Math 4320 : Introduction to Algebra

Geometry and groups - Isometries on Euclidean spaces

(due March 13, 2009, 1:25pm)

Part I. Problem 1- 2 show that the group Isom($\mathbb{R}^2$) is generated by reflections. (You can also use results proved in section 2.3.)

1. (i) Show that an isometry of the plane that fixes a point is a rotation if it preserves orientation and it is a reflection if it reverses orientation.
(ii) Show that an isometry of the plane that fixes no point is a translation if it preserves orientation and a glide reflection if it reverses orientation. (A glide reflection is a reflection about an axis l followed by a nontrivial translation parallel to l.)

2. (i) Show that the product of two reflections with parallel axes is a translation perpendicular to these axes by a distance twice the distance between two axes.
(ii) Show that the product of two reflections with axes meeting in a point is a rotation about that point through an angle twice the angle between two axes.
(iii) Deduce that every orientation preserving transformation is a product of two reflections, and every orientation reversing transformation is a product of three reflections. Therefore the group Isom($\mathbb{R}^2$) is generated by reflections.

3. (i) Show that if a subgroup G of the group Isom($\mathbb{R}^2$) contains no nontrivial translation, then G fixes a point.
(ii) Show that if G is a finite subgroup of Isom($\mathbb{R}^2$), then G is either cyclic or dihedral.

Part II. : Icosahedral group I is the group of rotational isometries of a regular dodecahedron. Problem 4-6 give a geometric proof that the alternating group A_5 is simple.

4. (i) Explain why the group I has 60 rotations. What are the possible rotation angles of the elements of I? What are the possible orders of the elements of I?
(ii) Explain why I is also the group of isometries of or a regular icosahedron. In other words, how is a dodecahedron related to a icosahedron?

5. (i) Using 4.(i), find the class equation of the group I by finding the conjugacy classes.
(ii) Using (i), show that the group I has no proper normal subgroup.

6. Show that the group I is isomorphic to the alternation group A_5.
(Hint. Imagine five cubes inscribed in the dodecahedron. The group I acts on the set of these cubes. Use the fact that I is simple.)