Homework set III

Question 1. Let σ_4 be the Bernouilli shift on 4 letters, and let μ_4 be the σ_4-invariant measure determined by giving equal probability $(1/4, \cdots, 1/4)$ to 1-cylinder sets. Show that (σ_4, μ_4) is isomorphic to σ_5 with invariant measure ν_5 determined by giving probability $(1/2, 1/8, 1/8, 1/8, 1/8)$ to 1-cylinder sets.

Question 2. Show that $h_\mu(T, \xi) = H_\mu(\xi | \vee_1^\infty T^{-i}\xi)$.

Question 3.

1. Show that the set Φ of equivalence classes of finite measurable partitions (up to measure zero) of $(X, \mathcal{B}, \mu)$ is a metric space with entropy metric
$$d(\alpha, \beta) = H(\alpha|\beta) + H(\beta|\alpha).$$
2. Show that $h(\alpha, T)$ is a continuous function of α.
3. Show that if α_i is an increasing sequence of finite partitions such that $\vee_1^\infty \alpha_k = \mathcal{B}$ (up to measure zero), then
$$\{\beta \in \Phi : \beta \leq \alpha_n, \exists n\}$$
is dense in Φ.
4. Prove Kolmogorov-Sinai theorem: in other words, show that if α is a generator, $\alpha_n = \vee_{-k}^{k} T^{-i}(\alpha)$, and $\beta \leq \alpha_n$ then $h(T, \beta) \leq h(T, \alpha)$.

(Reference : Peterson, Chapter 5)