \circ \bar{F}$, as required. □

Notice that the symplectic structure of the reduced space doesn't play a role in the proof of this theorem. Roughly speaking, the proof uses only the fact that the symplectic pieces are semialgebraic in suitable local coordinates and that the reduced space is locally a product of a ball with a lower-dimensional stratified space. So the method of the proof can also be used to show that the image $i(M/G)$ of the whole orbit space is a Whitney stratified subset of $\mathbf{R}^n$.

It follows from Mather's theory of control data (see [26]) that a Whitney stratified subset of Euclidean space is a stratified space in the sense of our Definition 1.7. An outline of the argument can be found in [12, page 40]. If Z is a Whitney stratified subset of $\mathbf{R}^n$ and p is a point in Z, one constructs a link L_p of p in Z as follows: Let S be the stratum of p. Choose a manifold N such that $N \cap S = \{p\}$ and N is transverse to all strata of Z. Then $N \cap Z$ is Whitney stratified by the intersections of N with strata in Z (cf. Gibson [9, Section 1.3]). (Here we may have to subdivide the intersections into their connected components in order not to violate the frontier condition.) Let ∂B be the sphere bounding a small ball in $\mathbf{R}^n$ round p. Set

$$L_p = N \cap Z \cap \partial B.$$

By Whitney's condition B, the sphere ∂B will intersect the strata of $N \cap Z$ transversely. Therefore L_p is again a Whitney stratified space. Furthermore, by Mather's tubular neighbourhood theorem (loc. cit.), S has an open neighbourhood T_S in Z with a topologically locally trivial fibration $\pi : T_S \to S$ such that $\pi^{-1}(p)$ is homeomorphic to the open cone over L_p.

Thus we obtain from Theorem 6.7 the following

6.11. Theorem. *The symplectic decomposition of the reduced space is a stratification.*

In fact, for a point in a reduced space one can make a very particular choice of a link and a conelike decomposition of a neighbourhood. At a point x

in M_0 we use the model space $\Phi_V^{-1}(0)/H$ of Theorem 5.1 and we embed it as in Example 6.5. We assume for simplicity that $V_H = 0$. Let S^{2p-1}, $p = 1/2 \dim V$, be the unit sphere in V with respect to some H-invariant Hermitian structure on V. Let d_i be the degree of the polynomial σ_i, $i = 1, \ldots, k$. We may assume that σ_1 is the distance squared to the origin in V. Then $d_1 = 2$ and all d_i's are ≥ 2. Define an action of $\mathbf{R}_+$ on $\mathbf{R}^k$ by

$$t \cdot (x_1, \ldots, x_k) = (t^{d_1} x_1, \ldots, t^{d_k} x_k),$$

for $t > 0$ and $(x_1, \ldots, x_k)$ in $\mathbf{R}^k$. For each symplectic piece S in $\Phi_V^{-1}(0)/H$, the image $\tilde{\sigma}(S) \subset \mathbf{R}^k$ is quasihomogeneous, i.e. invariant under the $\mathbf{R}_+$-action.

On the other hand, the image under σ of the set $L = S^{2p-1} \cap \Phi_V^{-1}(0)$ is the intersection of $\Phi_V^{-1}(0)/H$ with the hyperplane given by $x_1 = 1$. It is easy to see from the quasihomogeneity that the manifolds $\tilde{\sigma}(S)$ intersect the hyperplane $x_1 = 1$ transversely. It follows that the manifolds $\tilde{\sigma}(S) \cap \{x_1 = 1\}$ make up a Whitney stratification of $\tilde{\sigma}(L/H)$. The upshot of this discussion is:

6.12. Corollary. *The set $\tilde{\sigma}(L/H)$ is a Whitney stratified subset of $\mathbf{R}^k$ and $\tilde{\sigma}(\Phi_V^{-1}(0)/H)$ is the quasihomogeneous cone*

$$\mathbf{R}_+ \cdot \tilde{\sigma}(L/H) \cup \{0\}$$

over $\tilde{\sigma}(L/H)$. It follows that $\tilde{\sigma}(L/H)$ is a link of the vertex of $\tilde{\sigma}(\Phi_V^{-1}(0)/H)$.

6.13. Remark. From the recipe described in the discussion preceding Theorem 6.11 it follows that the set $S^{k-1} \cap \tilde{\sigma}(\Phi_V^{-1}(0)/H)$ is also a link of the vertex in $\tilde{\sigma}(\Phi_V^{-1}(0)/H)$. The reader might wonder what is the relation between this set and $\tilde{\sigma}(L/H)$. It is not hard to show that it can be homeomorphically mapped onto $\tilde{\sigma}(L/H)$ by a decomposition-preserving isotopy.

To conclude, we would like to call attention to the following conjecture, raised by Cushman (cf. also [2]).

6.14. Conjecture (Cushman, strong version). *There exists a Poisson structure on $\mathbf{R}^n$ that restricts to the Poisson structure on M_0. In other words, the bracket $\{\cdot, \cdot\}_{M_0}$ can be lifted to a bracket on $C^\infty(\mathbf{R}^n)$ via the surjection*

$$i^* : C^\infty(\mathbf{R}^n) \to C^\infty(M_0).$$

If the conjecture were true, the symplectic strata in M_0 would be symplectic leaves of the Poisson manifold $\mathbf{R}^n$, and Hamiltonian mechanics on M_0 could be studied by using coordinates on $\mathbf{R}^n$. In some cases, one might try to simplify the conjecture by finding a G-equivariant embedding of the Hamiltonian G-space M into a *symplectic* representation space for G. Quite possibly, the symplectic embedding theorems of Gromov and Tischler (cf. [14]) could be relevant in this context. The linear case of Cushman's Conjecture seems to us to deserve special mention.

6.15. Conjecture (Cushman, weak version). *Let V be a symplectic representation space for a compact Lie group H with quadratic momentum map Φ_V. Choose a set of generators of the algebra of invariant polynomials and embed the reduced space $\Phi_V^{-1}(0)/H$ into $\mathbf{R}^k$ by means of the Hilbert map. Then the Poisson bracket on $\Phi_V^{-1}(0)/H$ can be lifted to a Poisson bracket on $\mathbf{R}^k$.*

6.16. Example. In this example we prove Conjecture 6.15 in the case where the algebra $P(V)^H$ has a set of generators $(\sigma_1, \ldots, \sigma_k)$ of degrees ≤ 2. We may assume that this set is minimal.

First we treat the special case where all the generators have degree exactly two. For any pair f, g of homogeneous polynomials in $P(V)$ one has either

$$\deg\{f, g\}_V = \deg f + \deg g - 2$$

or $\{f, g\}_V = 0$. In particular, for each pair σ_i, σ_j of generators one has $\deg\{\sigma_i, \sigma_j\}_V = 2$ or $\{\sigma_i, \sigma_j\}_V = 0$. It follows that the set of all quadratic H-invariant polynomials, which is the linear span of the σ_i's, is a k-dimensional Lie subalgebra of $C^\infty(V)$. This Lie subalgebra has structure constants c_l^{ij} defined by

$$\{\sigma_i, \sigma_j\}_V = \sum_l c_l^{ij} \sigma_l.$$

We define a Poisson bracket on $C^\infty(\mathbf{R}^k)$ by putting

$$\{f, g\} = \sum_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} w_{ij},$$

where $w_{ij}(x) = \sum_l c_l^{ij} x_l$. By construction, the embedding $\tilde{\sigma} : \Phi^{-1}(0)/H \to \mathbf{R}^k$ is a Poisson map. The Hilbert map σ has a nice interpretation. The Hamiltonian vector fields

$$\Xi_{\sigma_1}, \ldots, \Xi_{\sigma_k}$$

are linear and linearly independent, and their span is a Lie subalgebra $\mathbf{s}$ of $\mathbf{sp}(V)$. The Poisson manifold $\mathbf{R}^k$ can be identified with the dual of $\mathbf{s}$, and the Hilbert map $\sigma : V \to \mathbf{R}^k$ is precisely the momentum map for the action of $\mathbf{s}$ on V. If some of the generators in the basis, say $\sigma_1, \dots, \sigma_r$, are linear, then the Hilbert map splits into a product of two maps,

$$(\sigma_1, \dots, \sigma_r) : V_H \xrightarrow{\sim} \mathbf{R}^r, \quad (\sigma_{r+1}, \dots, \sigma_k) : W \to \mathbf{R}^{k-r},$$

where W is the skew complement of V_H. Note that r has to be even. In order to turn the Hilbert map into a Poisson map, we can put a Poisson structure on $\mathbf{R}^{k-r}$ by means of the procedure described above, and a symplectic structure on $\mathbf{R}^r$.

7 A Symplectic Tubular Neighbourhood of a Stratum

The purpose of this section is to describe a normal form for a reduced space M_0 in the neighbourhood of a stratum $(M_0)_{(H)}$. We were led to the construction of this 'semiglobal' normal form by the following observation. We have seen in Section 5 that a point x in the reduced space has a small neighbourhood U which is the cartesian product, $U = C \times B$, of a ball B round the point inside its stratum with a cone C, obtained from reducing a certain linear Hamiltonian space at the zero level. The cartesian projection

$$U = C \times B \to B$$

retracts the neighbourhood U onto a small neighbourhood of x in its stratum. We are going to show that this trivial fibration is roughly speaking the restriction of a fibre bundle defined *globally* over the stratum of the point x. More precisely, we want to show that a neighbourhood of $(M_0)_{(H)}$ in M_0 is a symplectic fibre bundle over $(M_0)_{(H)}$ with typical fibre being a symplectic stratified space, a cone over the link of a point in $(M_0)_{(H)}$. The stratum $(M_0)_{(H)}$ will be the 'vertex section' of this bundle.

The proof is in two stages. First we will show that a neighbourhood in M of $Z_{(H)} := Z \cap M_{(H)}$ is a symplectic fibre bundle over $(M_0)_{(H)}$. At the second stage we will show that this bundle can be reduced fibre by fibre.

We start by reinterpreting the constant rank embedding theorem, Theorem 2.2, as a theorem about symplectic fibrations. Let us suppose first that

$Z_{(H)}$ is coisotropic. Let $\mathcal{V}^*$ be the dual of the vertical bundle of the fibration $Z_{(H)} \to (M_0)_{(H)}$. (The vertical bundle is, by Theorem 2.1, the null foliation of $Z_{(H)}$.) Then $\mathcal{V}^*$ is also a fibre bundle over $(M_0)_{(H)}$ with typical fibre being the cotangent bundle of the orbit $T^*(G/H)$. The existence part of the coisotropic embedding theorem asserts that there exists a closed two-form μ on $\mathcal{V}^*$ which restricts to the canonical symplectic form on the fibre $T^*(G/H)$ and is nondegenerate in a neighbourhood of $Z_{(H)}$, where $Z_{(H)}$ embeds into $\mathcal{V}^*$ as the zero section. Consequently,

$$T^*(G/H) \to (\mathcal{V}^*, \mu) \to (M_0)_{(H)}$$

is a symplectic fibration. The uniqueness part of the embedding theorem guarantees the existence of an equivariant symplectic diffeomorphism between neighbourhoods of $Z_{(H)}$ in $(\mathcal{V}^*, \mu)$ and in (M, ω).

In general, if $Z_{(H)}$ is not coisotropic, one must also use the symplectic normal bundle N of $Z_{(H)}$ in order to reconstruct a neighbourhood of $Z_{(H)}$ in M. Note that for a point $p \in Z_{(H)}$ the fibre N_p is the symplectic slice to the orbit through p modulo the space of the H-fixed vectors, which is the symplectic vector space W occurring in the proof of Theorem 2.1. In the existence part of the proof of the constant rank embedding theorem we constructed a closed two-form σ on the space $N^\#$, the pullback of the bundle $N \to Z_{(H)}$ along the projection $\mathcal{V}^* \to Z_{(H)}$. The space $N^\#$ can also be considered a vector bundle over $Z_{(H)}$ and so $Z_{(H)}$ embeds into $N^\#$ as the zero section.

Let us now describe the fibres of $N^\#$ considered as a bundle over $(M_0)_{(H)}$. A typical fibre of the fibration $N \to (M_0)_{(H)}$ is the the total space of the associated bundle $G \times_H W \to G/H$. The pullback of this bundle along the projection $T^*(G/H) \to G/H$ is a bundle $F = G \times_H ((\mathbf{g}/\mathbf{h})^* \times W)$ (cf. the local model Y of Proposition 2.5). The total space of F is a typical fibre of the fibration $N^\# \to (M_0)_{(H)}$.

Observe that the space F is symplectic. Indeed, the reduction at zero of the Hamiltonian H-space

$$T^*G \times W$$

gives precisely F. We leave it to the reader to check that this symplectic structure on F agrees with the restriction of the form σ on $N^\#$ to F, considered as a fibre of $N^\# \to (M_0)_{(H)}$.

We sum up these results in a lemma.

7.1. LEMMA. *There exists a neighbourhood of the submanifold $Z_{(H)} = Z \cap M_{(H)}$ of M that is symplectically and G-equivariantly diffeomorphic to a neighbourhood of the zero section of the vector bundle $N^{\#} \to Z_{(H)}$. The space $N^{\#}$ is a symplectic fibre bundle over the stratum $(M_0)_{(H)}$,*

$$F \to N^{\#} \to (M_0)_{(H)}, \tag{17}$$

with fibre F given by

$$F = G \times_H ((\mathbf{g}/\mathbf{h})^* \times W).$$

The group G acts on the fibration (17) by bundle automorphisms covering the identity map on the base. We would like to deduce that the reduction of $N^{\#}$ can be carried out fibre by fibre.

Since the fibration (17) is symplectic, it possesses a canonically defined connection. The corresponding horizontal distribution $\mathcal{H}$ is simply obtained by taking the perpendiculars to the vertical with respect to the presymplectic form σ. The connection is G-invariant and it is easy to see that the G-moment map $J : M \to \mathbf{g}^*$ is parallel. Indeed, let p be a point in $N^{\#}$, v a horizontal vector at p and ξ an element of $\mathbf{g}$. Then

$$\langle \xi, dJ_p(v) \rangle = \sigma(\xi_{N^{\#}}(p), v) = 0,$$

since $\xi_{N^{\#}}$ is tangent to the fibres of (17). Thus for any $v \in \mathcal{H}$, $v \cdot J = 0$. The parallel transport corresponding to the connection, if it exists, allows us to identify the intersections of the zero level set of J with the fibres of $N^{\#}$ in a G-equivariant fashion, thereby leading us to conclude that the reduced space $J^{-1}(0)/G$ is a fibration over $(M_0)_{(H)}$ with typical fibre being the singular space $(J^{-1}(0) \cap F)/G$. However, the fibres of (17) are not compact so it is not a priori clear that the connection integrates to a parallel transport. We will overcome this difficulty by showing that the structure group of the bundle (17) is a compact Lie group $\mathcal{G}$, in other words, that it is a bundle associated to a certain principal bundle $\mathcal{G} \to Q \to (M_0)_{(H)}$.

Here is the construction of the bundle Q. Consider the symplectic normal bundle of $Z_{(H)}$,

$$W \to N \to Z_{(H)}.$$

We may assume that the vector bundle N has a G-invariant Hermitian structure such that the imaginary part of the metric is the original symplectic form on the fibres (cf. the Appendix, Example 8.6). Let $U = U(W)$ be the corresponding unitary group of the fibre W and $U \to Fr \to Z_{(H)}$ the bundle of unitary frames of N. Note that since the group G acts on the bundle N, it also acts on the bundle of frames by bundle automorphisms and so the actions of G and U on Fr commute.

Composing the two projections maps we can realize Fr as a fibre bundle over $(M_0)_{(H)}$. The typical fibre of this bundle is the homogeneous space $(G \times U)/H = G \times_H U$. (Note that since the group H acts on the fibre W and preserves the Hermitian inner product, there exists a homomorphism $H \to U$.)

7.2. Lemma. *Suppose a compact Lie group $\mathcal{L}$ acts on a fibre bundle $C \to A \xrightarrow{p} B$ by bundle automorphisms covering the identity, that is, the action of $\mathcal{L}$ maps each fibre C into itself. Assume further that this action is transitive, i.e., $C \simeq \mathcal{L}/\mathcal{K}$ for some $\mathcal{K} < \mathcal{L}$. Then the transition maps of the bundle A can be chosen so that they take their values in $\mathcal{K}' = N_{\mathcal{L}}(\mathcal{K})/\mathcal{K}$, where $N_{\mathcal{L}}(\mathcal{K})$ denotes the normalizer of $\mathcal{K}$ in $\mathcal{L}$. Consequently A is a bundle associated to some principal $\mathcal{K}'$-bundle E, and the group $\mathcal{L}$ acts on E by bundle automorphisms.*

Proof. Since $\mathcal{L}$ is compact, there exists on A an $\mathcal{L}$-invariant metric. This metric defines a connection on the bundle $p : A \to B$. Since the fibres of the bundle are compact, a curve $\gamma : [0,1] \to B$ defines a parallel transport $\mathbf{P}_\gamma : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$. Since the connection is $\mathcal{L}$-invariant, the parallel transport is $\mathcal{L}$-equivariant. Thus we get a map from the space of loops based at $b \in B$ into the space of $\mathcal{L}$-equivariant diffeomorphisms of the fibre $p^{-1}(b) \simeq \mathcal{L}/\mathcal{K}$, the space $\mathit{Diff}(\mathcal{L}/\mathcal{K})^{\mathcal{L}}$, which is isomorphic to $N_{\mathcal{L}}(\mathcal{K})/\mathcal{K}$. □

The lemma implies that there exists a principal $\mathcal{G}$-bundle Q over $(M_0)_{(H)}$, where $\mathcal{G} = N_{G\times U}(H)/H$, such that $Fr \to (M_0)_{(H)}$ is a bundle associated to Q,

$$Fr = Q \times_{\mathcal{G}} (G \times_H U).$$

Now, by definition of the frame bundle, the manifold $Z_{(H)}$ is the orbit space Fr/U, so

$$Z_{(H)} = Q \times_{\mathcal{G}} ((G \times U)/H)\,/U = Q \times_{\mathcal{G}} (G/H).$$

It follows that the dual of the vertical bundle of $Z_{(H)} \to (M_0)_{(H)}$ satisfies

$$\mathcal{V}^*(Z_{(H)}) = Q \times_{\mathcal{G}} (T^*(G/H)) = Q \times_{\mathcal{G}} (G \times_H (\mathbf{g}/\mathbf{h})^*)\,.$$

Also by definition,

$$\begin{array}{rcl} N = Fr \times_U W & = & [Q \times_{\mathcal{G}} (((G \times U)/H) \times W)]/U \\ & = & Q \times_{\mathcal{G}} ((G \times U \times W)/(H \times U)) \\ & = & Q \times_{\mathcal{G}} (G \times_H W). \end{array}$$

Consequently the push-out $N^{\#}$,

$$\begin{array}{ccc} N^{\#} & \longrightarrow & N \\ \downarrow & & \downarrow \\ \mathcal{V}^*(Z_{(H)}) & \longrightarrow & Z_{(H)} \end{array}$$

considered as a bundle over the stratum $(M_0)_{(H)}$ is the associated bundle

$$N^{\#} = Q \times_{\mathcal{G}} [G \times_H ((\mathbf{g}/\mathbf{h})^* \times W)].$$

Indeed, the diagram

$$\begin{array}{ccc} G \times_H ((\mathbf{g}/\mathbf{h})^* \times W) & \longrightarrow & G \times_H W \\ \downarrow & & \downarrow \\ G \times_H (\mathbf{g}/\mathbf{h})^* & \longrightarrow & G/H \end{array}$$

commutes. This discussion results in the following lemma.

7.3. Lemma. *The structure group of the fibration $F \to N^{\#} \to (M_0)_{(H)}$ is the compact Lie group $\mathcal{G}$.*

By Remark 8.5 a choice of a connection one-form on the principal bundle

$$\mathcal{G} \to Q \to (M_0)_{(H)}$$

gives rise to a closed two-form on $N^{\#}$ which is nondegenerate near

$$Q \times_{\mathcal{G}} [G \times_H (\{0\} \times \{0\})] \simeq Z_{(H)}.$$

The uniqueness part of the constant rank embedding theorem furnishes us with a symplectic diffeomorphism between a neighbourhood of $Z_{(H)}$ in $N^{\#}$ and a neighbourhood in the manifold M, that is, we may consider the embedding $Z_{(H)} \hookrightarrow N^{\#}$ as a model for the embedding of $Z_{(H)}$ into M.

On the other hand, the fibre $F = G \times_H ((\mathbf{g}/\mathbf{h})^* \times W)$ of the bundle (17) is a Hamiltonian G-space, and this action of G commutes with the action of $\mathcal{G}$. By Theorem 8.7 the bundle $N^{\#}$ is a Hamiltonian G-space and reduction at zero gives a bundle over the stratum $(M_0)_{(H)}$ with typical fibre being F_0, the reduction of the fibre F. A computation similar to the one in the proof of Theorem 5.1 shows that F_0 is simply the vector space W reduced at 0 with respect to the action of H,

$$F_0 = \Phi_W^{-1}(0)/H,$$

in the notation of (10). This proves the main assertion of the section — a tubular neighbourhood theorem for a stratum of the reduced space M_0.

7.4. Theorem. *Given a stratum $(M_0)_{(H)}$ of the reduced space M_0, there exists a symplectic fibre bundle over $(M_0)_{(H)}$ with typical fibre being a conical stratified symplectic space such that a neighbourhood of the vertex section of this bundle is symplectically diffeomorphic to a neighbourhood of the stratum inside the reduced space.*

We conclude this section with the following observation. Using the tubular neighbourhood theorem, one can show that the symplectic links of the points in a stratum $(M_0)_{(H)}$ fit together to form a locally trivial bundle

$$\bigsqcup_{x \in (M_0)_{(H)}} \Sigma_x = \Sigma_{(H)} \longrightarrow (M_0)_{(H)},$$

called the *symplectic link bundle* of the stratum. Here is a sketch of the proof.

Multiplication by complex numbers of norm 1 defines an S^1-action on the Hermitian vector space W commuting with the actions of both H and U. It is the action generated by the Hamiltonian $w \mapsto \|w\|^2 - 1$. Going through all the steps of the proof of Theorem 7.4 one checks that this circle action defines uniquely a fibre-preserving Hamiltonian circle action on the bundle $(N^{\#})_0 \to (M_0)_{(H)}$. We define the bundle $\Sigma_{(H)}$as the S^1-reduction of $(N^{\#})_0$ at the zero level. Applying Theorem 8.7 we find that $\Sigma_{(H)}$ is a symplectic fibre bundle over $(M_0)_{(H)}$, whose fibre is the space obtained by reduction of $F_0 = \Phi_W^{-1}(0)/H$ with respect to the circle action. Using reduction-in-stages we see that a typical fibre of $\Sigma_{(H)}$ is the symplectic link of a point in the stratum $(M_0)_{(H)}$.

8 Appendix: Minimal Coupling of Sternberg and Weinstein

We briefly review minimal coupling, which was discovered by Sternberg [33], in the form due to Weinstein [35].

Let $K \to P \xrightarrow{\pi} B$ be a principal K-bundle over a symplectic base (B, ω). Let θ be a connection one-form on P and let $\langle \cdot, \cdot \rangle$ denote the pairing between $\mathbf{k}^*$ and $\mathbf{k}$. (Here $\mathbf{k}^*$ denotes the dual of the Lie algebra $\mathbf{k}$ of K.) Then $\langle pr_2, \theta \rangle$ is a real-valued K-invariant one-form on $P \times \mathbf{k}^*$. The corresponding *minimal coupling form* σ_θ is defined to be

$$\sigma_\theta = \pi^* \omega - d\langle pr_2, \theta \rangle.$$

8.1. Theorem. *The minimal coupling form σ_θ defines a presymplectic structure on the manifold $P \times \mathbf{k}^*$ which is nondegenerate near $P \times \{0\}$. Moreover, the action of K on $P \times \mathbf{k}^*$ given by*

$$a \cdot (p, \eta) = (pa^{-1}, Ad^*(a)\eta)$$

is Hamiltonian with the corresponding momentum map being minus the projection on the second factor,

$$-pr_2 : P \times \mathbf{k}^* \to \mathbf{k}^*.$$

If the base B is compact, then there exists a neighbourhood $\mathcal{U}$ of zero in $\mathbf{k}^*$ such that $P \times \mathcal{U}$ is a symplectic manifold.

Sketch of Proof. Let $\sigma = \sigma_\theta$. Clearly σ is closed and K-invariant. To show the nondegeneracy of σ at a point of the form $(p, 0)$ one writes first

$$\begin{aligned} \sigma_{(p,0)} &= (\pi^*\omega)_p - \langle d\, pr_2 \wedge \theta \rangle_{(p,0)} - \langle pr_2, d\theta \rangle_{(p,0)} \\ &= (\pi^*\omega)_p - \langle d\, pr_2 \wedge \theta_p \rangle. \end{aligned}$$

Now, if we use the connection to split the tangent space to P at p into horizontal and vertical spaces, $T_pP = \mathcal{H}_p \oplus \mathcal{V}_p$, we see that $(\pi^*\omega)_p$ is nondegenerate on $\mathcal{H}_p$ and that $\langle d\, pr_2 \wedge \theta_p \rangle$ is nondegenerate on $\mathcal{V}_p \oplus \mathbf{k}^* \simeq \mathbf{k} \oplus \mathbf{k}^*$.

Given an action of K on a manifold X and a vector $\xi \in \mathbf{k}$, let ξ_X denote the corresponding induced vector field on X.

The K-invariance of $\langle pr_2, \theta\rangle$ implies that for $\xi \in \mathbf{k}$

$$\begin{aligned} -\iota(\xi_P + \xi_{\mathbf{k}^*})\, d\langle pr_2, \theta\rangle &= d\,\iota(\xi_P + \xi_{\mathbf{k}^*})\langle pr_2, \theta\rangle \\ = d\langle pr_2, \theta(\xi_P)\rangle &= -\langle pr_2, \xi\rangle. \end{aligned}$$

It follows that $-pr_2$ is a momentum map for the action of K on $P \times \mathbf{k}^*$. $\square$

8.2. Remark. If θ_1 and θ_2 are two different connection one-forms on P, then the two corresponding minimal coupling forms σ_1 and σ_2 coincide at the points of $P \times \{0\}$. Consequently, by the Darboux–Moser–Weinstein Theorem, the forms σ_1 and σ_2 are equivalent on a neighbourhood of $P \times \{0\}$. If the base B is compact then this neighbourhood may be taken to be of the form

$$P \times \mathcal{U}$$

for some open subset $\mathcal{U}$ of $\mathbf{k}^*$.

8.3. Example. Suppose $K \to Q \to X$ is a principal K-bundle over an arbitrary manifold X. Let $Q^\#$ be the pullback of the bundle along the projection $T^*X \to X$. A connection on Q pulls back to a connection on $Q^\#$ and minimal coupling then defines a closed two-form on $Q^\# \times \mathbf{k}^*$ which is nondegenerate everywhere. Indeed, a choice of a connection on Q allows one to identify $Q^\# \times \mathbf{k}^*$ with the cotangent bundle T^*Q with its canonical symplectic form.

8.4. Theorem. *Let $K \to P \xrightarrow{\pi} (B, \omega)$ be a principal fibre bundle over a symplectic base, θ a connection one-form on P and $\sigma = \sigma_\theta$ the corresponding minimal coupling form. Suppose that there exists an open set $\mathcal{W} \subset \mathbf{k}^*$ such that the form σ is nondegenerate on $P \times \mathcal{W}$. Let (F, ω_F) be a Hamiltonian K-space with momentum map $J : F \to \mathbf{k}^*$ and suppose that the image of F under J is contained in $\mathcal{W}$,*

$$J(F) \subset \mathcal{W}.$$

Then the associated fibre bundle $P \times_K F$ is symplectic and the map $(F, \omega_F) \hookrightarrow P \times_K F$ is a symplectic embedding.

Sketch of Proof. The diagonal action of K on $(P \times \mathcal{W}) \times F$ is Hamiltonian with moment map Φ given by

$$\Phi(p, \eta, f) = J(f) - \eta.$$

Therefore, the product $P \times F$ is diffeomorphic to the zero level set of Φ by the K-equivariant diffeomorphism given by

$$P \times F \ni (p, f) \mapsto (p, J(f), f) \in P \times \mathcal{W} \times F.$$

So the associated bundle $P \times_K F$ is just the reduced space of $P \times \mathcal{W} \times F$ at the zero level. □

8.5. Remark. What happens if no such set $\mathcal{W}$ exists? Clearly, the associated bundle $P \times_K F$ still carries a closed two-form which restricts to the symplectic form on each fibre. Also, the set

$$\{(p, f) \in P \times F : \sigma_\theta \text{ is nondegenerate at } (p, J(f))\}$$

is an open K-invariant subset of $P \times F$ containing $P \times J^{-1}(0)$. Consequently, the presymplectic form on $P \times F$ is nondegenerate in a neighbourhood of $P \times_K J^{-1}(0)$.

8.6. Example. Let $V \to E \xrightarrow{\pi} B$ be a symplectic vector bundle over a symplectic base (B, ω_0). We may assume that the bundle E is complex and that the linear symplectic form on the fibre is the imaginary part of a Hermitian form (see [34]). Let $Fr(E) \to B$ be the bundle of unitary frames of E and let U denote the corresponding unitary group, the structure group of $Fr(E)$. Then E is an associated bundle,

$$E = Fr(E) \times_U V.$$

Since the symplectic action of U on (V, ω_V) is linear, it is Hamiltonian. Indeed, the map $\Phi_V : V \to \mathbf{u}^*$ given by

$$\langle \xi, \Phi_V(v) \rangle = 1/2\, \omega_V(\xi_V \cdot v, v)$$

for $\xi \in \mathbf{u} = Lie(U)$ is a momentum map. A choice of a connection one-form on $Fr(E)$ defines on E a closed two-form which, by Remark 8.5, is nondegenerate near the zero section of E. Given a different choice of a connection, the corresponding symplectic form on E is equivalent to the old form in a neighbourhood of the zero section.

8.7. THEOREM. *Let $K \to P \xrightarrow{\pi} (B, \omega)$, $J : F \to \mathbf{k}^*$ etc. be as in Theorem 8.4. Assume that there exists on F a Hamiltonian action of another group H with momentum map $\Phi_F : F \to \mathbf{h}^*$, and that the actions of K and H commute. Assume further that the reduction of F at zero with respect to the action of H makes sense. (For example, if the group H is compact, the reduction is well-defined by Theorem 2.1.) Then the associated bundle $P \times_K F$ is a Hamiltonian H-space, the corresponding momentum map $\Phi : P \times_K F \to \mathbf{h}^*$ sends the class of $(p, f) \in P \times F$ to $\Phi_F(f)$ and the H-reduced space $\Phi^{-1}(0)/H$ is the associated bundle*

$$\Phi^{-1}(0)/H = P \times_K \left(\Phi_F^{-1}(0)/H\right).$$

In other words, the H-reduction can be carried out on the bundle $P \times_K F$ fibre by fibre.

SKETCH OF PROOF. Use Theorem 8.4 and reduction in stages (see Section 4). □

References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics,* second edition, Benjamin/Cummings, Reading, 1978.

[2] J. Arms, R. Cushman, M. Gotay, A Universal Reduction Procedure for Hamiltonian Group Actions, preprint 591, University of Utrecht, the Netherlands, 1989.

[3] J. Arms, M. Gotay and G. Jennings, Geometric and Algebraic Reduction for Singular Momentum Maps, *Advances in Mathematics* **79** (1990), 43–103.

[4] W. Baily, The Decomposition Theorem for V-Manifolds, *Amer. J. of Math.* **78** (1956), 862–888.

[5] E. Bierstone, *The Structure of Orbit Spaces and the Singularities of Equivariant Mappings,* Monografias de matemática **35**, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1980.

[6] G. Bredon, *Introduction to Compact Transformation Groups,* Academic Press, New York, 1972.

[7] R. Cushman, private communication, 1989.

[8] R. Cushman and R. Sjamaar, On Singular Reduction of Hamiltonian Spaces, to appear in the proceedings of the Colloque international en l'honneur de Jean-Marie Souriau, 1990.

[9] C. G. Gibson et alii, *Topological Stability of Smooth Mappings,* Lecture Notes in Mathematics **552**, Springer Verlag, New York, 1976.

[10] J. Basto Gonçalves, Reduction of Hamiltonian Systems with Symmetry, preprint of the Universidade do Porto, Portugal, 1988.

[11] M. Goresky and R. MacPherson, Intersection Homology Theory, *Topology* **19** (1980), 135–162.

[12] M. Goresky and R. MacPherson, *Stratified Morse Theory,* Springer Verlag, New York, 1988.

[13] M. J. Gotay, On Coisotropic Embeddings of Presymplectic Manifolds, *Proc. Am. Math. Soc.* **84** (1982), 111–114.

[14] M. Gromov, *Partial Differential Relations,* Springer Verlag, New York, 1986.

[15] V. Guillemin and S. Sternberg, Geometric Quantization and Multiplicities of Group Representations, *Invent. Math.* **67** (1982), 515–538.

[16] V. Guillemin and S. Sternberg, A Normal Form for the Moment Map, in: *Differential Geometric Methods in Mathematical Physics,* editor S. Sternberg, Reidel Publishing Company, Dordrecht, 1984.

[17] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics,* Cambridge University Press, Cambridge, 1984.

[18] G. Kempf and L. Ness, *The Length of Vectors in Representation Space,* Lecture Notes in Mathematics **732**, Springer Verlag, New York, 1979.

[19] F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry,* Mathematical Notes **31**, Princeton University Press, Princeton, 1984.

[20] F. Kirwan, Convexity Properties of the Moment Mapping III, *Invent. Math.* **77** (1984), 547–552.

[21] E. Lerman, R. Montgomery and R. Sjamaar, Examples of Singular Reduction, in preparation.

[22] E. Lerman, *Symplectic Fibrations and Weight Multiplicities of Compact Groups,* PhD thesis, MIT, 1989.

[23] S. Łojasiewicz, *Ensembles sémi-analytiques,* IHES, 1965.

[24] C.-M. Marle, Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique, *Rendiconti del Seminario Matematico* **43** (1985), 227–251, Università e Politechnico, Torino.

[25] J. Marsden and A. Weinstein, Reduction of Symplectic Manifolds with Symmetry, *Rep. Math. Phys.* **5** (1974), 121–130.

[26] J. Mather, *Notes on Topological Stability,* Harvard University, 1970, unpublished.

[27] J. Mather, Stratifications and Mappings, in: *Dynamical Systems*, M. M. Peixoto, editor, Academic Press, New York, 1973.

[28] J. Mather, Differentiable Invariants, *Topology* **16** (1977), 145–155.

[29] I. Satake, On a Generalization of the Notion of Manifold, *Proc. Nat. Acad. Sc.* **42** (1956), 359–363.

[30] G. W. Schwarz, *Generalized Orbit Spaces,* revised version of PhD thesis, MIT, 1972, unpublished.

[31] G. W. Schwarz, Smooth Functions Invariant under the Action of a Compact Lie Group, *Topology* **14** (1975), 63–68.

[32] R. Sjamaar, *Singular Orbit Spaces in Riemannian and Symplectic Geometry,* PhD thesis, University of Utrecht, 1990.

[33] S. Sternberg, On Minimal Coupling and the Symplectic Mechanics of a Classical Particle in the Presence of a Yang–Mills Field, *Proc. Nat. Acad. Sci. USA* **74** (1977), 5253–5254.

[34] A. Weinstein, *Lectures on Symplectic Manifolds,* CBMS Regional Conf. Series in Math **29**, 1977.

[35] A. Weinstein, A Universal Phase Space for Particles in Yang-Mills Fields, *Lett. Math. Phys.* **2** (1978), 417–420.

[36] H. Whitney, Analytic Extensions of Differentiable Functions Defined in Closed Sets, *Trans. A.M.S.* **36** (1934), 63–89.

[37] H. Whitney, Tangents to an Analytic Variety, *Ann. of Math.* **81** (1965), 496–549.