COUNTING OVERLATTICES FOR POLYHEDRAL COMPLEXES

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Abstract. We investigate the asymptotics of the number of "overlattices" of a cocompact lattice Γ in $\mathrm{Aut}(X)$, where X is a locally finite polyhedral complex. We use complexes of groups to prove an upper bound for general X, and a lower bound for certain right-angled buildings.

1. Introduction

Let G be a locally compact topological group with Haar measure μ. A discrete subgroup $\Gamma \leq G$ is a *lattice* if the covolume $\mu(\Gamma \backslash G)$ is finite, and is *cocompact* if $\Gamma \backslash G$ is compact.

A theorem of Kazhdan–Margulis [KM] implies that for a given connected semisimple Lie group G, there is a positive lower bound on the covolume $\mu(\Gamma \backslash G)$ of lattices in G. In contrast, if G is the automorphism group of a locally finite tree, Bass–Kulkarni [BK] constructed infinite strictly ascending sequences of lattices

$$\Gamma_1 < \Gamma_2 < \cdots < \Gamma_i < \cdots$$

in G, hence the covolumes $\mu(\Gamma_i \backslash G)$ tend to zero. A question raised by Bass and Lubotzky ([BL], Section 0.7) is to find the asymptotic behavior of the number $u_\Gamma(n)$ of *overlattices* of Γ of index n, that is, the number of lattices $\Gamma' \leq G$ containing Γ with $[\Gamma' : \Gamma] = n$.

In this paper, we consider the asymptotics of $u_\Gamma(n)$ for Γ cocompact in the automorphism group G of a locally finite polyhedral complex X. The topology on G is described in Section 2.1 below. By arguments similar to those for tree lattices (Theorem 6.5, [BK]), for such lattices Γ the cardinality $u_\Gamma(n)$ is finite.

The case X is a tree is treated by Lim [L]. In higher dimensions, suppose X is the Bruhat–Tits building for a higher-rank semisimple Lie group $\mathcal{G}$ over a nonarchimedean local field of characteristic 0, for example $\mathcal{G} = SL_3(\mathbb{Q}_p)$. Then $\mathcal{G}$ has finite index in $\mathrm{Aut}(X)$ (Tits, [T]). It follows that for any cocompact lattice $\Gamma \leq \mathrm{Aut}(X)$, $u_\Gamma(n) = 0$ for large enough n, since the covolumes of lattices in $\mathcal{G}$ are bounded away from zero (Borel–Prasad [BP]). In contrast, if X is a right-angled building, such as Bourdon's building $I_{p,q}$ (see [Bo]), then Thomas [Th] showed that $\mathrm{Aut}(X)$ admits infinite ascending sequences of cocompact lattices. Hence there is a Γ such that $u_\Gamma(n) > 0$ for arbitrarily large n.

In order to further study the growth rate of $u_\Gamma(n)$, we apply results of [LT] on coverings of complexes of groups (see Section 2.2 for a summary of Haefliger's theory of complexes of groups [BH]). We define isomorphism of coverings of complexes of groups, and prove in [LT] that there is a bijection between isomorphism classes of n–sheeted coverings, and overlattices of index n (a precise statement is given in Theorem 3.1 below). Using this and deep results of finite group theory, in Section 4.1 we establish the following upper bound, for very general X:

Theorem 1.1. *Let X be a simply connected, locally finite polyhedral complex and $\Gamma \leq \mathrm{Aut}(X)$ a cocompact lattice. Then there are some positive constants C_0 and C_1, depending only on Γ, such that*

$$\forall\, n > 1, \qquad u_\Gamma(n) \leq (C_0 n)^{C_1 \log^2(n)}.$$

The lower bound, proved in Section 4.2, is for certain right-angled buildings. A special case of the lower bound we obtain is:

Theorem 1.2. *Let q be prime and let X be a Bourdon building $I_{p,2q}$. Then there is a cocompact lattice Γ in $\mathrm{Aut}(X)$, and constants C_0 and C_1, such that for any $N > 0$, there exists $n > N$ with*

$$u_\Gamma(n) \geq (C_0 n)^{C_1 \log n}.$$

The full statement, in Theorem 4.2, applies to more general right-angled buildings, including not only some hyperbolic buildings, but also buildings in arbitrarily high dimension which may be equipped with a piecewise Euclidean metric. The proof applies the Functor Theorem of [Th] to a construction for tree lattices in [L].

Theorems 1.1 and 4.2, together with the examples given above for buildings, are presently the only known behaviors for overlattice counting functions in higher dimensions.

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2. Background

2.1. **Lattices in automorphism groups of polyhedral complexes.** Let G be a locally compact topological group with left-invariant Haar measure μ. Let S be a left G–set such that for every $s \in S$, the stabilizer G_s is compact and open. Then if $\Gamma \leq G$ is discrete, the stabilizers Γ_s are finite. We define the *S–covolume* of Γ by

$$\mathrm{Vol}(\Gamma\backslash\backslash S) := \sum_{s \in \Gamma\backslash S} \frac{1}{|\Gamma_s|} \leq \infty.$$

A theorem of Serre [S] shows that Haar measure may be normalized so that $\mu(\Gamma\backslash G)$ equals the S–covolume.

Let M^n_κ be the complete, simply connected, Riemannian n–manifold of constant sectional curvature $\kappa \in \mathbb{R}$. An *M_κ–polyhedral complex* X is a finite-dimensional CW-complex such that:

(1) each open cell of dimension n is isometric to the interior of a compact convex polyhedron in M^n_κ; and
(2) for each cell σ of K, the restriction of the attaching map to each open codimension one face of σ is an isometry onto an open cell of K.

Let X be a connected, locally finite polyhedral complex, with first barycentric subdivision X'. Let $V(X')$ be the set of vertices of X'. Let $\mathrm{Aut}(X)$ be the group of cellular isometries of X. A subgroup of $\mathrm{Aut}(X)$ is said to act *without inversions* on X if its elements fix pointwise each cell that they preserve.

The group $G = \mathrm{Aut}(X)$ is naturally a locally compact topological group, with the compact-open topology. By the same arguments as for tree lattices ([BL], Chapter 1), it can be shown that if $G\backslash X$ is finite, then a discrete subgroup $\Gamma \leq G$ is a lattice if and only if its $V(X')$–covolume converges, and Γ is cocompact if and only if this sum has finitely many terms. Using Serre's normalization, we now normalize the Haar measure μ on $G = \mathrm{Aut}(X)$ so that for all lattices $\Gamma \leq G$, the covolume of Γ is

$$\mu(\Gamma\backslash G) = \mathrm{Vol}(\Gamma\backslash\backslash V(X')).$$

2.2. **Complexes of groups.** Complexes of groups may be thought of as analogues of "quotient orbifolds" for groups (with some torsion) acting on polyhedral complexes. We give here the definitions from the theory of complexes of groups, due to Haefliger in [BH], necessary for stating the results from [LT] that we will be applying.

Let Y be a polyhedral complex with barycentric subdivision Y'. Let $V(Y')$ be the set of vertices and $E(Y')$ the set of edges of Y'. Each $a \in E(Y')$ corresponds to cells $\tau \subset \sigma$ of Y, and so may be oriented from σ to τ. We write $i(a) = \sigma$ and $t(a) = \tau$. Two edges a and b of Y' are *composable* if $i(a) = t(b)$, in which case there exists an edge $c = ab$ of Y' such that $i(c) = i(b)$, $t(c) = t(a)$ and a, b and c form the boundary of a 2–simplex in Y'.

A *complex of groups* $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ over a polyhedral complex Y is given by:

(1) a group G_σ for each $\sigma \in V(Y')$, called the *local group* at σ;
(2) a monomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$ for each $a \in E(Y')$; and
(3) for each pair of composable edges a, b in Y', a "twisting" element $g_{a,b} \in G_{t(a)}$, such that

$$\mathrm{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where $\mathrm{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$, and for each triple of composable edges a, b, c the following cocycle condition holds

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}.$$

Next we define morphisms of complexes of groups. Let $G(Y) = (G_\sigma, \psi_a, g_{a,b})$ and $H(Z) = (H_\sigma, \psi_{a'}, g_{a',b'})$ be complexes of groups over polyhedral complexes Y and Z respectively. Let $f : Y' \to Z'$ be a simplicial map sending vertices to vertices and edges to edges (such an f is *nondegenerate*). A *morphism* $\phi : G(Y) \to H(Z)$ over f consists of:

(1) a homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in V(Y')$, and
(2) an element $\phi(a) \in H_{t(f(a))}$ for each $a \in E(Y')$, such that

$$\mathrm{Ad}(\phi(a)) \circ \psi_{f(a)} \circ \phi_{i(a)} = \phi_{t(a)} \circ \psi_a$$

where $\mathrm{Ad}(\phi(a))(g) = \phi(a)g\phi(a)^{-1}$, and for all pairs of composable edges (a, b) in $E(Y')$,

$$\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi_{f(a)}(\phi(b))g_{f(a),f(b)}.$$

If f is an isometry of simplicial complexes and each ϕ_σ is an isomorphism of groups then the morphism ϕ is an *isomorphism*.

A morphism of complexes of groups $\phi : G(Y) \to H(Z)$ is a *covering* if in addition to the conditions above,

(1) each ϕ_σ is injective (in which case we say ϕ is *injective on the local groups*), and

(2) for each $\sigma \in V(Y')$ and $b \in E(Z')$ such that $t(b) = f(\sigma)$, the map

$$\coprod_{\substack{a \in f^{-1}(b) \\ t(a)=\sigma}} G_\sigma / \psi_a(G_{i(a)}) \to H_{f(\sigma)} / \psi_b(H_{i(b)})$$

induced by $g \mapsto \phi_\sigma(g)\phi(a)$ is a bijection.

Note that an isomorphism of complexes of groups is a covering. From Condition (2) in the definition of covering, it follows that for all $\tau \in V(Z')$, all $b \in E(Z')$ such that $t(b) = \tau$, and all $\sigma \in f^{-1}(\tau)$,

$$\sum_{\substack{a \in f^{-1}(b) \\ t(a)=\sigma}} \frac{|G_\sigma|}{|G_{i(a)}|} = \frac{|H_\tau|}{|H_{i(b)}|}.$$

Since Y' is connected, the value of

$$n := \sum_{\sigma \in \ f^{-1}(\tau)} \frac{|H_\tau|}{|G_\sigma|} = \sum_{a \in f^{-1}(b)} \frac{|H_{i(b)}|}{|G_{i(a)}|}$$

is independent of the vertex τ and the edge b. A covering of complexes of groups with the above n is said to be *n–sheeted*.

Let G be a group acting without inversions on a polyhedral complex X, and let $Y = G\backslash X$. This action induces a complex of groups $G(Y)$, which is unique up to isomorphism of complexes of groups. A complex of groups is *developable* if it is isomorphic to a complex of groups induced by an action. There is a local condition, called nonpositive curvature of a complex of groups, which ensures developability:

Theorem 2.1 (Haefliger, [BH])**.** *A nonpositively curved complex of groups is developable.*

The *fundamental group* $\pi_1(G(Y), T)$ of $G(Y)$ is defined with respect to a choice of maximal tree T in the 1–skeleton of Y', so that if Y is simply connected and all $g_{a,b}$ are trivial, then $\pi_1(G(Y), T)$ is isomorphic to the direct limit of the family of groups G_σ and monomorphisms ψ_a.

If a complex of groups $G(Y)$ is developable, then its *universal cover* $D(G(Y), T)$, defined with respect to a choice of maximal tree T, is a connected, simply-connected polyhedral complex. Different choices of trees T result in isometric universal covers. The universal cover is equipped with a natural action of the fundamental group, so that the complex of groups induced by the action of $\pi_1(G(Y), T)$ on the universal cover $D(G(Y), T)$ is canonically isomorphic to $G(Y)$.

3. Covering theory for complexes of groups

To count overlattices, we use a one-to-one correspondence between isomorphism classes of coverings of complexes of groups and overlattices. Although this correspondence might be expected from general Galois theory of coverings, the covering theory of complexes of groups developed in [LT] is necessary for concrete problems, e.g., for counting overlattices, as we specify the choices we make to construct isomorphism classes of coverings induced by overlattices. All the results in this section are proved in [LT].

Theorem 3.1. *Let X be a simply connected, locally finite polyhedral complex, and let Γ be a cocompact lattice in $\mathrm{Aut}(X)$ (acting without inversions) which induces a complex of groups $G(Y)$ over $Y = \Gamma\backslash X$. Then there is a bijection between the set of overlattices of Γ of index n (acting without inversions) and the set of isomorphism classes of n–sheeted coverings of faithful developable complexes of groups by $G(Y)$.*

The definition of isomorphism of coverings is given at the end of this section.

We need the following lemma to ensure that the target complexes of groups constructed for the lower bound (in section 4.2) are also developable.

Lemma 3.2. *Let $\lambda : G(Y) \to H(Z)$ be a covering of complexes of groups, over a nondegenerate simplicial map $l : Y' \to Z'$. Suppose that for some $\kappa \leq 0$, Y and Z are M_κ–polyhedral complexes with finitely many isometry classes of cells, and that $l : Y' \to Z'$ is a local isometry on each simplex. If $G(Y)$ has nonpositive curvature (thus is developable), then $H(Z)$ also has nonpositive curvature, thus $H(Z)$ is developable.*

Let $G(Y)$ be a developable complex of groups over a polyhedral complex Y, with universal cover X and fundamental group Γ. We say that $G(Y)$ is *faithful* if the action of Γ on X is faithful. If a complex of groups $G(Y)$ is faithful, then Γ may be regarded as a subgroup of $\mathrm{Aut}(X)$. Moreover, Γ is a discrete subgroup if and only if all local groups of $G(Y)$ are finite, and Γ is a cocompact lattice if and only if Y is a finite polyhedral complex. We will need Proposition 3.3 below, which gives sufficient conditions for faithfulness.

If $G(Y)$ is developable, then for any choice of tree T, there is a canonical morphism of complexes of groups

$$\iota_T : G(Y) \to \pi_1(G(Y), T)$$

which is injective on each local group G_σ (here, the group $\pi_1(G(Y), T)$ is considered as a complex of groups over a single vertex).

Proposition 3.3. *Let $G(Y)$ be a developable complex of groups over a connected polyhedral complex Y. Choose a maximal tree T in the 1–skeleton of Y', and identify each local group G_σ with its image in $\pi_1(G(Y), T)$ under ι_T. Let*

$$N_T = \ker(\pi_1(G(Y), T) \to D(G(Y), T)).$$

Then

1. *N_T is a vertex subgroup, that is $N_T \leq G_\sigma$ for each $\sigma \in V(Y')$.*
2. *N_T is Y–invariant, that is $\psi_a(N_T) = N_T$ for each $a \in E(Y')$.*
3. *N_T is normal, that is $N_T \trianglelefteq G_\sigma$ for each $\sigma \in V(Y')$.*
4. *N_T is maximal: if N'_T is another Y–invariant normal vertex subgroup then $N'_T \leq N_T$.*

The following result appears as Proposition 2.5 in [LT], where the induced maps Λ_{T_1,T_2} and $L^\lambda_{T_1,T_2}$ are explicitly defined.

Proposition 3.4. *Let $\lambda : G(Y_1) \to G(Y_2)$ be a covering of complexes of groups over a nondegenerate simplicial map $l : Y'_1 \to Y'_2$, where Y_1 and Y_2 are connected polyhedral complexes. Assume $G(Y_1)$ and $G(Y_2)$ are developable. For any maximal trees T_1 and T_2 in the 1–skeletons of Y'_1 and Y'_2 respectively, there is an induced monomorphism of fundamental groups*

$$\Lambda_{T_1,T_2} : \pi_1(G(Y_1), T_1) \to \pi_1(G(Y_2), T_2)$$

and a Λ_{T_1,T_2}–equivariant isomorphism of universal covers

$$L^{\lambda}_{T_1,T_2} : D(G(Y_1), T_1) \to D(G(Y_2), T_2).$$

Proposition 3.4 is used to define isomorphism of coverings, as follows. Let $\lambda : G(Y_1) \to G(Y_2)$ and $\lambda' : G(Y_1) \to G(Y_3)$ be coverings of developable complexes of groups over connected polyhedral complexes. We say that λ and λ' are *isomorphic* coverings if for any choice of maximal trees T_1, T_2 and T_3 in Y_1, Y_2 and Y_3 respectively, there exists an isomorphism $\lambda'' : G(Y_2) \to G(Y_3)$ of complexes of groups such that the following diagram of induced isomorphisms of universal covers commutes

$$\begin{tikzcd} D(G(Y_1),T_1) \arrow[r, "L^{\lambda}_{T_1,T_2}"] \arrow[dr, "L^{\lambda'}_{T_1,T_3}"'] & D(G(Y_2),T_2) \arrow[d, "L^{\lambda''}_{T_2,T_3}"] \\ & D(G(Y_3),T_3). \end{tikzcd}$$

4. Counting overlattices

In this section we use Theorem 3.1 to establish upper and lower bounds on $u_\Gamma(n)$, the number of overlattices of Γ of index n.

4.1. **Upper bound.** We now prove the upper bound of Theorem 1.1, stated in the introduction. We will use the following deep results of finite group theory. Suppose G is a group of order m. Let $m = \prod_{i=1}^{t} p_i^{k_i}$ be the prime decomposition of m and let $\mu(m) = \max\{k_i\}$. We denote by $d(G)$ the minimum cardinality of a generating set for G, and by $f(m)$ the number of isomorphism classes of groups of order m. By results of Lucchini [Luc], Guralnick [G] and Sims [Si],

$$d(G) \leq \mu(m) + 1$$

and by work of Pyber [P] and Sims [Si], we obtain

$$f(m) \leq m^{\frac{2}{27}\mu(m)^2 + \frac{1}{2}\mu^{5/3}(m) + 75\mu(m) + 16}.$$

Let $g(m) = \frac{2}{27}\mu(m)^2 + \frac{1}{2}\mu^{5/3}(m) + 75\mu(m) + 16$, so that $f(m) \leq m^{g(m)}$.

Now let X and Γ be as in the statement of Theorem 1.1. Fix a quotient complex of groups $G(Y)$ for the action of Γ on X. From the definition of covering, if a covering $G(Y) \to H(Z)$ is defined over a nondegenerate simplicial map $l : Y' \to Z'$ then l must be onto. Since $Y = \Gamma\backslash X$ is finite, there exist only finitely many polyhedral complexes Z such that a covering (of any number of sheets) from $G(Y)$ to a complex of groups over Z may be defined. Thus it is enough to show the upper bound for the number of overlattices with a fixed quotient Z. We count the n–sheeted coverings of complexes of groups $\lambda : G(Y) \to H(Z) = (H_\tau, \psi_{a'}, g_{a',b'})$ over morphisms $l : Y' \to Z'$, where Z is fixed. Note that we do not insist on the complex of groups $H(Z)$ being faithful or developable.

For $\sigma \in V(Y')$, let $c_\sigma = |G_\sigma|$, and for $\tau \in V(Z')$ let

$$c_\tau = \left(\sum_{\sigma \in f^{-1}(\tau)} c_\sigma^{-1} \right)^{-1}.$$

By the definition of an n–sheeted covering, the cardinality $|H_\tau|$ is equal to nc_τ.

Let $c_0 = |V(Y')| \geq |V(Z')|$ and $c_1 = |E(Y')| \geq |E(Z')|$. Let us first count the number of possible complexes of groups $H(Z)$. There are at most $\prod_{\tau \in V(Z')}(c_\tau n)^{g(c_\tau n)}$ isomorphism classes of groups H_τ. There are at most $\prod_{b \in E(Z')}(c_{t(b)} n)^{\mu(c_{i(b)} n)+1}$ monomorphisms $\psi_b : H_{i(b)} \to H_{t(b)}$ determined by the images of generators of $H_{i(b)}$, and at most $\prod_{a \in E(Z')}(c_{t(a)} n)^{c_1}$ twisting elements $g_{a',b'}$. Now for a given complex of groups $H(Z)$, we count the number of possible coverings determined by local maps λ_σ and elements $\lambda(a)$. There are at most $\prod_{\sigma \in V(Y')}(c_{l(\sigma)} n)^{\mu(c_\sigma)+1}$ injections $\lambda_\sigma : G_\sigma \to H_{l(\sigma)}$, and at most $\prod_{a \in E(Y')} n c_{t(l(a))}$ choices for the $\lambda(a)$.

Let $M = \max\limits_{\sigma \in V(Y')} \max\{c_\sigma, c_{l(\sigma)}\}$ and $\mu = \mu(Mn)$. The number $u_\Gamma(n)$ is at most the product of the number of isomorphism classes of groups H_τ, the number of monomorphisms ψ_b, the number of twisting elements $g_{a',b'}$, the number of local maps λ_σ, and the number of elements $\lambda(a)$. Combining all of the estimates above, we get the following upper bound for $u_\Gamma(n)$:

$$
\begin{aligned}
u_\Gamma(n) \leq & \prod_{\tau \in V(Z')} (c_\tau n)^{g(c_\tau n)} \prod_{b \in E(Z')} (c_{t(b)} n)^{\mu(c_{i(b)} n)+1} \prod_{b \in E(Z')} (c_{t(b)} n)^{c_1} \\
& \prod_{\sigma \in V(Y')} (c_{l(\sigma)} n)^{\mu(c_\sigma)+1} \prod_{a \in E(Y')} n c_{t(l(a))} \\
\leq & \prod_{\tau \in V(Z')} (Mn)^{g(Mn)} \prod_{b \in E(Z')} (Mn)^{\mu(Mn)+1+c_1} \prod_{\sigma \in V(Y')} (Mn)^{\mu(M)+1} \prod_{a \in E(Y')} nM \\
\leq & (Mn)^{c_0 g(Mn) + c_1(\mu(Mn)+c_1+1) + c_0(\mu(M)+1) + c_1} \leq (Mn)^{C_1 \mu(Mn)^2} \\
\leq & (C_0 n)^{C_1'(\log n)^2}
\end{aligned}
$$

where $C_1 = c_0(2/27 + 1/2 + 75 + 16 + c_0 + 1) + c_1(1 + c_1 + 1)$ and $C_1' = 2C_1/(\log 2)^2$. This completes the proof of Theorem 1.1.

Note that the leading term comes from the number of isomorphism classes of local groups H_τ and that more careful counting of other morphisms or twisting elements does not change the asymptotics of the upper bound we obtain.

4.2. **Lower bound for right-angled buildings.** In this section we establish a lower bound on the number of overlattices, for certain right-angled buildings. See Theorem 4.2 below for a precise statement.

We first define right-angled hyperbolic buildings. Let P be a compact, convex polyhedron in n–dimensional hyperbolic space $\mathbb{H}^n$, with all dihedral angles $\frac{\pi}{2}$. Let (W, I) be the right-angled Coxeter system generated by reflections in the $(n-1)$–dimensional faces of P. A *right-angled hyperbolic building* of type (W, I) is a polyhedral complex X with a maximal family of subcomplexes called *apartments*, each isometric to the tesselation of $\mathbb{H}^n$ by copies of P, which satisfies the usual axioms for a building.

The dimension of a right-angled hyperbolic building is at most 4, and this bound is sharp (see [PV], [JS]). However, given any right-angled Coxeter system (W, I), there exist right-angled buildings with apartments isometric to the Davis complex for (W, I); thus right-angled buildings may be constructed in arbitrarily high dimensions [JS].

We now define general right-angled buildings, introducing terms which will be needed in the proof of Theorem 4.2. Let (W, I) be any right-angled Coxeter system.

Let N be the finite nerve of (W, I) and let P' be the simplicial cone with vertex x_0 on the barycentric subdivision N'. We write $\mathcal{S}^f$ for the set of $J \subset I$ such that the subgroup W_J of W generated by s_j, $j \in J$ is finite. By convention, $W_\phi = 1$, so the empty set ϕ is in $\mathcal{S}^f$. There is then a one-to-one correspondence between the vertices of P' and the types $J \in \mathcal{S}^f$. For each $i \in I$, the vertex of P' of type $\{i\}$ will be called an *i–vertex*, and the union of the simplices of P' which contain the i–vertex but not the cone point x_0 will be called an *i–face*.

A *right-angled building* of type (W, I) is then a polyhedral complex X equipped with a maximal family of subcomplexes, called *apartments*. Each apartment is polyhedrally isometric to the Davis complex for (W, I), and the copies of P' in X are called *chambers*. The apartments and chambers of X satisfy the usual axioms for a Bruhat–Tits building.

Each vertex of a right-angled building X has a type $J \in \mathcal{S}^f$, induced by the types of P'. For $i \in I$, an *$\{i\}$–residue* of X is a connected subcomplex consisting of all chambers which meet a given i–face.

If the Coxeter system (W, I) is in fact generated by reflections in the faces of a right-angled hyperbolic polyhedron P, then P' may be identified with the barycentric subdivision of P. For example, Bourdon's building $I_{p,q}$ has P a regular right-angled hyperbolic p–gon, $p \geq 5$, and all $q_i = q \geq 2$. Each $\{i\}$–residue of $I_{p,q}$ consists of q copies of P, glued together along a common edge, which is the i–face.

The following result classifies right-angled buildings.

Theorem 4.1 (Proposition 1.2, [HP])**.** *Let (W, I) be a right-angled Coxeter system and $\{q_i\}_{i \in I}$ a family of positive integers ($q_i \geq 2$). Then there exists a unique (up to isometry) building X of type (W, I), such that for each $i \in I$, the $\{i\}$–residue of X has cardinality q_i.*

In the 2–dimensional case, this result is due to Bourdon [Bo]. According to [HP], Theorem 4.1 was proved by M. Globus, and was known also to M. Davis, T. Januszkiewicz and J. Świątkowski.

We now prove the following:

Theorem 4.2. *Let X be a right-angled building of type (W, I) with parameters $\{q_i\}_{i \in I}$ ($q_i \geq 2$). Assume that for some $i_1, i_2 \in I$, $i_1 \neq i_2$,*

1. *$q_{i_1} = q_{i_2} = 2p$ where p is prime; and*
2. *the i_1– and i_2–faces of the chamber P' are non-adjacent (equivalently, $m_{i_1,i_2} = \infty$ in the Coxeter system associated to X).*

Then there is a cocompact lattice Γ in $\mathrm{Aut}(X)$, acting without inversions, such that for $n = p^k$, and $k \geq 3$,

$$u_\Gamma(n) \geq n^{\frac{1}{2}(k-3)}.$$

Proof. Let $T = T_{2p}$ be the $2p$–regular tree. In [L], Lim constructed many isomorphism classes of coverings of faithful graphs of groups with universal cover T, of the form
If Γ is the cocompact lattice in $\mathrm{Aut}(T)$ associated to the left-hand graph of groups in Figure 1, this yields the lower bound $u_\Gamma(n) \geq n^{\frac{1}{2}(k-3)}$, for $n = p^k$ and $k \geq 3$. We now take the "double cover" of the graphs of groups in Figure 1 above to obtain coverings of faithful graphs of groups with universal cover T, of the form shown in Figure 2.

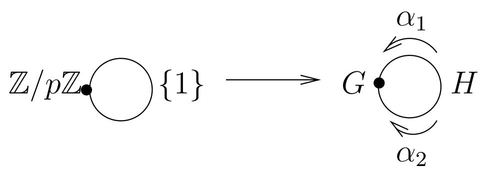

FIGURE 1. The tree coverings

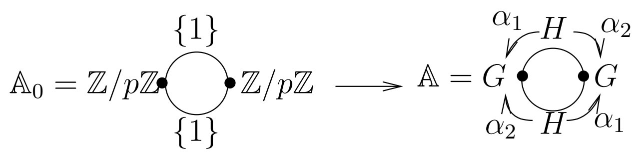

FIGURE 2. The subdivided tree coverings

We then carry out a special case of the Functor Theorem, [Th]. Let A be the graph with two edges underlying the graphs of groups in Figure 2. Let P'_1 and P'_2 be two copies of P'. Glue the i_1–face of P'_1 to the i_1–face of P'_2 in a type-preserving manner, and similarly with the i_2–faces, and let the resulting polyhedral complex be Y'. Each edge and each vertex of A may be identified to a unique vertex of Y'. Also, the vertices of Y' have types $J \subset I$ with W_J finite.

Let $\mathbb{A}_0$ and $\mathbb{A}$ be the graphs of groups in Figure 2. Then $\mathbb{A}$ induces a complex of groups $G(Y)$ over Y', as follows (the construction for $\mathbb{A}_0$ is similar). First fix the local groups induced by the identification of the graph A with some of the vertices of Y'. Each map from edge to vertex groups in $\mathbb{A}$ then induces a monomorphism ψ_a along an edge a of Y'. For each $i \in I$, let G_i be a group of order q_i.

Let J be a subset of I such that W_J is finite. If J does not contain i_1 or i_2, then we assign the local group at the vertices of Y' of type J to be

$$H \times \prod_{j \in J} G_j.$$

The monomorphisms between such local groups are natural inclusions. Now consider J containing one of i_1 and i_2 (since $m_{i_1,i_2} = \infty$, J cannot contain both i_1 and i_2). Without loss of generality suppose J contains i_1. Then the vertex of type J in Y' is contained in the glued i_1–face, and we assign the local group at the vertex of Y of type J to be

$$G \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j.$$

The monomorphism from G to this local group is inclusion onto the first factor. For each $J' \subset J$ with $i_1 \in J$, the monomorphism

$$G \times \prod_{\substack{j \in J' \\ j \neq i_1}} G_j \to G \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

is the natural inclusion. For each $J' \subset J$ with $i_1 \notin J$, the monomorphism

$$H \times \prod_{j \in J'} G_j \to G \times \prod_{\substack{j \in J \\ j \neq i_1}} G_j$$

is a monomorphism $H \to G$ from the graph of groups $\mathbb{A}$ on the first factor, and natural inclusions on the other factors. Put all $g_{a,b} = 1$ and we have a complex of groups $G(Y)$.

Let $G_0(Y)$ be the complex of groups induced in this way by $\mathbb{A}_0$. It is not hard to verify that $G_0(Y)$ has nonpositive curvature and is thus developable, and that its universal cover is the right-angled building X. Also, every covering as in Figure 2 induces a covering of the associated complexes of groups $G_0(Y) \to G(Y)$. By Lemma 3.2, since $G_0(Y)$ has nonpositive curvature, each $G(Y)$ is developable. Recall that the graphs of groups in [L] are faithful because there is no nontrivial subgroup of H whose images in G under α_1 and α_2 are the same. This condition clearly implies that there is no nontrivial N_T satisfying the conditions in Proposition 3.3, thus each $G(Y)$ is faithful. Moreover, in Lim's construction, the reason that distinct coverings in Figure 1 are non-isomorphic is that the corresponding local groups G and H are non-isomorphic groups. Hence the induced coverings of complexes of groups $G_0(Y) \to G(Y)$ are also non-isomorphic. By Theorem 3.1, this completes the proof. $\square$

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