Branching rules of unitary representations: Examples and applications to automorphic forms.

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Birgit Speh
Cornell University

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Example: $V = \{$ functions on G $\}$. G acts on V by

$$\pi(g)F(x) = F(gx)$$

Given 2 groups $H \subset G$

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Example 3: Littlewood–Richardson rules are combinatorial algorithms for finite dimensional branching laws.

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Example 2 : $S^2 = SO(3)/SO(2)$. So $L^2(S^2)$ induced representation from the trivial representation of a torus $H = SO(2)$ in the orthogonal group G=S0(3). Spherical functions allow to define representations of SO(3).

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$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
Ex: $\mathfrak{k}$ skew symmetric matrices, $\mathfrak{p}$ symmetric matrices

A look at unitary representations

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A representation $\pi : G \to U(V)$ is called **admissible** for every irreducible representation $\sigma : K \to U(V_\sigma)$ of the maximal compact subgroup K .

$$\dim(\sigma, \pi) < \infty$$

Denote by $V(\sigma) \subset V$ the isotypic subspace type σ.

An irreducible unitary representation π is admissible and

$$V_K = \oplus_\sigma V(\sigma)$$

of K-finite vectors is dense in V (Harish Chandra). The Lie algebra $\mathfrak{g}$ and the maximal compact subgroup K act compatibly on V_K , the **Harish Chandra module of** π.

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If π is an irreducible unitary representation then its $(\mathfrak{g}, K)$- module V_K is also irreducible and as a K-module isomorphic to a direct sum of irreducible K modules. A representation σ with

$$\mathrm{dim}(\sigma, \pi) \neq 0$$

is called a **K-type of** π.

An Example:

$H \subset \mathbb{C}$ upper half plane disc, G=SL(2,$\mathbb{R}$)

$$V = \left\{ \text{f analytic on H} |\ ||f||^2 = \int\int_{\Im z > 0} |f(z)|^2 dxdy < \infty \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (-bz+d)^{-2} f(\frac{az+d}{bz+d})$$

Here K =SO(2) so its characters are parametrized by $\mathbb{Z}$ and Harish Chandra module

$$V_K = \oplus_{n>2} V(n)$$

Branching Problem:

Let H semisimple subgroup of G with maximal compact subgroup $K_H = H \cap K$ and $\mathfrak{h}$ Lie algebra of H
Assume that H is the fix point set of an involution.
Ex 1: $H = Sp(2m, \mathbb{R}) \subset SL(2m, \mathbb{R})$ symplectic 2m $\times$ 2m matrices, max. compact subgroup U(n) unitary matrices

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Precisely:

$$\pi_{|H} = \int_M \pi_\nu \; d\nu$$

Understand the discrete part of this integral decomposition.

The restriction of unitary representations.

First case: The restriction of π to H is a Hilbert direct sum

$$\bigoplus_{m \in M} \pi_m$$

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Example: Take as H the max. compact sub group K, then every irreducible unitary representations is K-admissible.

Theorem (Kobayashi)
Suppose that π *is H-admissible for a subgroup H which is the fix point set of an involution of G. Then the underlying* $(\mathfrak{g}, K)$ *module is a direct sum of irreducible* $(\mathfrak{h}, K \cap H)$ *-modules ,(i.e* π *is infinitesimally H-admissible.)*

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Conclusion: If π is H- admissible then it is an algebraic problem to determine the branching law for the restriction of π to H

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Example: (joint with B. Orsted)
Let G=SL(4,$\mathbb{R}$). There are 2 conjugacy classes of symplectic subgroups. Let H_1 and H_2 be symplectic groups in different conjugacy classes.

There exists an unitary representation π of G which is H_1 admissible but not H_2 admissible .

Branching law for irreducible unitary H-admissible representation π is a formula for

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In special cases obtained by Kobayashi, Duflo, Vargas, Orsted , S. and many others.

It is conjectured to be of "Blattner type", i.e similar to the formula for the multiplicity of K-types..

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Conclusion: If the representation π is not H-admissible, Kobayashi's theorem implies that finding direct summands is an analysis problem and not an algebra problem concerning the underlying Harish Chandra module.

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Warning: There exists a unitary representation π of SL(2,$\mathbb{C}$) whose restriction to SL(2,$\mathbb{R}$) contains a direct summand σ but σ doesn't contain any smooth vectors of π. (joint with Venkataramana)

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Open Problem: For every irreducible unitary representation $\pi : G \to U(V)$ find a G invariant subspace V_o "which allows us to detect direct summands", (i.e so if $\sigma : H \to U(W)$ for a closed subspace $W \subset V$ is a direct summand then $W_o \cap V_o \neq 0$.

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Proposition: Let G acts by left translations on $C_c(G)$. Let σ be a irreducible unitary representation. There is a metric on $C_c(G)$ so that the completion of $C_c(G)$ is the direct sum of the Hilbert spaces

$$\sigma \oplus L^2(G)$$

How do find direct summands: An example (joint work with Venkataramana)

Let $G_{\mathbb{C}} = SL(2, \mathbb{C})$, and take $H = G_{\mathbb{R}} = Sl(2, \mathbb{R})$

We construct the representations as follows: Let $B_{\mathbb{C}}$ the Borel subgroup of upper triangular matrices in $G_{\mathbb{C}}$,
and

$$\rho(\begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}) = \mid a \mid^2 .$$

For $u \in \mathbb{C}$

$$\pi_u = \{f \in C^{\infty}(G) \mid \ f(bg) = \rho(b)^{1+u} f(g)$$

for all $b \in B(\mathbb{C})$ and all $g \in G(\mathbb{C})$ and in addition are $SU(2)$-finite. in other words we consider as sections of a line bundle over $B_{\mathbb{C}} \backslash G_{\mathbb{C}}$.

Take the completion of this space with respect to the inner product

$$< f_1, f_2 >= \int_K f_1(k)\bar{f}_2(k)dk$$

This defines a unitary principal series representation π_u if the parameter u is imaginary.

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For U real and $0 < u < 1$ completion to the unitary complementary series rep $\widehat{\pi}_u$ with respect to an inner product

$$< f_1, f_2 >_{\pi_u}= \int_K f_1(k)I_u(\bar{f}_2)(k)dk$$

for an integral operator I_u.

Similiar define the complementary series $\widehat{\sigma}_t$ of H=Sl(2,$\mathbb{R}$) for $0 < t < 1$.

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If u is real and $1/2 \leq u \leq 1$ then the restriction of π_u to $Sl(2,\mathbb{R})$ has direct summand π_v, a complementary series representation of $Sl(2,\mathbb{R})$. Proof by showing that the geometric restriction of functions to the equation is a continuous operator with respect to these inner products.

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Work by N.Bergeron on the Lefschetz properties of real and complex hyperbolic manifolds

Conjecture (Bergeron)
Let X be the real hyperbolic n-space and $\Gamma \subset$ SO(n, 1) a congruence arithmetic subgroup. Then non-zero eigenvalues λ of the Laplacian acting on the space $\Omega^i(X)$ of differential forms of degree i satisfy:

$$\lambda > \epsilon$$

for some $\epsilon > 0$ independent of the congruence subgroup Γ, provided i is strictly less than the middle" dimension (i.e. i ¡ [n/2]).

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For n=2 Selberg proved that Eigen values λ of the Laplacian on function satisfy $\lambda > 3/16$ and more generally Clozel showed there exists a lower bound on the eigenvalues of the Laplacian on functions independent of Γ.

Using restrictions of complementary series representations proves

Theorem(Joint with Venkataramana)
If the Bergeron's conjecture holds true for differential forms in the middle degree for all SO(n,1), then the conjecture holds for arbitrary degrees of the differential forms