INTERPRETING HASSON'S EXAMPLE

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ABSTRACT. We generalize Ziegler's fusion result [8] by relaxing the definability of degree requirement. As an application, we show that an example proposed by Assaf Hasson [3] has a rank and degree preserving interpretation in a strongly minimal set.

1. INTRODUCTION

Assaf Hasson [3] proved that any theory with finite Morley rank and the definable multiplicity property (DMP) has a rank and degree preserving interpretation in a strongly minimal set. He also proved a partial converse, showing that any theory admitting a rank preserving interpretation in a strongly minimal set has finite Morley rank and the weak definable multiplicity property (wDMP). As a test case, Hasson constructed an example with the wDMP but no rank-preserving expansion with the DMP.

Here we will prove that Hasson's example has a rank and degree preserving interpretation in a strongly minimal set. We will do this by relaxing the definable degree requirement in Ziegler's fusion [8] using an idea from [6]. Specifically, we will prove the following theorem.

Theorem 1.1. *Suppose T_1 and T_2 are countable complete theories with finite Morley rank, the same degree, disjoint languages, and nice codes. If K, v_1, v_2 are integers so that $K = v_1 RM(T_1) = v_2 RM(T_2)$ then there is a countable complete theory $T \supseteq T_1 \cup T_2$ with Morley rank K, nice codes, and*

$$RM_T(\phi(x;a)) = v_i RM_{T_i}(\phi(x;a)) \text{ and } dM_T(\phi(x;a)) = dM_{T_i}(\phi(x;a))$$

for all $\phi(x;y) \in L(T_i^{eq})$ and $i = 1, 2$.

We defer the definition of *nice codes* until later, when we discuss Hasson's example in more detail.

We should note that the amalgamation construction Hasson described in [6] does not quite work. For his purposes it does not matter, since he was able to obtain the same structure by alternative means in his thesis [4]. The proof of the theorem below contains the details required to repair his construction.

We are going to borrow most of the notation and conventions from Ziegler's paper [8].

2. CODES

Hrushovski's fusion machinery relies on a special notion of normal formula, called a *code*. In this section we will repeat the code construction of [2] and make a few minor adjustments.

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Recall that an ω-stable theory T has the *weak definable multiplicity property (wDMP)* if rank is definable and degree is uniformly bounded in T; i.e., for every formula $\phi(x;y) \in L(T^{eq})$ and consistent instance $\phi(x;a)$ there is a $\theta(y) \in \mathrm{tp}(a)$ and $D \in \mathbb{N}$ such that

$$\models \theta(a') \text{ implies } \mathrm{RM}(\phi(x;a')) = \mathrm{RM}(\phi(x;a)) \text{ and } \mathrm{dM}(\phi(x;a')) \leq D.$$

We fix a theory T with finite rank and the wDMP for the rest of this section.

A *code* c is a parameter-free formula $\phi_c(\mathbf{x};y)$ with the following properties.

(1) $\mathbf{x}$ is a tuple of real variables, $|\mathbf{x}| = n_c$, and $y \in T^{eq}$.

(2) Consistent $\phi_c(\mathbf{x};a)$ have rank k_c and degree at most D_c. If $\mathbf{b} \models \phi_c(\mathbf{x};a)$ then the elements of $\mathbf{b}$ are distinct and for each $S \subsetneq \{1,...,n_c\}$

$$\mathrm{RM}(\mathbf{b}/a\mathbf{b}_S) \leq k_{c,S}$$

with equality for generic $\mathbf{b}$. If a is generic in $\exists \mathbf{x} \phi_c(\mathbf{x};y)$ then $\phi_c(\mathbf{x};a)$ has degree 1. Lastly, $k_{c,\{i\}} < k_c$ for all i.

(3) If $\mathrm{RM}(\phi_c(\mathbf{x};a) \wedge \phi_c(\mathbf{x};a')) = k_c$ then $a = a'$.

(4) There is a $G_c \leq Sym(n_c)$ such that for each consistent $\phi_c(\mathbf{x};a)$ and $\sigma \in Sym(n_c)$,
   (a) $\sigma \in G_c$ implies $\phi_c(\mathbf{x};a) \equiv \phi_c(\mathbf{x}^\sigma;a)$.
   (b) $\sigma \notin G_c$ implies $\mathrm{RM}(\phi_c(\mathbf{x};a) \wedge \phi_c(\mathbf{x}^\sigma;a')) < k_c$ for all a'.

This definition of codes differs from the DMP case in one critical way. The degree of consistent instances $\phi_c(\mathbf{x};a)$ is not always 1. In fact, if $D_c = 1$, then the two definitions coincide.

A formula $\psi(\mathbf{x};d)$ is *simple* if it has degree 1, the components of its realizations are distinct, and the components of any generic realization lie outside $\mathrm{acl}(d)$. For any two formulas $\psi_1(\mathbf{x};d_1)$ and $\psi_2(\mathbf{x};d_2)$ with the same free variables, we write

$$\psi_1(\mathbf{x};d_1) \sim \psi_2(\mathbf{x};d_2)$$

when

$$\mathrm{RM}(\psi_1(\mathbf{x};d_1) \triangle \psi_2(\mathbf{x};d_2)) < \mathrm{RM}(\psi_1(\mathbf{x};d_1)) = \mathrm{RM}(\psi_2(\mathbf{x};d_2)).$$

If $\psi(\mathbf{x};d)$ is simple and $\phi_c(\mathbf{x};a) \sim \psi(\mathbf{x};d)$, then we say that c *encodes* $\psi(\mathbf{x};d)$. If $\psi(\mathbf{x};d)$ is simple and $\mathrm{RM}(\phi_c(\mathbf{x};a) \wedge \psi(\mathbf{x};d)) = k_c = \mathrm{RM}(\psi(\mathbf{x};d))$, then we say that c *covers* $\psi(\mathbf{x};d)$.

Lemma 2.1. *Every simple $\psi(\mathbf{x};d)$ is encoded by some code c.*

Proof. Let a be the canonical base of the global type isolated by $\psi(\mathbf{x};d)$ and let $\phi_c(\mathbf{x};y)$ be parameter-free so that $\phi_c(\mathbf{x};a) \sim \psi(\mathbf{x};d)$. We will strengthen $\phi_c(\mathbf{x};y)$ to meet the requirements above.

Let $\mathbf{b}$ be a generic realization of $\phi_c(\mathbf{x};a)$. Let $k_{c,S} = \mathrm{RM}(\mathbf{b}/a\mathbf{b}_S)$ for $S \subsetneq \{1,...,n_c\}$. Strengthening $\phi_c(\mathbf{x};y)$, we may assume

$$\mathrm{RM}(\phi_c(\mathbf{x};a) \wedge \mathbf{x}_S = \mathbf{b}_S) = k_{c,S}$$

for all S. Let $\theta(y)$ isolate $\mathrm{tp}(a)$ in its rank. Replace $\phi_c(\mathbf{x};y)$ with

$$\phi_c(\mathbf{x};y) \wedge \theta(y) \wedge \bigwedge_S \mathrm{RM}_{\mathbf{z}}(\phi_c(\mathbf{z};y) \wedge \mathbf{z}_S = \mathbf{x}_S) = k_{c,S}.$$

Now, the wDMP implies the existence of D_c, the choice of $\theta(y)$ forces $\phi_c(\mathbf{x};a')$ to have degree 1 for any a' generic in $\exists \mathbf{x} \phi_c(\mathbf{x};y)$, and $k_{c,\{i\}} < k_c$ follows from the simplicity of $\psi(\mathbf{x};d)$. Thus we have (2).

Let $p(y) = \mathrm{tp}(a)$ and note that since a is a canonical base,

$$p(y) \wedge p(y') \wedge \mathrm{RM}_{\mathbf{x}}(\phi_c(\mathbf{x}; y) \wedge \phi_c(\mathbf{x}; y')) = k_c \rightarrow y = y'.$$

By compactness there is some $\theta(y) \in p(y)$ which works in place of $p(y)$ above. If we replace $\phi_c(\mathbf{x}; y)$ with $\phi_c(\mathbf{x}; y) \wedge \theta(y)$ we get (3).

To achieve (4), first note that the collection of all $\sigma \in Sym(n_c)$ such that $\phi_c(\mathbf{x}; a) \sim \phi_c(\mathbf{x}^\sigma; a^\sigma)$ for some $a^\sigma \equiv a$ forms a subgroup $G_c \leq Sym(n_c)$. Replacing $\phi(\mathbf{x}; y)$ with

$$\bigwedge_{\sigma \in G_c} \phi_c(\mathbf{x}^\sigma; y) \wedge \mathrm{RM}_{\mathbf{x}} \left(\bigwedge_{\sigma \in G_c} \phi_c(\mathbf{x}^\sigma; y) \right) = k_c,$$

we have (4a). Since, for $\sigma \in Sym(n_c) \setminus G_c$,

$$p(y) \wedge p(y') \rightarrow \mathrm{RM}_{\mathbf{x}}(\phi(\mathbf{x}; y) \wedge \phi_c(\mathbf{x}^\sigma; y')) < k_c,$$

there is (by compactness) a $\theta(y) \in p(y)$ such that

$$\phi_c(\mathbf{x}; y) \wedge \theta(y)$$

satisfies (4b) as well. □

Lemma 2.2. *There exists a set of codes $\mathcal{C}$ such that*

(1) *Every simple formula is covered by a unique $c \in \mathcal{C}$.*
(2) *If $c \in \mathcal{C}$ and $\sigma \in Sym(n_c)$ there is a unique $c^\sigma \in \mathcal{C}$ with $\phi_c(\mathbf{x}^\sigma; y) \equiv \phi_{c^\sigma}(\mathbf{x}; y)$.*

Proof. We will build $\mathcal{C}$ as a limit of finite sets, starting with $\mathcal{C} = \emptyset$ and inductively maintaining (1)' and (2), where

(1)' Every simple formula is covered by at most one $c \in \mathcal{C}$.

Suppose $\psi(\mathbf{x}; d)$ is a simple formula not covered by some code in $\mathcal{C}$. Choose c which encodes $\psi(\mathbf{x}; d)$. Replace $\phi_c(\mathbf{x}; y)$ with

$$\phi_c(\mathbf{x}; y) \wedge \bigwedge_{c' \in \mathcal{C}'} \forall y' \ \mathrm{RM}_{\mathbf{x}}(\phi_{c'}(\mathbf{x}; y') \wedge \phi_c(\mathbf{x}; y)) < k_c,$$

where $\mathcal{C}' := \{c' \in \mathcal{C} : n_c = n_{c'} \text{ and } k_c = k_{c'}\}$, and note that this is still a code.

Choose representatives $\sigma_1, ..., \sigma_m$ of the right cosets of G_c and define, for $\sigma \in Sym(n_c)$, c^σ to be the code with $\phi_{c^\sigma}(\mathbf{x}; y) := \phi_c(\mathbf{x}^\sigma; y)$. Now $\mathcal{C} \cup \{c^{\sigma_1}, ..., c^{\sigma_m}\}$ satisfies (1)' and (2) and covers $\psi(\mathbf{x}; d)$. □

We call a collection of codes $\mathcal{C}$ satisfying the conclusion of the lemma above a *system of codes* for T.

Lemma 2.3. *For every code c there is a constant m_c and a $\emptyset$-definable partial function f_c so that if $\mathbf{b}_1, ..., \mathbf{b}_{m_c}$ are independent realizations of $\phi_c(\mathbf{x}; a)$, then $a = f_c(\mathbf{b}_1, ..., \mathbf{b}_{m_c})$.*

Proof. This is a standard stability fact. □

3. Free Fusion

Assumption 3.1. *The countable complete theories T_1 and T_2 have finite, definable Morley rank, degree 1, and quantifier elimination in relational languages L_1 and L_2. The languages are disjoint; i.e., $L_1 \cap L_2 = \emptyset$.*

In this section, we will describe the free fusion of T_1 and T_2 as laid out in [5,8]. Since those papers only required definable rank to develop their theories of the free fusion, we will not give proofs of things stated there.

Let K, v_1, v_2 be integers so that

$$K = v_1 \mathrm{RM}(T_1) = v_2 \mathrm{RM}(T_2).$$

For $A \subseteq B \models T_1^\forall \cup T_2^\forall$ with $B \setminus A$ finite, we define

$$\delta(B/A) := v_1 \mathrm{RM}_{T_1}(B/A) + v_2 \mathrm{RM}_{T_2}(B/A) - K|B \setminus A|.$$

Using δ, we define

$$\mathcal{K}_\infty := \{A \models T_1^\forall \cup T_2^\forall : \delta(B) \geq 0 \text{ for all finite } B \subseteq A\}.$$

Notation 3.2. *The letters A, B, C will always denote elements of $\mathcal{K}_\infty$.*

We say A is a *strong substructure* of B and write $A \leq_s B$ whenever $A \subseteq B$ and $\delta(A \cup C/A) \geq 0$ for all finite $C \subseteq B$.

Because rank is definable in T_i, it is also additive. It follows that δ is additive and submodular; i.e.,

$$\delta(C/A) = \delta(C/B) + \delta(B/A) \text{ whenever } A \subseteq B \subseteq C,$$

and

$$\delta(A/A \cap B) \geq \delta(A \cup B/B) \text{ whenever } A, B \subseteq C.$$

Many interesting properties of $\leq_s$ follow from these two properties of δ. For example,

$$A \leq_s B \leq_s C \text{ implies } A \leq_s C,$$

and

$$A, B \leq_s C \text{ implies } A \cap B \leq_s C.$$

In turn, these two properties of $\leq_s$ suffice to prove

$$\mathrm{cl}_B(A) := \bigcap \{A' \leq_s B : A' \supseteq A\} \leq_s B$$

is monotone, finite-character, and continuous as a function on the subsets of some fixed B.

We say that $A \leq_s B$ is *minimal* if there is no C with $A \lneq_s C \lneq_s B$.

Lemma 3.3. *If $A \leq_s B$ is minimal, then $B \setminus A$ is finite and one of the following holds.*

(1) *$A \leq_s B$ is* algebraic*: $\delta(B/A) = 0$, $B = A \cup \{b\}$, and for some $i = 1, 2$, $tp_{T_i}(b/A)$ is algebraic and $tp_{T_{2-i}}(b/A)$ generic.*
(2) *$A \leq_s B$ is* prealgebraic*: $\delta(B/A) = 0$ and $tp_{T_i}(b/A)$ is not algebraic for any $b \in B \setminus A$ and $i = 1, 2$.*
(3) *$A \leq_s B$ is* transcendental*: $N \geq \delta(B/A) > 0$ and $tp_{T_i}(b/A)$ is not algebraic for any $b \in B \setminus A$ and $i = 1, 2$.*

We will to need a finer version of closure. We write $A \leq_{s,m} B$ if $A \subseteq B$ and $\delta(A \cup C/A) \geq 0$ for all $C \subseteq B$ with $|C| < m$.

Lemma 3.4. *If $A \subseteq B$, then there is a $cl_{B,m}(A) \leq_{s,m} B$ such that $A \subseteq cl_{B,m}(A)$ and $cl_{B,m} \subseteq C$ whenever $A \subseteq C \leq_{s,m} B$.*

Proof. Call $A' \subseteq A''$ an *m-step* if $|A'' \setminus A| < m$, $\delta(A''/A') < 0$, and $\delta(A^*/A') \geq 0$ whenever $A' \subseteq A^* \subsetneq A''$. Choose some maximal chain $A = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n$ of m-steps. Set $\mathrm{cl}_{B,m}(A) := A_n$ and note that $\mathrm{cl}_{B,m} \leq_{s,m} B$.

Now, suppose $A \subseteq C \leq_{s,m} B$ and $\mathrm{cl}_{B,m} \nsubseteq C$. Let $i < n$ be least so that $A_{i+1} \nsubseteq C$. Then $0 > \delta(A_{i+1}/C \cap A_i) \geq \delta(A_{i+1} \cup C/C)$, which contradicts our assumption that $C \leq_{s,m} B$. □

For convenience, we extend δ and $\leq_{s,m}$ to imaginary elements. Define

$$\mathrm{acl}^{eq}(A) := \mathrm{acl}^{eq}{}_{T_1}(A) \times \mathrm{acl}^{eq}{}_{T_2}(A)$$

and include $A \subseteq \mathrm{acl}^{eq}(A)$ via $a \mapsto (a, a)$. If Σ is the home sort shared by T_1 and T_2 then for $X \subseteq Y \subseteq \mathrm{acl}^{eq}(C)$ define

$$\delta(Y/X) := v_1 \mathrm{RM}_{T_1}(\pi_1(Y)/\pi_1(X)) + v_2 \mathrm{RM}_{T_2}(\pi_2(Y)/\pi_2(X)) - N|(Y \setminus X) \cap \Sigma|.$$

For $A \subseteq B$ and $X \subseteq \mathrm{acl}^{eq}(B)$, write $X \leq_{s,m} A$ if $X \cap \Sigma \subseteq A$ and $\delta(X \cup C/X) \geq 0$ whenever $C \subseteq X$ and $|C| < m$.

Lemma 3.5. *If $A \subseteq B$ and $X \subseteq acl^{eq}(B)$, then there is a $cl_{A,m}(X) \subseteq A$ such that $X \cup cl_{A,m}(X) \leq_{s,m} A$ and $cl_{A,m}(X) \subseteq C$ whenever $C \subseteq A$ and $X \cup C \leq_{s,m} A$.*

Proof. Like the previous lemma. □

4. Prealgebraic Codes

Fix a system of codes $\mathcal{C}_i$ for each T_i. A *prealgebraic code* is a pair $c = (c_1, c_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ so that

- $n_{c_1} = n_{c_2}$,
- $v_1 k_{c_1} + v_2 k_{c_2} - K n_c = 0$,
- and $v_1 k_{c_1,S} + v_2 k_{c_2,S} - K(n_c - |S|) < 0$ for $\emptyset \subsetneq S \subsetneq \{1, ..., n_c\}$.

To each prealgebraic code c we associate the additional data

- $n_c := n_{c_1} (= n_{c_2})$.
- $\phi_c(\mathbf{x}; y) := \phi_{c_1}(\mathbf{x}; y_1) \wedge \phi_{c_2}(\mathbf{x}; y_2)$,
- $D_c := D_{c_1} \cdot D_{c_2}$,
- and $G_c := G_{c_1} \cap G_{c_2}$.

We say a prealgebraic code instance $\phi_c(\mathbf{x}; a)$ is *over* A if $a \in \mathrm{acl}^{eq}(A)$; i.e., if $a = (a_1, a_2) \in \mathrm{acl}^{eq}_{T_1}(A) \times \mathrm{acl}^{eq}_{T_2}(A)$.

Suppose $\phi_c(\mathbf{x}; a)$ is over A and $B, \mathbf{b} \subseteq A$. We say that $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ is a B-generic if $\mathrm{RM}_{T_i}(\mathbf{b}/Ba_i) = k_{c_i}$ for $i = 1, 2$. Thus a sequence of realizations $\mathbf{b}_1, ..., \mathbf{b}_N$ of $\phi_c(\mathbf{x}; a)$ is independent if and only if it is independent over a_i in each T_i.

Lemma 4.1. *If $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic there is a unique prealgebraic code c and parameter $a \in acl^{eq}(A)$ such that $\mathbf{b}$ is an A-generic realization of $\phi_c(\mathbf{x}; a)$.*

On the other hand, if $\mathbf{b} \nsubseteq A$, $a \in acl^{eq}(A)$, and $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ then $\delta(\mathbf{b}/A) \leq 0$. Moreover $\delta(\mathbf{b}/A) = 0$ if and only if $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic if and only if $\mathbf{b}$ is an A-generic realization of $\phi_c(\mathbf{x}; a)$.

Proof. This is proved in [2], but we include a proof here because it helps explain the purpose of prealgebraic codes.

Suppose $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic. Since $\mathrm{tp}_{T_i}(\mathbf{b}/A)$ is not algebraic, there is a simple $\psi_i(\mathbf{x}; d_i) \in L_i$ such that $d_i \in \mathrm{acl}^{eq}{}_{T_i}(A)$ and $\mathbf{b}$ is an A generic realization of $\psi_i(\mathbf{x}; d_i)$. Now choose $c_i \in \mathcal{C}_i$ and $a_i \in \mathrm{acl}^{eq}{}_{T_i}(A)$ such that

$$\mathrm{RM}_{T_i}(\psi_i(\mathbf{x}; d_i) \wedge \phi_{c_i}(\mathbf{x}; a)) = \mathrm{RM}_{T_i}(\psi_i(\mathbf{x}; d_i)) = k_c.$$

Because $A \leq_s A \cup \{\mathbf{b}\}$ is prealgebraic, $\delta(\mathbf{b}/A) = 0$ and $\delta(\mathbf{b}/A\mathbf{b}_s) < 0$ whenever $\emptyset \subsetneq S \subsetneq \{1, ..., n_c\}$. It follows that $v_1 k_{c_1} + v_2 k_{c_2} - K n_c = 0$ and $v_1 k_{c_1,S} + v_2 k_{c_2,S} - K(n_c - |S|) < 0$ whenever $\emptyset \subsetneq S \subsetneq \{1, ..., n_c\}$. Thus $c = (c_1, c_2)$ is a prealgebraic code and $\mathbf{b}$ is an A-generic realization of $\phi_c(\mathbf{x}; a)$ where $a = (a_1, a_2) \in \mathrm{acl}^{eq}(A)$.

For the second part, note that if $A \cap \{\mathbf{b}\} \neq \emptyset$, then $\delta(\mathbf{b}/A) \leq v_1 k_{c_1,S} + v_2 k_{c_2,S} - K(n_c - |S|) < 0$, where $S = \{i \mid b_i \in A\}$. Furthermore, if $A \cap \{\mathbf{b}\} = \emptyset$, then $\delta(\mathbf{b}/A) \leq v_1 k_{c_1} + v_2 k_{c_2} - K n_c = 0$. $\square$

Lemma 4.2. *For each prealgebraic code c we can find an integer $m_c \geq n_c$ so that if $A \leq_{s,m_c} B$, $a \in acl^{eq}(B)$, and $a \notin dcl^{eq}(A)$, then fewer than m_c distinct realizations of $\phi_c(\mathbf{x}; a)$ intersect A. Moreover, for any distinct $\mathbf{b}_1, .., \mathbf{b}_{m_c}$ there is at most one parameter a such that $\mathbf{b}_i \models \phi_c(\mathbf{x}; a)$ for all $i \leq m_c$.*

Proof. It suffices to prove the lemma for set-wise distinct realizations.

Suppose $\mathbf{b}_1, ..., \mathbf{b}_m \models \phi_c(\mathbf{x}; a)$ and $\mathbf{b}_i \nsubseteq \bigcup_{j<i} \mathbf{b}_j$ for all $i < m$. By the additivity of δ,

$$\delta(\mathbf{b}_1 ... \mathbf{b}_m) \leq \delta(a) + \sum_{i \leq m} \delta(\mathbf{b}_i / a\mathbf{b}_1 ... \mathbf{b}_{i-1}).$$

By Lemma 4.1, $\mathbf{b}_i$ is a non-generic realization of $\phi_c(\mathbf{x}; a)$ over $a\mathbf{b}_1 ... \mathbf{b}_{i-1}$ if and only if $\delta(\mathbf{b}_i / a\mathbf{b}_1 ... \mathbf{b}_{i-1}) < 0$. Since $\delta(\mathbf{b}_1 ... \mathbf{b}_N) \geq 0$, $\mathbf{b}_i$ must be $a\mathbf{b}_1 ... \mathbf{b}_{i-1}$-generic for all but at most $\delta(a)$ of the $i < m$. Moreover, $\delta(a)$ is bounded uniformly in c.

The above paragraph shows that given a sufficiently long sequences $\mathbf{b}_1, ..., \mathbf{b}_m$ of set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$, more than half of the length m_{c_i} ($i = 1, 2$) subsequences are independent. Thus given a sufficiently long sequence, a_i is the consensus value of f_{c_i} on the length m_{c_i} subsequences. Hence a is uniquely determined.

Suppose $A \leq_{s,m_c} B$, $a \in \mathrm{acl}^{eq}(B)$, and $a \notin \mathrm{dcl}(A)$. Since $|\mathrm{cl}_{B,2n_c}(a)| < 2n_c\delta(a)$ there is a finite bound M_c on the number of $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ with $\mathbf{b} \subseteq A$ or $\mathbf{b} \subseteq \mathrm{cl}_{B,2n_c}(a)$. By Lemma 4.1, any two set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$ which are not contained in $\mathrm{cl}_{B,2n_c}(a)$ are disjoint. Thus if $\mathbf{b}_1, ..., \mathbf{b}_m$ are set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$ with $\mathbf{b}_i \cap A \neq \emptyset$, then

$$0 \leq \delta(\mathbf{b}_1 ... \mathbf{b}_k a/A) \leq \delta(a/A) - (m - M_c).$$

Thus we can increase m to the desired m_c. $\square$

We say that a prealgebraic code instance $\phi_c(\mathbf{x}; a)$ is *strongly based* on a set A if A contains at least m_c distinct realizations of $\phi_c(\mathbf{x}; a)$.

Choose an injective function $c \mapsto s_c$ on the prealgebraic codes such that

$$s_c > (m_c n_c + 1)! + 2m_c\delta(a)$$

for all consistent $\phi_c(\mathbf{x}; a)$.

We say a prealgebraic code instance $\phi_c(\mathbf{x}; a)$ over A is *long in A* if and there are more than s_c distinct realizations of $\phi_c(\mathbf{x}; a)$ in A. If $\mathbf{b}_1, ..., \mathbf{b}_N$ are distinct realizations of some $\phi_c(\mathbf{x}; a)$ and $N > s_c$, then we say that $\{\mathbf{b}_i\}$ is a *long sequence in $\phi_c(\mathbf{x}; a)$*.

Lemma 4.3. *(Decomposition) Suppose $A \leq_s B$ and $B \setminus A$ is finite. We can find*

$$A \leq_s X \subsetneq B$$

such that if

$$Z := \{\mathbf{b} \subseteq B \mid \mathbf{b} \nsubseteq X \text{ is an element of a long sequence strongly based on } X\},$$

then

(1) $\delta(\mathbf{b}\mathbf{b}'/X) = 0$ *for all* $\mathbf{b}, \mathbf{b}' \in Z$.
(2) *For every long* $\phi_c(\mathbf{x}; a)$ *either*
  (a) $\phi_c(\mathbf{x}; a)$ *is strongly based on* X *and* $cl_{B,m_c}(a) \subseteq X$,
  or (b) *there is a* $\mathbf{b} \in Z$ *such that* $X \cup \{\mathbf{b}\}$ *contains every realization of* $\phi_c(\mathbf{x}; a)$.

Proof. We will build X in stages starting with $X = A$ and inductively maintaining the following conditions.

- $\delta(\mathbf{b}\mathbf{b}'/X) = 0$ for all $\mathbf{b}, \mathbf{b}' \in Z$.
- If (2) fails for $\phi_c(\mathbf{x}; a)$, then
  - $X \leq_{s,m_c} B$,
  - $X \cup \{\mathbf{b}\} \leq_{s,m_c} B$ for all $\mathbf{b} \in Z$,
  - and $||Z|| > 2m_c\delta(X/A)$ where $||Z||$ is the number of set-wise distinct elements in Z.

Choose a $\phi_c(\mathbf{x}; a)$ that witnesses the failure of (2). Since $X \leq_{s,m_c} B$, it can not be the case that $\phi_c(\mathbf{x}; a)$ is strongly based on X. In fact, fewer than m_c realizations of $\phi_c(\mathbf{x}; a)$ intersect X by Lemma 4.2. Since $c \mapsto s_c$ is injective, we may choose $\phi_c(\mathbf{x}; a)$ which maximizes m_c.

If there is a $\mathbf{b} \in Z$ with $\phi_c(\mathbf{x}; a)$ is strongly based on $X \cup \{\mathbf{b}\}$, then set $\tilde{X} := X \cup \{\mathbf{b}\}$. Otherwise, choose $\mathbf{b}_1, ..., \mathbf{b}_{m_c} \models \phi_c(\mathbf{x}; a)$ and set $\tilde{X} := X \cup \bigcup_i \{\mathbf{b}_i\}$. By the proof of Lemma 4.2, we can select the $\mathbf{b}_i$ which include all the realizations of $\phi_c(\mathbf{x}; a)$ which intersect X. Moreover, we can select the $\mathbf{b}_i$ such that set-wise distinct realizations of $\phi_c(\mathbf{x}; a)$ not contained in $\tilde{X}$ are pairwise disjoint.

Define

$$\tilde{Y} := \{\mathbf{b} \in \tilde{Z} \mid \mathbf{b} \in Z \text{ or } \mathbf{b} \models \phi_c(\mathbf{x}; a)\}$$

and note that $||\tilde{Y}|| > 2m_c\delta(\tilde{X}/A)$, because $s_c > (m_c n_c + 1)! + 2m_c\delta(a)$.

Now, close $\tilde{X}$ under the following three operations.

- If $\tilde{X} \not\leq_{s,m_c} B$ then set $\tilde{X} := \mathrm{cl}_{B,m_c}(\tilde{X})$.
- If $\tilde{X} \cup \{\mathbf{b}\} \not\leq_{s,m_c} B$ for some $\mathbf{b} \in \tilde{Z}$ then set $\tilde{X} := \mathrm{cl}_{B,m_c}(\tilde{X} \cup \{\mathbf{b}\})$.
- If there are $\mathbf{b}, \mathbf{b}' \in \tilde{Z}$ with $\delta(\mathbf{b}\mathbf{b}'/X) < 0$ then set $\tilde{X} := \tilde{X} \cup \{\mathbf{b}, \mathbf{b}'\}$.

By the maximality of m_c and induction, each closure step reduces $||\tilde{Y}||$ by at most $2m_c$ and reduces $\delta(\tilde{X}/A)$ by at least 1. It follows that after closing, we have

$$||\tilde{Z}|| \geq ||\tilde{Y}|| > 2m_c\delta(\tilde{X}/A)$$

and the rest of the induction hypothesis. Moreover, $\phi_c(\mathbf{x}; a)$ no longer witnesses the failure of (2).

Iteration of this process must stop because $B \setminus A$ is finite. Once finished, (1) and (2) must hold and $||Z|| > 0$ implies $X \subsetneq B$. □

5. Weak Closure

Given prealgebraic code instance $\phi_c(\mathbf{x};a)$ over some A, we are going to define a first-order approximation $\mathrm{wcl}_A(\phi_c(\mathbf{x};a)) \subseteq \mathrm{cl}_A(a)$.

For each prealgebraic code c, define

$$\Phi_c(\mathbf{x}_1, ..., \mathbf{x}_{m_c+1}) := \bigwedge_{i<j} \mathbf{x}_i \neq \mathbf{x}_j \wedge \bigwedge_i \phi_c(\mathbf{x}_i; y),$$

and

$$\Gamma_c := \{\Phi_{c'} : s_c > s_{c'}\}.$$

Lemma 5.1. *We may assume that if $\phi_c(\mathbf{x};a)$ is over A and $\mathbf{b}, \mathbf{b}' \models \phi_c(\mathbf{x};a)$ are A-generic, then $qftp_{\Gamma_c}(\mathbf{b}/A) = qftp_{\Gamma_c}(\mathbf{b}'/A)$.*

Proof. The easiest way to obtain this is to redo the code constructions in each T_i. Make sure that the lemma is true in T_i for $\Gamma_{c_i} := \{\Phi_{c_i'} : n_{c_i} > m_{c_i'} \cdot n_{c_i'}\}$. Now, since $s_c > s_{c'}$ implies $n_{c_i} > m_{c_i'} \cdot n_{c_i'}$ for $i = 1, 2$, the lemma follows. ☐

Lemma 5.2. *For any prealgebraic code instance $\phi_c(\mathbf{x};a)$ over A, there is a unique minimal subset $W \subseteq A$ with the following properties.*

(1) *Suppose for some A-generic $\mathbf{b} \models \phi_c(\mathbf{x};a)$ there is a $\phi_{c'}(\mathbf{x}';a')$ with a long sequence in $\mathbf{b}$. If*

$$Y := \{\mathbf{b}' \subseteq A \cup \{\mathbf{b}\} \mid \mathbf{b}' \models \phi_{c'}(\mathbf{x}';a')\},$$

then $A \cap \bigcup Y \subseteq W$.

(2) *If $\mathbf{b} \subseteq A$, $\mathbf{b} \models \phi_c(\mathbf{x};a)$, and $qftp_{\Gamma_c}(\mathbf{b}/W)$ is not generic, then $\mathbf{b} \subseteq W$.*

Moreover, W is contained in $cl_{A,n_c}(a)$, and first-order definable.

Proof. First we show $\mathrm{cl}_{A,n_c}(a)$ satisfies (1) and (2).

Condition (2) is easy, because if $\mathrm{qftp}_{\Gamma_c}(\mathbf{b}/\mathrm{cl}_{A,n_c}(a))$ fails to be generic, then $\delta(\mathbf{b}/\mathrm{cl}_{A,n_c}(a)) < 0$. This contradicts the assumption $\mathrm{cl}_{A,n_c}(a) \leq_{s,n_c} A$.

For condition (1), suppose $\mathbf{b} \models \phi_c(\mathbf{x};a)$, $\phi_{c'}(\mathbf{x}';a')$ is long in $\mathbf{b}$, $\mathbf{b}' \subseteq A \cup \{\mathbf{b}\}$, $\mathbf{b}' \not\subseteq \mathbf{b}$, and $\mathbf{b}' \models \phi_{c'}(\mathbf{x}';a')$. Since $A \cap \{\mathbf{b}'\} \downarrow_a^{T_i} a'$ and $a' \notin \mathrm{acl}^{eq}(a)$, we have $\mathbf{b}' \subseteq \mathrm{cl}_{A,n_c}(a)$ by Lemma 4.1.

The class of sets satisfying (1) and (2) is closed under intersection. Thus uniqueness and containment in $\mathrm{cl}_{A,n_c}(a)$ follows from the fact that $\mathrm{cl}_{A,n_c}(a)$ is finite (recall $|\mathrm{cl}_{A,n_c}(a)| < n_c\delta(a)$).

Since checking condition (1) and (2) is first-order for a set of fixed size and we have a bound on the size of W, W is first-order definable. ☐

With W as in the lemma above, we define

$$\mathrm{wcl}_A(\phi_c(\mathbf{x};a)) := W,$$

and call it the *weak closure of $\phi_c(\mathbf{x};a)$ in A.*

Lemma 5.3. *If $\phi_c(\mathbf{x};a)$ is over A, $\mathbf{b} \models \phi_c(\mathbf{x};a)$ is A-generic and $\phi_{c'}(\mathbf{x}';a')$ is long in $\mathbf{b}$, then $wcl_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}';a')) \subseteq wcl_A(\phi_c(\mathbf{x};a)) \cup \{\mathbf{b}\}$.*

Proof. Note that by Lemma 5.1, we can restrict condition (1) above to a single generic realization.

Because $\phi_{c'}(\mathbf{x}';a')$ is long in $\mathbf{b}$, there is a $\mathbf{b}' \subseteq \mathbf{b}$ such that $\mathbf{b}' \models \phi_{c'}(\mathbf{x}';a')$ is $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}';a'))$-generic. Since $\Gamma_{c'} \subseteq \Gamma_c$, $\mathrm{wcl}_A(\phi_c(\mathbf{x};a)) \cup \{\mathbf{b}\}$ satisfies conditions (1) and (2) for $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}';a'))$. ☐

6. Nice Codes

In this section, we temporarily move back to the context of a single theory T with the wDMP. We need to make additional assumptions about the codes in T in order to progress further. We find these assumptions by looking more closely at our intended application.

Hasson's example [3] is rank and degree preserving biinterpretable with a theory T that has an equivalence relation E such that:

(1) T/E is strongly minimal with the DMP.
(2) The structure of each E-class has rank 1, degree $\leq D$, and the DMP,
(3) Distinct E-classes are orthogonal.
(4) Generic E-classes are pure sets.

For the rest of this section, fix such a theory T. We write $[a]$ for the equivalence class coded by an imaginary $a \in T/E$. Thus, we write $Th([a])$ for the induced structure on the equivalence class a represents. We assume $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$.

Let $\{a_n\}$ enumerate $\mathrm{dcl}^{eq}(\emptyset) \cap (T/E)$. For each n let $d_n := \mathrm{dM}([a_n])$ and add predicates $\{P_{n,k} : k \leq d_n\}$ which partition $[a_n]$ into strongly minimal sets.

Lemma 6.1. *There is a system of codes $\mathcal{C}$ with the following two properties.*

(1) *If $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ is generic, $b_i \in P_{n,k}$, and $\phi_c(\mathbf{x}; a) \not\models P_{n,k}(x_i)$, then $\phi_c(\mathbf{x}; a) \models \bigvee_{j \leq d_n} P_{n,j}(x_i)$ and for any $j \leq d_n$ we can change b_i so that $b_i \in P_{n,j}$ while maintaining $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ generic.*
(2) *If $\psi(\mathbf{x}; d)$ is simple and covered by c, there is a parameter a and a conjuction $\theta(\mathbf{x})$ of atoms $P_{n,k}(x_i)$ such that $\psi(\mathbf{x}; d) \sim \phi_c(\mathbf{x}; a) \wedge \theta(\mathbf{x})$.*

Proof. Suppose we are building a code for the simple formula $\psi(\mathbf{x}; d)$. Since $\psi(\mathbf{x}; d)$ is simple, we may assume it implies a complete atomic E-type $\xi(\mathbf{x})$. Let $S_1 \cup \cdots \cup S_m = \{1, ..., |\mathbf{x}|\}$ be a partition such that $\xi(\mathbf{x})$ implies $x_i E x_j$ if and only if $i, j \in S_k$ for some k. By the orthogonality condition (3),

$$\psi(\mathbf{x}; d) \sim \bigwedge_k \exists \mathbf{x}_{\{1,\ldots,|x|\} \setminus S_k} \psi(\mathbf{x}; d).$$

If we choose codes c_k which encode $\exists \mathbf{x}_{\{1,\ldots,|x|\} \setminus S_k} \psi(\mathbf{x}; d)$, then

$$\phi_c(\mathbf{x}; y) := \xi(\mathbf{x}) \wedge \bigwedge_k \phi_{c_k}(\mathbf{x}_{S_k}; y_k)$$

is a code which encodes $\psi(\mathbf{x}; d)$. Thus we may assume $\psi(\mathbf{x}; d) \to \bigwedge_{i<j} x_i E x_j$.

Case 1: If b_1/E is generic over d for generic $\mathbf{b} \models \psi(\mathbf{x}; d)$, then, since generic E-classes are pure sets, we must have $\psi(\mathbf{x}; d) \sim \bigwedge_{i<j} x_i E x_j$. In this case, $\phi_c(\mathbf{x}) := \bigwedge_{i<j} x_i E x_j \wedge x_i \neq x_j$ is a code which encodes $\psi(\mathbf{x}; d)$. Since $\phi_c(\mathbf{x})$ has degree 1, properties (1) and (2) are trivial.

Case 2: If $b_1/E \in \mathrm{acl}(d)$ for generic $\mathbf{b} \models \psi(\mathbf{x}; d)$, then we can strengthen $\psi(\mathbf{x}; d)$ such that $\psi(\mathbf{x}; d) \to \mathbf{x} \subseteq [a]$ for some $a \in (T/E) \cap \mathrm{acl}(d)$.

Case 2a: If $\mathrm{RM}(a) = 0$, then we may assume $a \in \mathrm{dcl}(\emptyset)$ and choose a $Th([a])$-code $\phi_c(\mathbf{x}; y)$ which encodes $\psi(\mathbf{x}; d)$. Since $Th([a])$ has the DMP, all instances of ϕ_c have degree 1. Thus (1) and (2) are again trivial.

Case 2b: If $\mathrm{RM}(a) = 1$, then $[a]$ is a pure set and $\psi(\mathbf{x}; d) \sim \mathbf{x} \subseteq [a]$. Thus the code $\phi_c(\mathbf{x}; y) \equiv \mathbf{x} \subseteq [y] \wedge \bigwedge_{i<j} x_i \neq x_j$ works. Note that $\mathrm{dM}(\phi_c(\mathbf{x}; a)) = \mathrm{dM}([a])^{n_c}$.

In particular, $\phi_c(\mathbf{x}; a_n)$ is partitioned into $(d_n)^{n_c}$ degree 1 sets by the formulas

$$\{\phi_c(\mathbf{x}; a_n) \wedge \bigwedge_{i \le n_c} P_{n,k_i}(x_i) : \mathbf{k} \in \{1, ..., d_n\}^{n_c}\}.$$

From this (1) and (2) follow. □

If $\mathcal{C}$ is a system of codes and there are disjoint predicates $\{P_{n,k} \mid k \le d_n\}$ which make the above lemma true, we say that $\mathcal{C}$ is a *nice system of codes.* Note that any system of codes for a DMP theory is nice via $d_n = 1$ and $P_{n,1} = \emptyset$.

Suppose $\mathcal{C}$ is a nice system of codes. Write Σ_n for the set of complete $\{P_{m,k} : m < n, k \le d_n\}$-formulas. Given a code $c \in \mathcal{C}$ and $\theta(\mathbf{x}) \in \Sigma_n$ with $|\mathbf{x}| = n_c$, let $c \wedge \theta$ be the code with

$$\phi_{c\wedge\theta}(\mathbf{x}; y) \equiv \phi_c(\mathbf{x}; y) \wedge \theta(\mathbf{x}) \wedge \mathrm{RM}_{\mathbf{x}}(\phi_c(\mathbf{x}; y) \wedge \theta(\mathbf{x})) = k_c.$$

We will call $c \wedge \theta$ a Σ_n*-specialization* of c. Note that by Lemma 6.1, $c \wedge \theta \in \mathcal{C}$ if and only if $\phi_c(\mathbf{x}; y) \models \theta(\mathbf{x})$ already.

7. The Class $\mathcal{K}_\mu$

Assumption 7.1. *Each theory T_i has a nice system of code $\mathcal{C}_i$ via the predicates $\{P^i_{n,k} : n \in \mathbb{N}$ and $k \le d^i_n\}$.*

We write $\Sigma_n := \Sigma^1_n \times \Sigma^2_n$. For a prealgebraic code c and a $\theta \in \Sigma_n$, write $c \wedge \theta$ for the Σ_n*-specialized prealgebraic code* $(c_1 \wedge \theta_1, c_2 \wedge \theta_2)$. Note that specializations $c \wedge \theta$ still code prealgebraic extensions in the sense of Lemma 4.1.

We are going to define a class $\mathcal{K}_\mu \subseteq \mathcal{K}_\infty$ by saying that $A \in \mathcal{K}_\mu$ when

$$\dim_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) \le \mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$$

for all specialized prealgebraic codes $c \wedge \theta$ and $a \in \mathrm{acl}^{eq}(A)$. Of course, we still need to define $\dim_A$ and μ_A.

If $\phi_{c\wedge\theta}(\mathbf{x}; a))$ a specialized prealgebraic instance over A, then let $\dim_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ be the cardinality of the set

$$\{\mathbf{b} \subseteq A : \mathbf{b} \not\subseteq \mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \text{ and } \mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)\};$$

i.e., the number of realizations outside of the weak closure.

For unspecialized prealgebraic codes c, let

$$\mu_A(\phi_c(\mathbf{x}; a)) = (D_c!)^{D_c} \cdot (s_c + m_c + 1).$$

For Σ_n-specializations $c \wedge \theta$, we will simultaneously define $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ and first-order approximations $\mathcal{K}_{c,n} \subseteq \mathcal{K}_\infty$ to the final $\mathcal{K}_\mu$.

Suppose $c \wedge \theta$ is a Σ_n-specialization of c. We inductively assume μ_A has been defined for instances of specialized prealgebraic codes $c' \wedge \theta'$ whenever $s_{c'} < s_c$ or $\theta' \in \Sigma_{n-1}$. Using the induction hypothesis, let $\mathcal{K}_{c,n}$ be the class of all $A \in \mathcal{K}_\infty$ such that

$$\dim_A(\phi_{c'\wedge\theta'}(\mathbf{x}'; a')) \le \mu_A(\phi_{c'\wedge\theta'}(\mathbf{x}'; a'))$$

for $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ over A with $s_{c'} < s_c$ and $\theta' \in \Sigma_n$. If $A \in \mathcal{K}_{c,n}$ and $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is over A, we say that $\phi_{c\wedge\theta}(\mathbf{x}; a)$ *extendible over* A when there is an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ so that $A \cup \{\mathbf{b}\} \in \mathcal{K}_{c,n}$. For A-extendible $\phi_{c\wedge\theta}(\mathbf{x}; a)$ define

$$\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) := \mu_A(\phi_{c\wedge\theta^-}(\mathbf{x}; a))/D,$$

where $\theta^- \in \Sigma_{n-1}$, $\theta \to \theta^-$, and D is the number of $\theta' \in \Sigma_n$ with $\theta' \to \theta^-$ and $\phi_{c\wedge\theta'}(\mathbf{x}; a)$ extendible over A. For non-A-extendible $\phi_{c\wedge\theta}(\mathbf{x}; a)$ define

$$\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) := 0.$$

Lemma 7.2. *If $A \in \mathcal{K}_{c,n}$ and $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-extendible, then $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) > s_c + m_c$.*

Proof. The degree of any prealgebraic code instance $\phi_c(\mathbf{x}; a)$ is bounded by D_c. Thus each time we divide by D in the definition of μ_A, we have $D \leq D_c$. Moreover, we divide by a number greater than 1 at most D_c times. □

Lemma 7.3. *If $A \in \mathcal{K}_{c,n}$, $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is over A, and $\theta \in \Sigma_n$ then $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ depends only on $qftp_{\Sigma_n\cup\Gamma_c}(wcl_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\})$ for A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$.*

Proof. The quantifier-free type above is uniquely determined by Lemma 5.1.

Suppose $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-generic and $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ witnesses $A \cup \{\mathbf{b}\} \notin \mathcal{K}_{c,n}$. Note that all of the realizations of $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ are contained in $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$. By induction, we know that $\mu_{A\cup\{\mathbf{b}\}}(\phi_{c'\wedge\theta'}(\mathbf{x}'; a'))$ is completely determined by $\mathrm{qftp}_{\Sigma_n\cup\Gamma_c}(\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a')) \cup \{\mathbf{b}'\})$ for some (any) $A \cup \{\mathbf{b}\}$-generic $\mathbf{b}' \models \phi_{c'\wedge\theta'}(\mathbf{x}'; a')$.

Note that $\mathrm{wcl}_{A\cup\{\mathbf{b}\}}(\phi_{c'}(\mathbf{x}'; a')) \subseteq \mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$, every realization of $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ is contained in $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$, and $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\}$ computes the same value for $\mu_{c'\wedge\theta'}(\mathbf{x}'; a'))$ as $A \cup \{\mathbf{b}\}$. It follows that the failure $A \cup \{\mathbf{b}\} \notin \mathcal{K}_{c,n}$ is encoded in $\mathrm{qftp}_{\Sigma_n\cup\Gamma_c}(\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\})$ and that $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is not A-extendible.

Thus the A-extendibility of $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is encoded in $\mathrm{qftp}_{\Sigma_n\cup\Gamma_c}(\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\})$. Unrolling the definition of $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ we see that it too is encoded. □

Lemma 7.4. *If $A \in \mathcal{K}_{c,n}$, $\theta \in \Sigma_n$, $\mathbf{b} \subseteq A$, $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$, and $\mathbf{b} \nsubseteq wcl_A(\phi_c(\mathbf{x}; a))$ then $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is extendible over A.*

Proof. Note that $\mathbf{b}$ has the same quantifier-free $\Sigma_n \cup \Gamma_c$ type over $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a))$ as any A-generic $\mathbf{b}' \models \phi_{c\wedge\theta}(\mathbf{x}; a)$. Since $\mathrm{wcl}_A(\phi_c(\mathbf{x}; a)) \cup \{\mathbf{b}\} \subseteq A \in \mathcal{K}_{c,n}$ we can apply the proof of the previous lemma to get $A \cup \{\mathbf{b}'\} \in \mathcal{K}_{c,n}$. □

Lemma 7.5. *For all prealgebraic codes c and $n \in \mathbb{N}$, $\mathcal{K}_{c,n+1} \subseteq \mathcal{K}_{c,n}$.*

Proof. This an easy consequence of the previous lemma and the definition of μ_A. □

In the following lemma we use the Decomposition Lemma and nice code assumption to show that our first order approximations $\mathcal{K}_{c,n} \supseteq \mathcal{K}_\mu$ are well-behaved.

Lemma 7.6. *Suppose $A \in \mathcal{K}_{c,n+1}$, $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-extendible, and $\theta \in \Sigma_n$. There is a $\theta^* \in \Sigma_{n+1}$ such that $\theta^* \to \theta$ and $\phi_{c\wedge\theta^*}(\mathbf{x}; a)$ is A-extendible.*

Proof. We induct on $S \subseteq \{1, ..., n_c\}$ to prove the following claim.

Claim. *There exists an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ such that $A \cup \{\mathbf{b}_S\} \in \mathcal{K}_{c,n+1}$.*

Suppose $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-generic and $S \subseteq \{1, ..., n_c\}$. Applying the Decomposition Lemma to $A \leq_s B = A \cup \{\mathbf{b}_S\}$, we get $A \leq_s X \subsetneq B$ and Z at stated there. Since $\mathbf{b} \models \phi_c(\mathbf{x}; a)$ being A-generic completely determines $\mathrm{qftp}_{\Gamma_c}(\mathbf{b}/A)$ and the values of δ on subsets of $A \cup \{\mathbf{b}\}$, the decomposition is the same for all A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$. Thus we may assume that $X \in \mathcal{K}_{c,n+1}$ by induction.

If $\mathbf{b}' \in Z$, then $\mathbf{b}'$ is an X-generic realization of some Σ_n-specialized prealgebraic code instance $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ strongly based on X. Since $X \in \mathcal{K}_{c,n+1}$ and $X \cup \{\mathbf{b}'\} \in \mathcal{K}_{c,n}$, we know that $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ is extendible over X. Because $s_{c'} < s_c$ we can use this lemma to find a $\theta'' \in \Sigma_{n+1}$ so that $\theta'' \to \theta$ and $\phi_{c'\wedge\theta''}(\mathbf{x}'; a')$ is X-extendible. By Lemma 6.1, we may assume that $\mathbf{b}' \models \phi_{c'\wedge\theta''}(\mathbf{x}'; a')$. Because $\mathrm{wcl}_B(\phi_{c'}(\mathbf{x}; a')) \subseteq X$ and $\mathbf{b}'$ is X-generic we have $X \cup \{\mathbf{b}'\} \in \mathcal{K}_{c,n+1}$.

Since the set-wise distinct elements of Z are pairwise disjoint, we can do this for all $\mathbf{b}' \in Z$ simultaneously.

Now, if $B \notin \mathcal{K}_{c,n+1}$ it must be because some Σ_n-specialized prealgebraic code instance $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ has a further Σ_{n+1}-specialization with too many realizations. By the above, we must have $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ strongly based on X.

Let $c'\wedge\theta_1, ..., c'\wedge\theta_D$ enumerate the X-extendible Σ_{n+1}-specializations of c' which further specialize $c' \wedge \theta'$. We may assume

$$\dim_B(\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')) > \mu_B(\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')) = \mu_X(\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')).$$

Since $\phi_{c'\wedge\theta'}(\mathbf{x}'; a')$ doesn't have too many realizations in B, we may assume that

$$\dim_B(\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')) < \mu_B(\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')) = \mu_X(\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')).$$

Since $X \in \mathcal{K}_{c,n+1}$, there is a $\mathbf{b}' \in Z$ realizing $\phi_{c'\wedge\theta_1}(\mathbf{x}'; a')$. Using Lemma 6.1 we can change $\mathbf{b}'$ into a realization of $\phi_{c'\wedge\theta_2}(\mathbf{x}'; a')$.

If $\phi_{c''\wedge\theta''}(\mathbf{x}''; a'')$ is any other Σ_{n+1}-specialized prealgebraic code instance over X, then its dimension is unchanged by this operation unless

$$\phi_{c''\wedge\theta''}(\mathbf{x}'; a'') \equiv \phi_{c'\wedge\theta_i}((\mathbf{x}')^\sigma; a')$$

for some $\sigma \in Sym(n_c)$ and $i = 1, 2$. If this latter condition holds, then $|x''| = |x'|$ and

$$\mu_X(\phi_{c''\wedge\theta''}(\mathbf{x}'; a'')) = \mu_X(\phi_{c'\wedge\theta_i}(\mathbf{x}'; a')).$$

Thus the net effect of changing $\mathbf{b}'$ is to reduce the total number of violations to the multiplicity rules. Iterating this process, we eventually get $B \in \mathcal{K}_{c,n+1}$. □

Lemma 7.7. *Suppose $A \in \mathcal{K}_\mu$, $\phi_{c\wedge\theta}(\mathbf{x}; a)$ is A-extendible, and $\dim_A(\phi_{c\wedge\theta}(\mathbf{x}; a)) < \mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$. There is an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ such that $A \cup \{\mathbf{b}\} \in \mathcal{K}_\mu$.*

Proof. Suppose $\theta \in \Sigma_n$. By the previous lemma, there is at least one $\theta^* \in \Sigma_{n+1}$ so that $\theta^* \to \theta$ and $\phi_{c\wedge\theta^*}(\mathbf{x}; a)$ is A-extendible. Since $\mu_A(\phi_{c\wedge\theta}(\mathbf{x}; a))$ is divided evenly amongst these θ^*, we can choose θ^* such that $\dim_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) < \mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$. Iterating this process, we can find an A-generic $\mathbf{b} \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ so that $A \cup \{\mathbf{b}\} \in \mathcal{K}_{c,n'}$ for all $n' > n$.

If $A \cup \{\mathbf{b}\} \notin \mathcal{K}_\mu$, then it must be the case that

$$\dim_{A\cup\{\mathbf{b}\}}(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) > \mu_{A\cup\{\mathbf{b}\}}(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$$

for some $\theta^* \in \Sigma_{n'}$ with $n' > n$ and $\mathbf{b} \models \phi_{c\wedge\theta^*}(\mathbf{x}; a)$. But $\mu_{A\cup\{\mathbf{b}\}}(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) = \mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$ and we constructed $\mathbf{b}$ so that $\mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) > \dim_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$. Thus $\dim_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a)) = \mu_A(\phi_{c\wedge\theta^*}(\mathbf{x}; a))$, a contradiction. □

8. The Theory T_μ

Lemma 8.1. *If $A \leq_s A \cup \{b\}$ is algebraic or transcendental, then $A \in \mathcal{K}_\mu$ implies $A \cup \{b\} \in \mathcal{K}_\mu$.*

Proof. Suppose $\mathbf{b}_1, ..., \mathbf{b}_N \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ witnesses $A\cup\{\mathbf{b}\} \notin \mathcal{K}_\mu$. Since $a \in \mathrm{acl}^{eq}(A)$, A and $A \cup \{b\}$ compute the same value for $\mu(\phi_{c\wedge\theta}(\mathbf{x}; a))$. Thus it can not be the case that $\mathbf{b}_i \subseteq A$ for all $i \leq N$, so we may assume $b \in \mathbf{b}_1$. This contradicts Lemma 4.1 and the assumption that $A \leq_s A \cup \{b\}$ is not prealgebraic. □

Lemma 8.2. *The class $\mathcal{K}_\mu$ has the amalgamation property with respect to $\leq_s$.*

Proof. Suppose $A \leq_s B, C \in \mathcal{K}_\mu$. We need to find a $D \in \mathcal{K}_\mu$ with $A \leq_s C \leq_s D$ and a $B' \leq_s D$ such that $B' \equiv_A B$. By induction, we may assume that both $A \leq_s B$ and $A \leq_s C$ are minimal.

Suppose $A \leq_s B$ is algebraic, say because $B = A\cup\{b\}$ and $\mathrm{tp}_{T_1}(b/A)$ is algebraic. If $\mathrm{tp}_{T_1}(b/A)$ is realized by $c \in C \setminus A$, then $B \equiv_A C$. Otherwise, we may assume $\mathrm{tp}_{T_1}(b/C)$ is some extension of $\mathrm{tp}_{T_1}(b/C)$ which implies $b \notin C$ and $\mathrm{tp}_{T_2}(b/C)$ is generic. It is then easy to check $C \leq_s C \cup \{b\}$, so $D = C \cup \{b\}$ works by the previous lemma.

Thus we may assume neither $A \leq_s B$ nor $A \leq_s C$ are algebraic. We compute the free fusion of B and C over A by assuming $\mathrm{tp}_{T_i}(B/C)$ is some non-forking extension of $\mathrm{tp}_{T_i}(B/A)$ and letting $D = B\cup C$. By the submodularity of δ, we have $B, C \leq_s D$.

Suppose $D \notin \mathcal{K}_\mu$ is witnessed by distinct $\mathbf{b}_1, ..., \mathbf{b}_N \models \phi_{c\wedge\theta}(\mathbf{x}; a)$ with N too large. We may assume $\phi_{c\wedge\theta}(\mathbf{x}; a)$ has degree 1.

By Lemma 4.2, we may assume that $a \in \mathrm{acl}^{eq}(B)$ and thus $\mathrm{cl}_D(a) \subseteq B$. It follows that B and D compute the same value for $\mu(\phi_{c\wedge\theta}(\mathbf{x}; a))$. Since $B \in \mathcal{K}_\mu$, we may assume $\mathbf{b}_1 \not\subseteq B$. By Lemma 4.1, $C = A \cup \{\mathbf{b}_1\}$. Since $B \downarrow_A^{T_i} C$, we must have $a \in \mathrm{acl}^{eq}(A)$ and thus $\mathrm{cl}_D(a) \subseteq A$. By repeating the argument just given, we may assume $B = A \cup \{\mathbf{b}_2\}$.

Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are both A-generic realizations of a degree 1 prealgebraic code instance over A, we must have $\mathbf{b}_1 \equiv_A \mathbf{b}_2$. Thus $B \equiv_A C$. □

We call an $M \in \mathcal{K}_\mu$ *rich* if for all finite $A \leq_s M$ and finite $A \leq_s B \in \mathcal{K}_\mu$ there is a $C \leq_s M$ with $B \equiv_A C$. The amalgamation property shows that for every $A \in \mathcal{K}_\mu$ we can find a rich $M \in \mathcal{K}_\mu$ with $A \leq_s M$.

Assumption 8.3. *If $K > 1$, then $RM(T_1) \leq RM(T_2)$, in T_1 every element is interalgebraic with infinitely many elements, and in T_2 there are infinitely many disjoint unary predicates of rank $RM(T_2) - 1$.*

Let T_μ be the theory which says, for $M \models T_\mu$, that

(1) $M \in \mathcal{K}_\mu$,
(2) $M \restriction L_i \models T_i$ for $i = 1, 2$,
(3) there is no prealgebraic extension $M \leq_s N \in \mathcal{K}_\mu$.

Note that axiom (3) is first order by Lemma 7.7.

Theorem 8.4. *The theory T_μ is consistent, complete, and the ω-saturated models of T_μ are exactly the rich structures on $\mathcal{K}_\mu$. Moreover, T_μ has rank K, nice codes, and*

$$RM_T(\phi(x; a)) = v_i RM_{T_i}(\phi(x; a)) \text{ and } dM_T(\phi(x; a)) = dM_{T_i}(\phi(x; a))$$

for all $\phi(x; y) \in L(T_i^{eq})$ and $i = 1, 2$.

Proof. We have set up the machinery required to run the proof of the corresponding theorem in [8]. The only thing that needs mention is that the pairs of predicates $P^1_{n,k} \wedge P^2_{n',k'}$ provide nice codes for T_μ. □

9. Wrap-up

Proof of Theorem 1.1. This has the same proof as the corresponding theorem in [8]. The main point is that if we are willing to expand the language, i.e., $L(T) \supsetneq L(T_1) \cup L(T_2)$, then we can obtain assumption 8.3 and apply Theorem 8.4. □

Corollary 9.1. *Theories with nice codes have rank and degree preserving intepretations in a strongly minimal sets.*

Proof. Same as Corollary 1.3 in [8]. □

I can currently do a little better than the above results. In particular, I can write down weaker versions of *nice codes* which are still sufficient for the main theorem. However, I still can not prove (or disprove) that any two finite rank, degree 1, wDMP theories have a fusion.

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