Homework set II

Question 1. Is there a Rényi type theorem for weak-mixing systems? (Recall : Rényi theorem says that T is strong -mixing iff $\mu(T^{-n}A \cap A) \to \mu(A)^2$, $\forall A \in \mathcal{B}$.)

Question 2. Give an alternative proof of the following: if T has no non-constant measurable eigenfunction, then T is weak-mixing.

(Hint: show that if T has no non-constant measurable eigenfunction, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} | < \mathcal{U}_T^n f, f > |^2 \to 0.)$$

Question 3. Let R_α be the circle rotation by an irrational number α. (We saw that it is uniquely ergodic.)

1. Show equidistribution property :
$$\lim \frac{1}{N} \sum_{i=0}^{N-1} \chi_{[a,b)}]((R_\alpha)^n(t)) \to (b-a).$$

2. Consider the first digit of 2^n, i.e. $1, 2, 4, 8, 1, 3, 6, \cdots$. What is the density of the first digit being k $(0 \le k \le 9)$?

Question 4. Let X be a compact metrizable group.

1. Show that there is a bi-invariant metric on X defining the topology on X.

2. Show that $d_Y(x) = \min\{d(x,) : y \in Y\}$ is a non-constant function on X which is constant on each coset of Y.

(Reference : Einsiedler-Ward, Chapter 1, 2, 4, 7)