1 Homework set I

Question 1. Prove that the followings are equivalent.

1. $(X, \mathcal{B}, \mu, T)$ is ergodic.
2. $\forall B \in \mathcal{B}$, if $\mu(T^{-1}B \triangle B) = 0$, then $\mu(B) = 0$ or 1.
3. $\forall A \in \mathcal{B}$, if $\mu(A) > 0$, then $\mu(\cup_{i=1}^{\infty} T^{-n}A) = 1$.
4. $\forall A, B \in \mathcal{B}$, if $\mu(A)\mu(B) > 0$, then $\exists n \geq 1$ such that $\mu(T^{-n}A \cap B) > 0$.

Question 2.

1. Prove that $(X, \mathcal{B}, \mu, T)$ is ergodic iff $\frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) \to \mu(A)\mu(B)$ for all A, B in a semialgebra generating $\mathcal{B}$.
2. Use part 1. to prove that Bernoulli shifts are ergodic. (We will see later that the same argument shows that Bernoulli shifts are strong-mixing.)

Question 3. Let A be a $d \times d$ integer matrix with $\det(A) \neq 0$. Consider the surjective endomorphism $T_A : \mathbb{T}^d \to \mathbb{T}^d$ induced by A (multiplication by A), which preserves the Lebesque measure $m_{\mathbb{T}^d}$. Show that T is ergodic if and only if no eigenvalue of A is a root of unity. (We will generalize this to a continuous surjective homomorphism of a compact abelian groups.)