Pseudo–Eisenstein forms and cohomology of arithmetic groups II

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Introduction.

Let Γ be a torsion free arithmetic subgroup of a semi simple Lie group $G(\mathbf{R})$, K a maximal compact subgroup, and $X = G(\mathbf{R})/K$ the corresponding symmetric space. Denote by $\Gamma\backslash X$ the associated locally symmetric space. The group cohomology $H^*(\Gamma, \mathbf{C})$ of the arithmetic group Γ coincides with the cohomology $H^*(\Gamma\backslash X, \mathbf{C})$ of the topological space $\Gamma\backslash X$. In this paper we use a geometric approach to $H^*(\Gamma\backslash X, \mathbf{C})$ via modular symbols.

Suppose $H \subset G$ is a $\mathbf{Q}$–rational reductive subgroup such that $K \cap H(\mathbf{R})$ is maximal compact in $H(\mathbf{R})$. Then the inclusion

$$H(\mathbf{R})/H(\mathbf{R}) \cap K = X_H \to X$$

induces a map

$$j : \Gamma \cap X_H\backslash \longrightarrow \Gamma\backslash X.$$

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Assume that $\Gamma \cap H \backslash X_H$ is a compact and oriented manifold. If a closed d–form ω represents $[\omega] \in H^d(\Gamma\backslash X_\infty, \mathbf{C}), d = \dim X_H$, then

$$\int_{\Gamma\cap H\backslash X_H} j^*\omega$$

is defined. This means that $\Gamma \cap H\backslash X_H$ determines a map

$$H^d(\Gamma\backslash X, \mathbf{C}) \to \mathbf{C},$$

which is called the *modular symbol* attached to H. We drop the assumption that $H \cap \Gamma\backslash X_H$ is compact and let $[\varphi]$ be an element in the i–th cohomology with compact supports $H_c^i(\Gamma \cap H\backslash X_H, \mathbf{C})$. Suppose that $[\varphi]$ is represented by a closed compactly supported i–form φ and that $i + k = \dim_{X_H}$. Then $[\varphi]$ determines a map

$$H^k(\Gamma\backslash X, \mathbf{C}) \to \mathbf{C}$$

by

$$[\omega] \to \int_{\Gamma\cap H\backslash X_H} \varphi \wedge j^*\omega$$

We call this map the *modular symbol attached to* $([\varphi], H)$. If now $\Gamma\backslash X$ is oriented we use Poincaré duality and identify the modular symbol $([\varphi], H)$ with an element in $H_c^*(\Gamma\backslash X, \mathbf{C})$. If we can find an ω such that $([\varphi], H)([\omega]) \neq 0$ then $([\varphi], H)$ is a nontrivial modular symbol.

It is as difficult to construct classes in $H_c^*(\Gamma\backslash X, \mathbf{C})$ as it is to find classes in $H^*(\Gamma\backslash X, \mathbf{C})$. However, if P is a proper $\mathbf{Q}$–rational parabolic subgroup of G and if we know a compactly supported i–form φ on $\Gamma \cap P\backslash X$ the *pseudo–Eisenstein form*

$$E_P(\varphi) := \sum_{\gamma\in\Gamma/P\cap\Gamma} (\gamma^{-1})^*\varphi$$

defines a class $[E_P(\varphi)] \in H_c^i(\Gamma\backslash X, \mathbf{C})$. We denote the map

$$H_c^i(\Gamma \cap P\backslash X, \mathbf{C}) \to H_c^i(\Gamma\backslash X, \mathbf{C})$$

induced from

$$\varphi \longmapsto E(\varphi)$$

by cor_P. We view the class $[E(\varphi)] \in H_c^i(\Gamma\backslash X, \mathbf{C})$ by Poincaré–duality as a linear map from $H^{\dim X - i}(\Gamma\backslash X, \mathbf{C})$ to $\mathbf{C}$. We denote the map by $(G, [E_P(\varphi)])$ and call it also a *modular symbol.*

In 2.2 we show that cor_P is the adjoint of the restriction map res_P with respect to Poincaré duality. Here res_P is the map given by the covering $j : \Gamma \cap P\backslash X \longrightarrow \Gamma\backslash X$. Then 2.2 implies that

$$\int_{\Gamma\backslash X} E(\varphi) \wedge \omega = \int_{\Gamma\cap\Gamma\backslash X} \varphi \wedge j^*\omega\,.$$

The classes in

$$H^*(\Gamma\backslash X, \mathbf{C})_{cc} := \cap_P \ker\ (\mathrm{res}_P)$$

where P runs in the set of proper $\mathbf{Q}$–rational parabolic subgroups are called *cohomologically cuspidal*. We show in Theorem 2.5 that the space of cohomologically cuspidal classes is the the orthogonal complement with respect to Poincaré duality of the subspace spanned by the modular symbols $(G, [E_P(\varphi)])$.

In general it is difficult to see if $[E(\varphi)] \neq 0$. In § 3 we investigate the restriction of $[E(\varphi)]$ to certain sub symmetric spaces. For this let P and Q be proper standard parabolic subgroups. We consider the restriction $\mathrm{res}_Q[E_P(\varphi)]$ of $[E_P(\varphi)]$ to the standard Levi component $L(Q)$ of Q. It is shown in Theorem 3.7. that this restriction is a sum of Pseudo–Eisenstein classes attached to φ with respect to parabolic subgroups whose Levi factors are conjugate to $L(Q)$. The formula is similar to the classical formula for the Fourier coefficient along Q of an Eisenstein series for P. There are however no convergence problems.

For $Q = P$ and a cohomologically cuspidal form φ the formula for $\mathrm{res}_P \circ \mathrm{cor}_P(\varphi)$ simplifies, see 3.14. and the image of $\mathrm{res}_P \circ \mathrm{cor}_P$ is determined in 3.15. In particular we obtain a generalization of the result of A.Ash and A.Borel on the non vanishing of the modular symbol attached to the fundamental class of the Levi factor of a parabolic subgroup, see 3.17.

Since all sums in the construction of pseudo–Eisenstein classes are locally finite we use in this paper algebraic methods. Crucial is the relation between the cohomology with compact support and the cohomology with coefficients in the Steinberg representations of G and its parabolic subgroups. The Steinberg representation has been used already by Ash and Reeder in a related context, see [A 2], [Re].

The results of the paper are independent of the ones in [R–Sp]. The translation of the analytical definition of cor_P which we have used in this introduction and in [R–Sp] to the purely algebraic definition of cor_P in 2.1. is explained in 2.2 and 2.3.

In contrast to the introduction we work in the paper with a reductive group G in the adelic context. Moreover we work with congruence subgroups $\Gamma \subset G(\mathbf{Q})$ of all levels at the same time, i.e. if for example G simply connected then $H^*(\Gamma\backslash X, \mathbf{C})$ is replaced by $\varinjlim_{\Gamma \subset G(\mathbf{Q})} H^*(\Gamma\backslash X, \mathbf{C})$. We also consider more general coefficient systems. In § 1 we recall the corresponding notation and results.

1 Preliminaries

In this chapter we fix our notation concerning adelic symmetric spaces and their cohomology. In particular we describe Poincaré–duality, the connection with Borel–Serre–duality and properties of the Steinberg representation of the $\mathbf{Q}$–rational points of an algebraic group. Details can be found in [B–S], [Ha 1,2], [Re] and [Ro].

1.1. Let G be a connected reductive group defined over $\mathbf{Q}$ of semi–simple $\mathbf{Q}$–rank $\ell > 0$. By K we denote a maximal compact subgroup of the group $G(\mathbf{R})$ of real points of G. We observe that $G(\mathbf{R})$ is non compact. Let A_G be the connected component of 1 of the group of real points of a maximal central $\mathbf{Q}$–split torus of G. Put $X_\infty := G(\mathbf{R})/A_G K$. Endowed with the quotient topology of the real Lie group $G(\mathbf{R})$ then X_∞ is a globally symmetric space. Let $\mathbf{A}_f \subset \mathbf{A}$ be the finite adeles of the adele ring $\mathbf{A}$ over $\mathbf{Q}$. We give $G(\mathbf{A}_f)$ the topology induced by the topology of $\mathbf{A}_f$. We define $X := X_\infty \times G(\mathbf{A}_f)$ and give X the product topology. We call X the *adelic symmetric space* attached to G. The group $G(\mathbf{Q})$ of $\mathbf{Q}$–rational points of G acts by left translation freely and discontinuously on X. We endow $G\backslash X =: S_G$ with the quotient topology. It is called the *adelic locally symmetric space* attached to G.

1.2 The space X_∞ depends on the choice of the point x_0 given by K. We choose x_0 such that the Levi components of all standard parabolic subgroups are θ_{x_0}–stable, where θ_{x_0} is the Cartan involution determined by K. To fix our notation we recall the argument from [A–B: 4.2]. We fix a minimal $\mathbf{Q}$–rational parabolic subgroup B of G and a maximal $\mathbf{Q}$–split torus S of G such that $S \subset B$. Then $B = Z(S)N$ where N is the unipotent radical of B and $Z(S)$ is the centralizer of S in G. All Levi subgroups of $B(\mathbf{R})$ are $B(\mathbf{R})$–conjugate, and given $x \in X_\infty$ there is exactly one Levi subgroup $L_x \subset B(\mathbf{R})$ which is θ_x–stable, where θ_x is the Cartan involution determined by x, see [B–S: § 1]. Hence if $L(\mathbf{R}) = Z(S)(\mathbf{R})$ there is a $p \in B(\mathbf{R})$ such that $L(\mathbf{R}) = pL_x p^{-1}$. We choose $x_0 := px$ and see that $L(\mathbf{R})$ is θ_{x_0}–stable. But then $S(\mathbf{R}) \cap K_{x_0} = \{1\}$ and $\theta_{x_0}(t) = t^{-1}$ for all $t \in S(\mathbf{R})$.

Let $\triangle$ be the set of simple $\mathbf{Q}$–roots with respect to (B, S). If $\psi \subset \Delta$ then $S^\psi := (\bigcap_{\alpha\in\psi} \ker\alpha)^0$ is a torus and its centralizer $Z(S^\psi)$ is the Levi component of the standard parabolic subgroup $Z(S^\psi)N = P_\psi$. It follows that $Z(S^\psi)$ is defined over $\mathbf{Q}$ and that $Z(S^\psi)(\mathbf{R})$ is θ_{x_0}–stable. We write $\theta = \theta_{x_0}$ and x_0 for the point $(x_0, 1) \in X = X_\infty \times G(\mathbf{A}_f)$.

1.3 Let $P \supset B$ be a standard parabolic subgroup of G with standard Levi part L_P. Let $x_0 \in X$ be as in (ii). We consider the orbit of the point x_o under $L_P(\mathbf{A})$ in the globally symmetric space X. We see that

$$X_{L_P} := (L_P(\mathbf{R})/(L_P(\mathbf{R}) \cap K)A_G) \times L_P(\mathbf{A}_f) \xrightarrow{\sim} L_P(\mathbf{A})x_0.$$

Since $L_P(\mathbf{R})$ is θ_{x_0}–stable, $L_P(\mathbf{R})/(L_P(\mathbf{R}) \cap K)A_G$ is a symmetric space and

$$S^\natural_{L_P} := L_P(\mathbf{Q})\backslash X_{L_P}$$

is a locally symmetric space. Moreover the above isomorphism is a homeomorphism with respect to the natural topologies on both spaces and the orbit X_{L_P} is a closed subspace of X. We have an induced continuous injection

$$S^\natural_{L_P} \longrightarrow S_G ,$$

It is known that the inclusion $S^\natural_L \longrightarrow S_G$ identifies $L_P(\mathbf{Q})/(L_P(\mathbf{R}) \cap K)A_G \times L_P(\mathbf{A}_f)$ with a closed subspace of S_G. This follows as in [A 1: 2.7]. We call $S^\natural_{L_P}$ the *modular manifold* attached to P.

Let A_{L_P} be as in 1.1 for L_P instead of G. We define the locally symmetric space S_{L_P} by

$$S_{L_P} := (L_P(\mathbf{Q})\backslash L_P(\mathbf{R})/(L_P(\mathbf{R}) \cap K)A_{L_P}) \times L_P(\mathbf{A}_f).$$

We have a fibration $f : S^{\natural}_{L_P} \to S_{L_P}$ with fibers isomorphic to A_{L_P}/A_G.

1.4 Let V be a finite dimensional $\mathbf{C}$–vector space and let $\rho : G(\mathbf{C}) \longrightarrow GL(V)$ be a representation. Then V determines a locally constant sheaf $\tilde{V}$ of $\mathbf{C}$–vector spaces with fibres V on S_G. By $H^*(S_G, \tilde{V})$ we denote the smooth sheaf cohomology of S_G with coefficients $\tilde{V}$. The group $G(\mathbf{A}_f)$ acts by right translation on S_G and $H^j(S_G, \tilde{V})$ is a smooth $G(\mathbf{A}_f)$–module, i.e. if K_f runs in the set of compact open subgroups of $G(\mathbf{A}_f)$ and if $H^j(S_G, \tilde{V})^{K_f}$ denotes the K_f–invariants in $H^j(S_G, \tilde{V})$ then

$$\bigcup_{K_f} H^j(S_G, \tilde{V})^{K_f} = H^*(S_G, \tilde{V}).$$

Moreover

$$H^j(S_G, \tilde{V})^{K_f} = H^j(S_G/K_f, \tilde{V})$$

where S_G/K_f is the topological quotient of S_G by the K_f–action, $\tilde{V}$ is the local system on S_G/K_f determined by V and $H^j(S_G/K_f, \tilde{V})$ is the sheaf–cohomology of S_G/K_f with coefficients in the sheaf $\tilde{V}$. One has a canonical isomorphism of $G(\mathbf{A}_f)$–modules

$$H^j(S_G, \tilde{V}) \xrightarrow{\sim} H^j(G(\mathbf{Q}), C^\infty(G(\mathbf{A}_f), V)).$$

Here $H^j(G(\mathbf{Q}), C^\infty(G(\mathbf{A}_f), V))$ denotes the group cohomology of $G(\mathbf{Q})$ acting on $C^\infty(G(\mathbf{A}_f), V)$. If P is a proper $\mathbf{Q}$–rational parabolic of G we also write $\tilde{V}$ for the locally constant sheaf with fibres V attached to V on $S_P := P(\mathbf{Q})\backslash X$, and one can see

$$H^j(S_P, \tilde{V}) = H^j(P(\mathbf{Q}), C^\infty(G(\mathbf{A}_f), V)).$$

1.5 Let B be a minimal $\mathbf{Q}$–rational parabolic subgroup of G. Let $\mathbf{Z}[H]$ denote the group algebra of a group H. Then we have a natural projection

$$r_P : \mathbf{Z}[G(\mathbf{Q})/B(\mathbf{Q})] \cong \mathbf{Z}[G(\mathbf{Q})] \otimes_{\mathbf{Z}[B(\mathbf{Q})]} \mathbf{Z} \longrightarrow \mathbf{Z}[G(\mathbf{Q})] \otimes_{\mathbf{Z}[P(\mathbf{Q})]} \mathbf{Z}.$$

By definition $St_G := \bigcap_{P \underset{+}{\subset} B} \ker r_P$, where P runs in the set of minimal parabolic subgroups which contain B properly. Then St_G is a $G(\mathbf{Q})$–module. For the following remarks, see [Re: § 1].

The Steinberg representation St_G does not depend on the choice of B. One has $\sum_{w \in W} (-1)^{|w|} w =: \tau_G \in St_G$, where $|w|$ denotes the length of w

in the $\mathbf{Q}$–rational Weyl group W of $G(\mathbf{Q})$. Moreover τ_G generates St_G as $B(\mathbf{Q})$–module.

If $P \supset B$ is a parabolic subgroup, then $St_P \subset \mathbf{Z}[P(\mathbf{Q})/B(\mathbf{Q})]$ denotes the Steinberg representation of $P(\mathbf{Q})$. It coincides with the Steinberg representation $St_{L_P(\mathbf{Q})}$ of the standard Levi part L_P of P and is generated as L_P–module by $\tau_P := \sum_{w \in W_P} (-1)^{|w|} w$, where W_P is the $\mathbf{Q}$–rational Weyl group of $P(\mathbf{Q})$ or $L_P(\mathbf{Q})$. The obvious surjection $\mathbf{Z}[G(\mathbf{Q})/B(\mathbf{Q})] \longrightarrow \mathbf{Z}[P(\mathbf{Q})/B(\mathbf{Q})]$ induces a $P(\mathbf{Q})$–linear surjection $s(P,G) : St_G \longrightarrow St_P$. Moreover, there is a $L_P(\mathbf{Q})$–linear section $\sigma(G,P) : St_P \longrightarrow St_G$ of $s(P,G)$ which induces an isomorphism

$$\sigma_P : \mathbf{Z}[P(\mathbf{Q})] \otimes_{L_P(\mathbf{Q})} St_P \xrightarrow{\sim} St_G$$

of $P(\mathbf{Q})$–modules. One has $\sigma_P(1 \otimes \tau_P) = \tau_G$.

1.6 By $\omega : G(\mathbf{R}) \to \{\pm 1\}$ we denote the orientation character of $G(\mathbf{R})$, i.e. if $g \in G(\mathbf{R})$ then $\omega(g) = 1$ resp. $\omega(g) = -1$ if left translation with g is orientation preserving, resp. orientation reversing on X_∞.

(i) We define

$$H_c^j(S_G, \tilde{V}) := H^{j-\ell}(G(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))$$

where ℓ is the semi simple $\mathbf{Q}$–rank of G. For motivation let $K_f \subset G(\mathbf{A}_f)$ be an open and compact subgroup. Then

$$H_c^j(S_G, \tilde{V})^{K_f} = H^{j-\ell}(G(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f)/K_f, V)))\,.$$

We can write $G(\mathbf{A}_f) = \bigcup_{i=1}^k G(\mathbf{Q}) a_i K_f, a_i \in G(\mathbf{A}_f)$, as finite disjoint union.

Put $\Gamma_i = G(\mathbf{Q}) \cap a_i K_f a_i^{-1}$ and assume that K_f is so small that all Γ_i are torsionfree. Then

$$H_c^j(S_G, \tilde{V})^{K_f} = \bigoplus_{i=1}^k H^{j-\ell}(\Gamma_i,\ \mathrm{Hom}(St_G, V))\,.$$

By Borel–Serre duality, [B–S: 15.1], and Poincaré–duality on the manifold $\Gamma_i \backslash X_\infty$ then

$$H_c^j(S_G, \tilde{V})^{K_f} = H_c^j(S_G/K_f, \tilde{V})\,.$$

where on the right side we have cohomology with compact supports of the manifold S_G/K_f with coefficients in the locally constant sheaf $\tilde{V}$ given by the $G(\mathbf{Q})$ action on X/K_f.

(ii) We define

$$H_c^j(S_P, \tilde{V}) := H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V)))\,.$$

We write $G(\mathbf{A}_f) = \bigcup_{t=1}^h P(\mathbf{Q}) b_t K_f, b_t \in G(\mathbf{A}_f)$, as finite disjoint union. One takes K_f as in (i). We get as in (i)

$$H_c^j(S_P, \tilde{V})^{K_f} = \bigoplus_{t=1}^h H_c^j(\Gamma_{P,t} \backslash X_\infty, \tilde{V}) = H_c^j(S_P/K_f, \tilde{V})$$

where on the right side we have cohomology with compact supports with coefficients in the locally constant sheaf determined by V on $\Gamma_{P,t} \backslash X_\infty$ and on S_P/K_f .

1.7. On S_G and on S_P Poincaré–duality holds. For this let $V^\vee$ be the contragredient representation of V . Then there is a non degenerate pairing

$$\langle\ ,\ \rangle_G : H^j(S_G, \tilde{V}) \times H_c^{d-j}(S_G, \widetilde{\omega \otimes V^\vee}) \to \mathbf{C}$$

which induces an isomorphism of smooth $G(\mathbf{A}_f)$–modules

$$H_c^{d-j}(S_G, \widetilde{\omega \otimes V^\vee}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{C}}^\infty(H^j(S_G, \tilde{V}), \mathbf{C}) \, .$$

Here $d = \dim X_\infty$ and if M is a smooth $G(\mathbf{A}_f)$–module then $\mathrm{Hom}_{\mathbf{C}}^\infty(M, \mathbf{C})$ denotes the smooth $G(\mathbf{A}_f)$–submodule of $\mathrm{Hom}_{\mathbf{C}}(M, \mathbf{C})$. The corresponding result holds for S_P instead of S_G.

2 Corestriction and modular symbols

Let P be a standard proper $\mathbf{Q}$–rational parabolic subgroup of G . In [R–Sp] we have attached to a class $[\varphi] \in H_c^j(S_P, \tilde{V})$ a class $\mathrm{cor}_P([\varphi]) \in H_c^j(S_G, \tilde{V})$. If a compactly supported V–valued differential form φ represents $[\varphi]$ then $\mathrm{cor}_P[\varphi]$ is represented by the differential form

$$\sum_{G(\mathbf{Q})/P(\mathbf{Q})} g^{*^{-1}} \varphi.$$

In this chapter we describe a group–cohomological construction of $\mathrm{cor}_P([\varphi])$. This algebraic description of $\mathrm{cor}_P[\varphi]$ has technical advantages, which will be useful in § 3. Using Poincaré–duality we consider

$$\mathrm{cor}_P([\varphi]) \in \mathrm{Hom}_{\mathbf{C}}^\infty(H^*(S_G, \widetilde{\omega \otimes V^\vee}), \mathbf{C}) \, .$$

In 2.5 we determine the subspace of $\mathrm{Hom}_{\mathbf{C}}^\infty(H^*(S_G, \widetilde{\omega \otimes V^\vee}), \mathbf{C})$ which is generated by all modular symbols $\mathrm{cor}_P([\varphi])$.

2.1. (i) We recall that the map $s(P, G) : St_G \longrightarrow St_P$ of the Steinberg representations of $G(\mathbf{Q})$ and $P(\mathbf{Q})$ for a proper $\mathbf{Q}$–rational parabolic subgroup P is induced by the natural restriction map $\mathbf{Z}[G(\mathbf{Q})/B(\mathbf{Q})] \longrightarrow \mathbf{Z}[P(\mathbf{Q})/B(\mathbf{Q})]$ of free $\mathbf{Z}$–modules generated by $G(\mathbf{Q})/B(\mathbf{Q})$ resp. $P(\mathbf{Q})/B(\mathbf{Q})$. If $t \in St_G$

then $s(P,G)(g^{-1}(t)) \neq 0$ only for finitely many classes $gP(\mathbf{Q}), g \in G(\mathbf{Q})$, see [Re: Lemma, p. 310].

(ii) Let $0 \longrightarrow C^\infty(G(\mathbf{A}_f), V) \longrightarrow A^*$ be a resolution by $G(\mathbf{Q}) \times G(\mathbf{A}_f)$ modules, which are acyclic as $G(\mathbf{Q})$–modules and smooth as $G(\mathbf{A}_f)$–modules. Then

$$H_c^j(S_G, \tilde{V}) \xrightarrow{\sim} H^{j-\ell}(G(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))$$

is computed as $j-\ell$–th cohomology of the complex $\mathrm{Hom}_{G(\mathbf{Q})}(St_G, A^*)$. If $C_* \longrightarrow \mathbf{C} \longrightarrow 0$ is a resolution of $\mathbf{C}$ by projective $G(\mathbf{Q})$–modules, we can take $A^* = \mathrm{Hom}_{\mathbf{C}}(C_*, C^\infty(G(\mathbf{A}_f), V))$. For the convenience of the reader we give an explicite construction of A^*. For this let V_{X_∞} be the constant sheaf on X_∞ with fibre V and denote by $V_{X_\infty} \longrightarrow \Omega^*$ the standard resolution of V_{X_∞} by the complex of sheaves of smooth V–valued differential forms on X_∞. If $\mathcal{C}^\infty$ denotes the sheaf of smooth $\mathbf{C}$–valued functions on $G(\mathbf{A}_f)$ then the exterior tensor product $\Omega^* \boxtimes \mathcal{C}$ is a resolution of $V_X := V_{X_\infty} \boxtimes \mathcal{C}^\infty$ on $X = X_\infty \times G(\mathbf{A}_f)$, by sheaves with $G(\mathbf{Q}) \times G(\mathbf{A}_f)$–action. Now X is paracompact and $\Omega^0 \boxtimes \mathcal{C}^\infty$ is a fine sheaf. Hence $\tilde{V} \longrightarrow \Omega^* \otimes \mathcal{C}^\infty =: B^*$ is a soft resolution of V_X, see [Go: II, 3.7.3]. Softness is a local property, see [Go: II 3.4.1], and $G(\mathbf{Q})$ acts freely and discontinuously on X. Hence the $G(\mathbf{Q})$–invariant direct image $f_*^{G(\mathbf{Q})} B^j$ is soft and $f_*^{G(\mathbf{Q})}$ is an exact functor. Here $f : X \longrightarrow G(\mathbf{Q})\backslash X$ is the natural projection. Therefore $f_*^{G(\mathbf{Q})} B^*$ is a soft resolution of $\tilde{V} = f_*^{G(\mathbf{Q})} V_X$. In particular $H^j(G(\mathbf{Q})\backslash X, f_*^{G(\mathbf{Q})} B^i) = H^j(G(\mathbf{Q}), B^i(X)) = 0$ if $j \geq 1$. Here we use a standard spectral sequence argument, see [Gr: 5.2.4]. Hence for $A^i := B^i(X)$ the resolution

$$0 \longrightarrow C^\infty(G(\mathbf{A}_f), V) \longrightarrow A^*$$

has the desired properties. Moreover, the same type of result holds if G is replaced by P.

The cohomology $H_c^j(S_P, \tilde{V})$ is the $j-\ell$–th cohomology of the complex $\mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*)$. If

$$\varphi \in \ \mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^{j-\ell}),$$

then

$$\varphi \circ s(P,G) \in \ \mathrm{Hom}_{P(\mathbf{Q})}(St_G, A^{j-\ell})$$

and by (i)

$$\sum_{G(\mathbf{Q})/P(\mathbf{Q})} {}^g(\varphi \circ s(P,G)) \in \ \mathrm{Hom}_{G(\mathbf{Q})}(St_G, A^{j-\ell})$$

is well defined. The map

$$\varphi \longrightarrow \sum_{G(\mathbf{Q})/P(\mathbf{Q})} {}^g(\varphi \circ s(P,G))$$

induces a map denoted by

$$\begin{aligned} \mathrm{cor}(G,P) : H^{j-\ell}&(P(\mathbf{Q}),\ \mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V))) \\ &\longrightarrow H^{j-\ell}(G(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V))). \end{aligned}$$

Hence

$$\mathrm{cor}(G,P) : H_c^j(S_P,\tilde V) \longrightarrow H_c^j(S_G,\tilde V)$$

is defined. We recall that both cohomology groups are smooth $G(\mathbf{A}_f)$–modules and that $\mathrm{cor}(G,P)$ is automatically a map of $G(\mathbf{A}_f)$–modules. By

$$\mathrm{res}(P,G) : H^*(S_G,\tilde V) \longrightarrow H^*(S_P,\tilde V)$$

we denote the restriction map induced by natural surjection $S_P \to S_G$. For short we write $\mathrm{cor}(P,G) = \mathrm{cor}$ and $\mathrm{res}(P,G) = \mathrm{res}$.

2.2. Proposition. *The map* cor(P,G) *is the adjoint of* res(P,G) *with respect to Poincaré–duality on* S_G *and* S_P*, i.e. if* $[\varphi] \in H_c^*(S_P,\tilde V)$ *and* $[\psi] \in H^*(S_G,\widetilde{\omega\otimes V^\vee})$ *then*

$$\langle \mathrm{cor}[\varphi],[\psi]\rangle_G = \langle [\varphi],\mathrm{res}[\psi]\rangle_P.$$

Proof. We use 1.6 to reduce the claim to the corresponding one where S_G is replaced by a finite union of connected oriented locally symmetric manifolds of the form $\Gamma\backslash X$ for an arithmetic group Γ, and where S_P is replaced by $\Gamma\cap P\backslash X$. Together with the isomorphisms in 1.6 the claim in this situation follows from [Re: 4.9 (1)]. q.e.d.

Next, we indicate the connection of the definition of cor with the one used in [R – Sp].

2.3. Let $\psi \subset \Delta$ be a set of simple $\mathbf{Q}$–roots and assume that $P := P_\psi$ is the standard parabolic subgroup of type ψ with Levi part $Z(S^\psi)$, see 1.2. For each place v of $\mathbf{Q}$ then $\alpha \in \Delta - \psi$ defines a homomorphism $\alpha_v : S^\psi(\mathbf{Q}_v) \longrightarrow \mathbf{Q}_v^*$, where $\mathbf{Q}_v$ is the completion of $\mathbf{Q}$ with respect to the normalized norm $|| \, ||_v$ attached to v. Put $|\alpha|_v(t) = ||\alpha(t)||_v$, $t \in S^\psi(\mathbf{Q}_v)$. Then $|\alpha|_v$ extends to a homomorphism $|\alpha|_v : P_\psi(\mathbf{Q}_v) \longrightarrow \mathbf{R}^*_{>0} = \{r \in \mathbf{R}, s > 0\}$ which is trivial on $N_P(\mathbf{Q}_v)$. We define $|\alpha| : P_\psi(\mathbf{A}) \longrightarrow \mathbf{R}^*_{>0}$ by $|\alpha|(p) = \prod_v |\alpha|_v(p_v)$ if $p = (\cdots,p_v,\cdots) \in P_\psi(\mathbf{A})$. We use the product formula for the norms $|| \quad ||_v$ and see that $|\alpha|$ is trivial on $P(\mathbf{Q})$. Of course $|\alpha|$ is trivial on all compact subgroups of $P(\mathbf{A})$. We chose a compact and open subgroup $K_f \subset G(\mathbf{A}_f)$ such that $P(\mathbf{A})K_\infty K_f = G(\mathbf{A})$. This is possible since it is locally possible. Now we can extend $|\alpha|$ to a smooth map again denoted by $|\alpha| : G(\mathbf{A}) \longrightarrow \mathbf{R}^*_{>0}$ such that $|\alpha|(au) = |\alpha|(a)$ for all $a \in P(\mathbf{A}), u \in K_\infty K_f$. By construction $|\alpha|$ is a smooth function $|\alpha| : X \longrightarrow \mathbf{R}^*_{>0}$ which does not depend on the choice of K_f. The maps $|\alpha|$ for $\alpha \in \Delta - \psi$ induce a smooth map $p_1 : X \longrightarrow \prod_{\Delta-\psi} \mathbf{R}^*_{>0}$. We put $X(1) := \{x \in X | p_1(x) = 1\}$ and get a decomposition

$$X \xrightarrow{\sim} \left(\prod_{\Delta-\psi} \mathbf{R}^*_{>0}\right) \times X(1).$$

We choose an order $\{\alpha_1, \ldots, \alpha_\ell\} = \Delta$ of the simple roots. Then $\Delta - \psi = \{\alpha_{i_1}, \cdots \alpha_{i_{\ell(P)}}\}$, $\ell(P) = |\Delta - \psi|$, where $i_j > i_k$ for $j > k$. Now $da := d|\alpha_{i_1}| \wedge \cdots \wedge d|\alpha_{i_{\ell(P)}}|$ is a $\ell(P)$–form on X. Put

$$e[P](1) := P(\mathbf{Q})\backslash X(1)$$

and denote by $p_2 : S_P \longrightarrow e[P](1)$ the obvious projection.

Let $f : \prod_{\Delta-\psi} \mathbf{R}^*_{>0} \longrightarrow \mathbf{R}_{>0}$ be a smooth compactly supported function such that $\int f(t_1, \cdots, t_{\ell(P)}) dt_1 \wedge \ldots \wedge dt_{\ell(P)} = 1$ and put for $x \in S_P$

$$\omega_P(x) = f(p_1(x)) da(x).$$

Then ω_P is a smooth $\ell(P)$–form on S_P. Given a compactly supported V–valued differential form φ on $e[P](1)$ then

$$E(\varphi) := \sum_{g \in P(\mathbf{Q})\backslash G(\mathbf{Q})} g^{*-1}(\omega_P \wedge p_2^* \varphi)$$

is a smooth compactly supported form on S_G. The map $\varphi \longmapsto \omega_P \wedge p_2^* \varphi$ induces an isomorphism

$$H_c^{j-\ell(P)}(e[P](1), \tilde{V}) \xrightarrow{\sim} H_c^j(S_P, \tilde{V}).$$

It follows then from [R–Sp: 2.2.] and 2.2 that

$$\mathrm{cor}([\omega_P \wedge p_2^* \varphi]) = [E(\varphi)].$$

2.4 (i) We define

$$H^j(S_G, \tilde{V})_{cc} = \cap_P \ker \mathrm{res}(P, G),$$

where P runs in the set of all proper $\mathbf{Q}$–rational parabolic subgroups of G. Obviously here it suffices to take the intersection over all proper standard maximal parabolic subgroups of G.

Let $e[P]$ be the face determined by P in the Borel–Serre boundary $\partial(\overline{S}_G)$ of the Borel–Serre compactification $\overline{S}_G$ of S_G and suppose that the differential form φ is representing the cohomology class $[\varphi]$. The restriction of $[\varphi] \in H^j(S_G, V)$ to the face $e[P]$ is determined by the constant Fourier coefficient φ^P along the unipotent radical of P, see [Sch: § 4]. Now the form φ is called cuspidal if φ^P is zero for all proper $\mathbf{Q}$–rational parabolic subgroups P. In analogy we call $[\varphi]$ *cohomologically cuspidal* if all $[\varphi^P]$ represent a trivial cohomology class on the face $e[P]$ of the Borel Serre compactification. We use the subscript cc to indicate the the subspace of cohomologically cuspidal classes.

We observe that the image $H^j_!(S_G, \tilde{V}))$ of the cohomology with compact supports $H_c^j(S_G, \tilde{V})$ in $H^j(S_G, \tilde{V})$ is contained in $H^j(S_G, \tilde{V})_{cc}$. The space

$$H^j_!(S_G, \tilde{V})\backslash H^j(S_G, \tilde{V})_{cc}$$

usually is called a space of *ghost classes*.

(ii) Let $V^\vee$ be the contragredient representation to V. Define $\mathcal{C}$ to be the subspace of $H_c^*(S_G, \widetilde{\omega \otimes \tilde{V}^\vee})$ spanned by all $\operatorname{cor}(G,P)(\varphi), \varphi \in H_c^*(S_P, \widetilde{\omega \otimes V^\vee}), P$ proper parabolic in G.

(iii) If $\Gamma \subset G(\mathbf{Q})$ is an arithmetic group and if $H \subset G$ is a subgroup, which is defined over $\mathbf{Q}$, we choose a maximal compact subgroup $K \subset G(\mathbf{R})$ such that $K \cap H(\mathbf{R})$ is maximal compact in $H(\mathbf{R})$. Then the inclusion of symmetric spaces

$$H(\mathbf{R})/K \cap H(\mathbf{R}) =: (X_H)_\infty \hookrightarrow X_\infty = G(\mathbf{R})/K$$

induces a map

$$j : \Gamma \cap H \backslash (X_H)_\infty \to \Gamma \backslash X_\infty .$$

Assume that $\Gamma \cap H \backslash (X_H)_\infty$ is compact. If $d = \dim(X_H)_\infty$ and a closed d–form ω represents $[\omega] \in H^d(\Gamma \backslash X_\infty, \mathbf{C})$, then $\int_{\Gamma \cap H \backslash (X_H)_\infty} j^*\omega$ is defined. This means that $\Gamma \cap H \backslash (X_H)_\infty$ determines a map $H^d(\Gamma \backslash X, \mathbf{C}) \longrightarrow \mathbf{C},$ which is called the *modular symbol* attached to H. We drop the assumption that $H \cap \Gamma \backslash (X_H)_\infty$ is compact and let $[\varphi] \in H_c^i(\Gamma \cap H \backslash (X_H)_\infty, \mathbf{C})$. If $[\varphi]$ is represented by a closed compactly supported i–form φ and $k = \dim\, (X_H)_\infty - i$, then $[\varphi]$ determines a map $H^k(\Gamma \backslash X, \mathbf{C}) \longrightarrow \mathbf{C}$ by

$$[\omega] \longrightarrow \int_{\Gamma \cap H \backslash (X_H)_\infty} \varphi \wedge j^*\omega.$$

As in the introduction we call this map the *modular symbol attached to* $([\varphi], H)$. Similarly for cohomology with coefficients. By Poincaré duality we consider $\operatorname{cor}(P,G)([\varphi])$ as defined in 2.1 (ii) as element of $\operatorname{Hom}_{\mathbf{C}}^\infty(H^*(S_G, \tilde{V}), \mathbf{C})$. Then 2.2 gives the motivation also to call this map a modular symbol. The next result describes the space generated by these modular symbols.

2.5. Theorem. *There is a natural isomorphism of* $G(\mathbf{A}_f)$*–modules*

$$\mathcal{C} \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{C}}^\infty(H^*(S_G, \tilde{V})/H^*(S_G, \tilde{V})_{cc}, \mathbf{C}))$$

which sends $\operatorname{cor}(G,P)([\varphi]) \in \mathcal{C}$ *to the map*

$$(G, \operatorname{cor}(G,P)([\varphi])) : H^*(S_G, V) \to \mathbf{C}$$

given by

$$(G, \operatorname{cor}(G,P)([\varphi]))([\psi]) = \langle \operatorname{cor}(G,P)([\varphi]), [\psi] \rangle_G$$

Here $[\varphi] \in H_c^*(S_P, \widetilde{\omega \otimes V^\vee})$, $[\psi] \in H^*(S_G, V)$, *and* $\langle\ ,\ \rangle_G$ *is the pairing given by Poincaré–duality.*

Proof. By Poincaré–duality we have a non degenerate pairing

$$\langle\ ,\ \rangle_G : H_c^*(S_G, \widetilde{\omega \otimes V^\vee}) \times H^*(S_G, \tilde{V}) \to \mathbf{C}$$

which identifies $H_c^*(S_G, \widetilde{\omega \otimes V^\vee})$ with the smooth dual of $H^*(S_G, \tilde{V})$. Let

$$\mathcal{C}^\perp := \left\{ [\psi] \in H^*(S_G, \tilde{V}) \mid \langle [m], [\psi] \rangle_G = 0 \text{ for all } [m] \in \mathcal{C} \right\}.$$

Since $[m] \in \mathcal{C}$ can be written as a finite sum $[m] = \sum_P \operatorname{cor}(G,P)([\varphi_P])$ where $[\varphi_P] \in H_c^*(S_P, \widetilde{\omega \otimes V^\vee})$ and P runs over all proper parabolic subgroups. We have by 2.2

$$\langle [m], [\psi] \rangle_G = \sum_P \langle \operatorname{cor}(G,P)([\varphi_P]), [\psi] \rangle_G = \sum_P \langle [\varphi_P], \operatorname{res}(P,G)[\psi] \rangle_P.$$

By Poincaré duality on S_P we deduce that $[\psi] \in \mathcal{C}$ implies $\operatorname{res}(P,G)[\psi] = 0$ for all P, i.e. $[\psi] \in H^*(S_G, \tilde{V})_{cc}$. It follows that the map in 2.5. induces an injection

$$\mathcal{C} \longrightarrow \operatorname{Hom}^\infty_{\mathbf{C}}(H^*(S_G, \tilde{V})/H^*(S_G, \tilde{V})_{cc}, \mathbf{C}).$$

Let $\alpha \in \Delta$ and denote by P_α the maximal standard parabolic subgroup of G given by $\{\alpha\} \subset \Delta$. Then we have an inclusion

$$\oplus_\alpha \operatorname{res}_{P_\alpha} : H^*(S_G, \tilde{V})/H^*(S_G, \tilde{V})_{cc} \hookrightarrow \oplus_{\alpha \in \Delta} H^*(S_{P_\alpha}, \tilde{V}),$$

see 2.4 (i). Let $\lambda \in \operatorname{Hom}^\infty_{\mathbf{C}}(H^*(S_G, \tilde{V})/H^*(S_G, \tilde{V})_{cc}, \mathbf{C})$. Then we can extend λ to a smooth map $\lambda' : \oplus_{\alpha \in \Delta} H^*(S_{P_\alpha}, \tilde{V}) \to \mathbf{C}$. By Poincaré–duality on the S_{P_α} there are $[\varphi_\alpha] \in H_c^*(P_\alpha, \widetilde{\omega \otimes V^\vee})$ such that for $[\xi] \in H^*(S_G, V)$

$$\lambda(\xi) = \lambda'([\xi]) = \sum_{\alpha \in \Delta} \langle [\varphi_\alpha], \operatorname{res}(P_\alpha, G)[\xi] \rangle_{P_\alpha}.$$

Then

$$\lambda([\xi]) = \sum_{\alpha \in \Delta} \langle \operatorname{cor}(G, P_\alpha)([\varphi_\alpha]), [\xi] \rangle_G$$

Hence λ is the image of $[m] = \sum_{\alpha \in \Delta} \operatorname{cor}(G, P_\alpha)([\varphi_\alpha]) \in \mathcal{C}$. q.e.d.

Theorem 2.5. means that all cohomology classes with non trivial restriction to faces of the Borel–Serre boundary can be detected by modular symbols in $\mathcal{C}$. This applies in particular to the classes which are constructed as values of Eisenstein series in [Sch].

3 Algebraic Restriction of the Cohomology with compact support and Modular Symbols

Let Q be a $\mathbf{Q}$–rational parabolic subgroup of G. The restriction map to Q in group cohomology induces the *algebraic restriction*

$$\begin{aligned}\operatorname{res}^c(Q,G) : H^{j-\ell}(G(\mathbf{Q}), \operatorname{Hom}(St_G,\ (C^\infty(G(\mathbf{A}_f), V))) \\ \to H^{j-\ell}(Q(\mathbf{Q}), \operatorname{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V))).\end{aligned}$$

In 3.2 we give a topological interpretation of

$$H^{j-\ell}(Q(\mathbf{Q}), \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V))).$$

As main result we show in 3.7 that the algebraic restriction of $\mathrm{cor}(G,P)([\varphi])$ is essentially a sum of modular symbols coming from modular submanifolds of Weyl–group conjugates of L_Q . In 3.16 we discuss the relationship between the algebraic restriction and the usual geometrically defined restriction.

3.1. To fix our notation, we recall some properties of induction. Here P is a parabolic subgroup with $\mathbf{Q}$–rational Levi part L_P and unipotent radical N_P. If no confusion is possible we will drop the subscript P. Let E be a smooth $L(\mathbf{A}_f)$–module.

We denote by $\mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{F}_f)} E$ the set of maps $\varphi : G(\mathbf{A}_f) \to E$ such that for every φ there is an open and compact subgroup K_f of $G(\mathbf{A}_f)$ so that for all $\ell \in L(\mathbf{A}_f), a \in G(\mathbf{A}_f), u \in K_f$ we have $\varphi(\ell a u) = \ell\varphi(a)$. Let $G(\mathbf{A}_f)$ act on φ by right–translation.

The assignment

$$E \to \mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{A}_f)} E$$

induces an exact functor from the category of smooth left $L(\mathbf{A}_f)$–modules to the category of smooth $G(\mathbf{A}_f)$–modules. If E is a $L(\mathbf{Q}) \times L(\mathbf{A}_f)$ module then $\mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{A}_f)} E$ is a $L(\mathbf{Q}) \times G(\mathbf{A}_f)$–module. Here for $\ell \in L(\mathbf{Q})$ the action of ℓ on $\varphi \in \mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{A}_f)} E$ is defined by ${}^{\ell}\varphi(a) = \ell\varphi(a)$. We will apply this functor to the module

$$E := C^\infty(L(\mathbf{A}_f), V),$$

where $\ell \times b \in L(\mathbf{Q}) \times L(\mathbf{A}_f)$ acts on $\psi \in C^\infty(L(\mathbf{A}_f), V)$ by

$$(\ell, b)\varphi(a) = \ell\psi(\ell^{-1}ab), a \in L(\mathbf{A}_f)\ .$$

3.2. Lemma *Let ℓ be the $\mathbf{Q}$–rank of G and P, L, N be as above.*

(i) *The $G(\mathbf{A}_f)$–modules $H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))$ and $H^{j-\ell}(L(\mathbf{Q}),\ \mathrm{Hom}(St_L, C^\infty(G(\mathbf{A}_f), V)))$ are isomorphic.*

(ii) *The $G(\mathbf{A}_f)$ –modules $H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))$ and $\mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^j(S_L^\natural, \tilde{V})$ are isomorphic.*

Proof: Since

$$St_G \xrightarrow{\sim} \mathbf{Z}[P(\mathbf{Q})] \otimes_{\mathbf{Z}[L(\mathbf{Q})]} St_L$$

as $P(\mathbf{Q})$–modules, see 1.5, the first claim follows from Shapiro's Lemma for group cohomomology.

To prove the second claim, we investigate $H_c^j(S_L^\natural, \tilde{V})$. The connected component A_L of the set of real points of the maximal $\mathbf{Q}$–split central torus of L

can be written as $A_G \times A'_P$, where $A'_P \xrightarrow{\sim} \mathbf{R}^{*\ell(P)}_{>0}$ and where $\ell - \ell(P)$ is the semisimple $\mathbf{Q}$–rank of L. Since A_G acts trivially on $X_{L\infty} \subset X_\infty$ we get a fibration $f : S^\natural_L \longrightarrow S_L$ with fibres A'_P. Hence there is a spectral sequence

$$H^i_c(S_L, R^j f_! \tilde{V}) \Longrightarrow H^{i+j}_c(S^\natural_L, \tilde{V}) .$$

Since A'_P is isomorphic to $\mathbf{R}^{\ell(P)}$ we get for $j < \ell(P)$ $R^j f_! \tilde{V} = 0$. With respect to a choice of an orientation on A'_P we see $R^{\ell(P)} f_! \tilde{V} \xrightarrow{\sim} \tilde{V}$ where now $\tilde{V}$ is the locally constant sheaf on S_L determined by the representation of L on V. Hence

$$H^{j-\ell(P)}_c(S_L, \tilde{V}) \xrightarrow{\sim} H^j_c(S^\natural_L, \tilde{V}).$$

Now we apply Borel–Serre duality for the reductive group L, see 1.6, and get

$$H^{j-\ell(P)}_c(S_L, \tilde{V}) \xrightarrow{\sim} H^{j-\ell}(L(\mathbf{Q}), \mathrm{Hom}(St_L, C^\infty(L(\mathbf{A}_f), V))) .$$

By induction in stages we have

$$C^\infty(G(\mathbf{A}_f), V) = C^\infty_{L(\mathbf{A}_f)}(G(\mathbf{A}_f), C^\infty(L(\mathbf{A}_f), V)) = \mathrm{Ind}^{G(\mathbf{A}_f)}_{L(\mathbf{A}_f)} C^\infty(L(\mathbf{A}_f), V).$$

Since $\mathrm{Ind}^{G(\mathbf{A}_f)}_{L(\mathbf{A}_f)}$ is an exact functor, the second claim holds. q.e.d.

3.3. Remarks. (i) The isomorphism in 3.2 (ii) depends on the choice of an orientation on A'_P, i.e. it is unique up to a sign. Since P is a standard parabolic subgroup this sign can be fixed by a choice of the order of the simple $\mathbf{Q}$–roots, see 2.3.
(ii) If Q is an arbitrary parabolic subgroup, there exists a standard parabolic subgroup P and a $g \in G(\mathbf{Q})$ such that $Q = gPg^{-1}$. We put $L_Q := gL_Pg^{-1}$. Then $A'_Q = gA'_Pg^{-1}$ and g is unique up to $p \in P(\mathbf{Q})$. But $P(\mathbf{R})$ fixes the orientation on A_P. Hence the isomorphism in 3.3 (ii) is uniquely determined by a choice of the order of the simple $\mathbf{Q}$–roots of G. We will apply this to the parabolic group ${}^{w^{-1}}Q, w \in W$, with θ–stable Levi–part $L_{{}^{w^{-1}}Q}$.
(iii) With 3.2 (ii) and 3.16 the algebraic restriction map $\mathrm{res}^c(P, G)$ can be interpreted as an *Oda*–restriction map to the modular manifold $S^\natural_P$, see [C–V].

3.4. (i) Let Q be a proper standard $\mathbf{Q}$–rational parabolic subgroup of G. Then the closed embedding

$$P \cap L_{{}^{w^{-1}}Q}(\mathbf{Q}) \backslash X_{(L_{{}^{w^{-1}}Q})} \hookrightarrow S_P$$

induces a map of cohomology with compact supports. We will describe a version of this map directly in group–theoretical terms, i.e. we define a natural map of complexes of $G(\mathbf{A}_f)$–modules

$$\begin{aligned} \mathrm{res}(P \cap L_{{}^{w^{-1}}Q}, P) : \mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*) \\ \to \mathrm{Hom}_{P \cap L_{{}^{w^{-1}}Q}(\mathbf{Q})}(St_{P \cap L_{{}^{w^{-1}}Q}}, A^*), \end{aligned}$$

where $0 \to C^\infty(G(\mathbf{A}_f), V) \to A^*$ is a resolution by $G(\mathbf{Q}) \times G(\mathbf{A}_f)$ modules, which are acyclic as $G(\mathbf{Q})$–modules and smooth as $G(\mathbf{A}_f)$–modules.

To simplify the notation we write Q in this section instead of ${}^{w^{-1}}Q$, i.e. Q is not necessarily a standard parabolic subgroup but one which contains a fixed $\mathbf{Q}$ split torus S, see 1.2.

For the translation of this map to the topological setting see also remark 3.16 (i).

(ii) Now $H_c^j(S_P, \tilde{V})$ is computed as $j-\ell$ the cohomology of the complex $\mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*)$. Let $N_P(\mathbf{Q})$ be the set of $\mathbf{Q}$–rational points of the unipotent radical N_P of P. Denote by $(A^*)^{N_P}$ the set of $N_P(\mathbf{Q})$–invariants of A^*. Since St_P is a trivial $N_P(\mathbf{Q})$–module we have the isomorphism

$$\mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*) \xrightarrow{\sim} \mathrm{Hom}_{L_P(\mathbf{Q})}(St_{L_P}, (A^*)^{N_P}).$$

We use that as P–modules $St_G \xrightarrow{\sim} \mathbf{Z}[P(\mathbf{Q})] \otimes_{\mathbf{Z}[L_P(\mathbf{Q})]} St_{L_P}$, see 1.5, to get an isomorphism

$$\mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*) \xrightarrow{\sim} \mathrm{Hom}_{P(\mathbf{Q})}(St_G, (A^*)^{N_P}))$$

given by

$$\varphi \longmapsto \varphi \circ s(P,G).$$

We have the $L_Q(\mathbf{Q})$–linear map

$$\sigma(G,Q) : St_{L_Q} \longrightarrow St_G$$

and the $L_{P\cap L_Q}(\mathbf{Q})$–linear map

$$\sigma(L_Q, P\cap L_Q) : St_{L_{P\cap L_Q}} \longrightarrow St_{L_Q},$$

where $L_{P\cap L_Q} = L_P \cap L_Q$ is a Levi part of the parabolic subgroup $P\cap L_Q$. For $\varphi \in \mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*)$ we consider the map

$$\psi := \varphi \circ s(P,G) \circ \sigma(G,Q) \circ \sigma(L_Q, P\cap L_Q)$$

in $\mathrm{Hom}_{L_P(\mathbf{Q})\cap L_Q(\mathbf{Q})}(St_{L_P(\mathbf{Q})\cap L_Q(\mathbf{Q})}, (A^*)^{N_P})$. The unipotent radical of the parabolic subgroup $P\cap L_Q$ is contained N_P, see [H–Ch: Lemma 2, a)]. Hence ψ is contained in

$$\mathrm{Hom}_{(P\cap L_Q)(\mathbf{Q})}(St_{P\cap L_Q}, A^*)\,.$$

We now replace Q by ${}^{w^{-1}}Q$, where Q is a standard parabolic subgroup and we see that $\mathrm{res}(P\cap L_{w^{-1}Q}, P)$ is defined.

(iii) The map induced by $\mathrm{res}(P\cap L_{w^{-1}Q}, P)$ in group cohomology is denoted by $\mathrm{res}^c(P\cap L_{w^{-1}Q}, P)$. By the considerations in 3.2. its target space is identified with

$$\mathrm{Ind}_{L_{w^{-1}Q}(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^*((P\cap L_{w^{-1}Q})(\mathbf{Q})\backslash X_{L_{w^{-1}Q}}, \tilde{V})$$

3.5. We proceed as in 2.1 and using the map

$$\varphi \longmapsto \sum_{g \in L_{w^{-1}Q}(\mathbf{Q}) \backslash (P(\mathbf{Q}) \cap L_{w^{-1}Q}(\mathbf{Q})} {}^{g}(\varphi \circ s(P \cap L_{w^{-1}Q}, L_{w^{-1}Q}))$$

for $\varphi \in \operatorname{Hom}_{P(\mathbf{Q}) \cap L_{w^{-1}Q}(\mathbf{Q})}(St_{P \cap L_{w^{-1}Q}}, A^*)$ we define a map $\operatorname{cor}(L_{w^{-1}Q}, P \cap L_{w^{-1}Q})$ from

$$H^{j-\ell}(P(\mathbf{Q}) \cap L_{w^{-1}Q}(\mathbf{Q}),\ \operatorname{Hom}(St_{P \cap L_{w^{-1}Q}}, C^\infty(G(\mathbf{A}_f), V)))$$

to

$$H^{j-\ell}(L_{w^{-1}Q}(\mathbf{Q}),\ \operatorname{Hom}(St_{L_{w^{-1}Q}}, C^\infty(G(\mathbf{A}_f), V))).$$

We use 3.2 and see that the target space of $\operatorname{cor}(L_{w^{-1}Q}, P \cap L_{w^{-1}Q})$ is identified with

$$H^{j-\ell}({}^{w^{-1}}Q(\mathbf{Q}),\ \operatorname{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V))).$$

3.6. For $w \in W$ let n_w represent w in the rational points of the normalizer $N(S)$ of the torus S. Let $\varphi \in \operatorname{Hom}_{w^{-1}Q(\mathbf{Q})}(St_G, A^*)$ and define

$$T(w)\varphi \in \operatorname{Hom}_{Q(\mathbf{Q})}(St_G, A^*)$$

by

$$(T(w)\varphi)(s) = n_w \varphi(n_w^{-1} s) =: {}^{w}\varphi(s)\,.$$

Here $s \in St_G$. We notice that ${}^{w}\varphi = T(w)\varphi$ does not depend on the choice of n_w. We use 3.2 and see that $T(w)$ defines a map

$$T(w) : \operatorname{Ind}_{L_{w^{-1}Q}(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^*(S^\natural_{L_{w^{-1}Q}}, \tilde{V}) \longrightarrow$$
$$\longrightarrow \operatorname{Ind}_{L_Q(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^*(S^\natural_{L_Q}, \tilde{V})\,.$$

Finally we can formulate the main result of this chapter

3.7. Theorem. *Let P and Q be standard $\mathbf{Q}$–rational proper parabolic subgroups of G. Then*

$$\operatorname{res}^c(Q, G) \circ \operatorname{cor}(G, P)$$

and

$$\sum_{w \in W_Q \backslash W / W_P} T(w) \circ \operatorname{cor}(L_{w^{-1}Q}, P \cap L_{w^{-1}Q}) \circ \operatorname{res}^c(P \cap L_{w^{-1}Q}, P)$$

are equal in $\operatorname{Hom}(H_c^*(S_P, \tilde{V}),\ \operatorname{Ind}_{L_Q(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^*(S^\natural_Q, \tilde{V}))$.

Proof. Let $[\varphi] \in H_c^j(S_P, \tilde{V})$ be represented by $\varphi \in \mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^{j-\ell})$. We use 1.5, 2.1 and the definition of res^c to deduce that $\mathrm{res}^c(Q,P) \circ \mathrm{cor}(G,P)([\varphi])$ is represented by

$$\psi := \sum_{g \in G(\mathbf{Q})/P(\mathbf{Q})} {}^g(\varphi \circ s(P,G)) \circ \sigma(G,Q)) \, .$$

Since this map is $L_Q(\mathbf{Q})$–linear and since St_{L_Q} is generated by τ_Q as $L_Q(\mathbf{Q})$–module the map ψ is determined by its value at τ_Q. Since $\sigma(G,Q)(\tau_Q) = \tau_G$ we have to compute

$$\sum_{G(\mathbf{Q})/P(\mathbf{Q})} {}^g(\varphi \circ s(P,G))(\tau_G).$$

Now

$${}^g(\varphi \circ s(P,G))(\tau_G) = g\varphi(s(P,G)(g^{-1}\tau_G)) = 0$$

unless $g^{-1} = pn_u^{-1}$ where $p \in P$ and $u^{-1} \in W$, see [Re: Lemma, p. 310]. If $g^{-1} = pn_u^{-1}$ then $g = n_u p^{-1}$ and since φ and $s(P,G)$ are P–linear, since $s(P,G)(\tau_G) = \tau_P$ and $u\tau_G = (-1)^{|u|}\tau_G$ we get

$$g\varphi(s(P,G)(g^{-1}\tau_G) = (-1)^{|u|} n_u \varphi(\tau_P) \, .$$

Hence

$$\psi(\tau_Q) = \sum_{W/W_P} {}^w(\varphi \circ s(P,G))(\tau_G).$$

Next we investigate

$$[\phi] := T(w) \circ \mathrm{cor}(L_{{}^{w^{-1}}Q}, P \cap L_{{}^{w^{-1}}Q}) \circ \mathrm{res}^c(P \cap L_{{}^{w^{-1}}Q}, P)([\varphi]).$$

For $\varphi \in \mathrm{Hom}_{P(\mathbf{Q})}(St_P, A^*)$ we put

$$\psi := \varphi \circ s(P,G) \circ \sigma(G, L_{{}^{w^{-1}}Q}) \circ \sigma(L_{{}^{w^{-1}}Q}, P \cap L_{{}^{w^{-1}}Q}) \, .$$

We use 3.2 ii, 3.4, 3.5 and 3.6 to deduce that $[\phi]$ is represented by

$${}^w\left(\sum_{q \in L_{{}^{w^{-1}}Q}(\mathbf{Q})/P(\mathbf{Q}) \cap L_{{}^{w^{-1}}Q}(\mathbf{Q})} {}^q(\psi \circ s(P \cap L_{{}^{w^{-1}}Q}, L_{{}^{w^{-1}}Q})) \circ s({}^{w^{-1}}Q, G) \right)$$

in $\mathrm{Hom}_{Q(\mathbf{Q})}(St_G, A^*)$. Since this map is $Q(\mathbf{Q})$–linear and since St_G is generated as $Q(\mathbf{Q})$–module by τ_G it suffices to compute the value of the map at τ_G. We have $w^{-1}\tau_G = n_w^{-1}\tau_G = (-1)^{|w|}\tau_G$ and $s({}^{w^{-1}}Q, G)(\tau_G) = \tau_{{}^{w^{-1}}Q}$ and thus we deduce that $\phi(\tau_G)$ is equal to

$$(-1)^{|w|} n_w \sum_{q \in L_{{}^{w^{-1}}Q}(\mathbf{Q})/P(\mathbf{Q}) \cap L_{{}^{w^{-1}}Q}(\mathbf{Q})} {}^q(\psi \circ s(P \cap L_{{}^{w^{-1}}Q}, L_{{}^{w^{-1}}Q}))(\tau_{{}^{w^{-1}}Q}).$$

We use again [Re: p. 310] and see that ${}^q(\psi\circ s(P\cap L_{w^{-1}Q}, L_{w^{-1}Q}))(\tau_{w^{-1}Q}) = 0$ unless $q^{-1} = pn_v^{-1}$, where $p \in P(\mathbf{Q}) \cap L_{w^{-1}Q}(\mathbf{Q})$ and where n_v represents $v \in W_{w^{-1}Q}$. If $q^{-1} = pn_v^{-1}$ then

$$s(P\cap L_{w^{-1}Q}, L_{w^{-1}Q})(q^{-1}\tau_{w^{-1}Q}) = (-1)^{|v|} n_v p(\tau_{P\cap L_{w^{-1}Q}})$$

and we get

$${}^q(\psi\circ s(P\cap L_{w^{-1}Q}, L_{w^{-1}Q}))(\tau_{w^{-1}Q}) = (-1)^{|v|} n_v \psi(\tau_{P\cap L({}^{w^{-1}}Q)}).$$

Hence

$$\begin{aligned}\phi(\tau_G) &= (-1)^{|w|} n_w \sum_{v\in W_{w^{-1}Q}} (-1)^{|v|} n_v \varphi(\tau_P)\\ &= \sum_{v\in W_Q} (-1)^{|v|}(-1)^{|w|} n_v n_w \varphi(\tau_P)\\ &= \sum_{v\in W_Q} {}^{vw}(\varphi\circ s(P,G))(\tau_G).\end{aligned}$$

Since

$$W/W_P = \bigcup_{w\in W_Q\backslash W/W_P} W_Q w$$

as disjoint union our claim holds. q.e.d.

3.8. Remarks. (i) The cohomology $H^*(P(\mathbf{Q}), \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))$ is analogous to the cohomology $H^*(P(\mathbf{Q}), C^\infty(G(\mathbf{A}_f), V))$, which can be identified with the cohomology of the face $e[P]$ attached to P of the Borel–Serre boundary of S_P. We view $\mathrm{cor}(G,P)(\varphi)$ as a version of the Eisenstein–series construction. Then the formula 3.7. is similar to the formula for the restriction of Eisenstein classes to faces of the boundary.

(ii) The formula in 3.7 tells, that the restriction to Q of the modular symbol $(G, \mathrm{cor}(G,P)([\varphi]))$ is a sum over $W_Q\backslash W/W_P$ of conjugates of modular symbols for the groups $L_{w^{-1}Q}, w \in W_Q\backslash W/W_P$. The definition of the involved maps and the formula 3.7 hold over arbitrary rings. However the topological interpretation involving $\mathrm{Ind}_{L_Q(\mathbf{A}_f)}^{G(\mathbf{A}_f)}$ requires that the ring over which the representation V is defined contains $\mathbf{Q}$.

(iii) We have worked throughout in the group–cohomological setting with coefficients $C^\infty(G(\mathbf{A}_f), V)$. This has the advantage that the $G(\mathbf{A}_f)$–module structure is gotten for free and more important that thanks to the occurrence of the Steinberg representation we can check 3.7 on the level on complexes. The complex $\mathrm{Hom}_{P(\mathbf{Q})}(St_G, A^*)$ contains information on the vanishing of restrictions to $S^\natural_{L_Q}$ of cocycles which in an argument with compactly supported differential forms is not easily accessible. For $G = GL_2|\mathbf{Q}$ a look of the computation in [R–Sp I: § 4] will explain the technical problem.

3.9. (i) Let P be a standard parabolic subgroup. We saw in 3.2 that

$$H^{j-\ell}(P(\mathbf{Q}), \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))$$

and

$$H^{j-\ell}(L_P(\mathbf{Q}),\ \mathrm{Hom}(St_{L_P}, C^\infty(G(\mathbf{A}_f), V)))$$

are isomorphic. If $R \subset L_P$ is a proper $\mathbf{Q}$–rational parabolic subgroup we have the restriction map

$$\begin{aligned}\mathrm{res}^c(R, L_P) : H^{j-\ell}(L_P(\mathbf{Q}),\ &\mathrm{Hom}(St_{L_P}, C^\infty(G(\mathbf{A}_f), V)))\\ &\longrightarrow H^{j-\ell}(R(\mathbf{Q}), \mathrm{Hom}(St_{L_P}, C^\infty(G(\mathbf{A}_f), V))).\end{aligned}$$

As in section 2.4 we define

$$H^{j-\ell}(L_P(\mathbf{Q}),\ \mathrm{Hom}(St_{L_P}, C^\infty(G(\mathbf{A}_f), V)))_{cc} := \bigcap_R \ker(\mathrm{res}(R, L_P))$$

where R runs in the set of proper $\mathbf{Q}$–rational parabolic subgroups of L_P .

(ii) We define the restriction

$$\begin{aligned}\mathrm{res}^c(L_P, P) : H^{j-\ell}(P(\mathbf{Q}), &\mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V)))\\ &\to H^{j-\ell}(L_P(\mathbf{Q}),\ \mathrm{Hom}(St_{L_P}, C^\infty(G(\mathbf{A}_f), V))).\end{aligned}$$

by the map

$$\begin{aligned}H^{j-\ell}(P(\mathbf{Q}), &\mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V)))\\ &\longrightarrow H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_G, C^\infty(G(\mathbf{A}_f), V)))\end{aligned}$$

induced by $s(P, G)$. and 3.2(i) Denote by

$$H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V))_{cc}$$

the preimage under $\mathrm{res}^c(L_P, P)$ of $H^{j-\ell}(L_P(\mathbf{Q}), \mathrm{Hom}(St_{L_P}, C^\infty(G(\mathbf{A}_f), V)))_{cc}$. This space corresponds with respect to the isomorphism

$$H_c^j(S_P, \tilde{V}) = H^{j-\ell}(P(\mathbf{Q}), \mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V)))$$

to a $G(\mathbf{A}_f)$–submodule $H_c^j(S_P, \tilde{V})_{cc}$ which is called the submodule of *cohomologically cuspidal* classes.

(iii) Now $R' := RN_P \subset P$ is a parabolic subgroup contained in P, and

$$\mathrm{Hom}_{R(\mathbf{Q})}(St_{L_P}, (A^*)^{N_P}) = \mathrm{Hom}_{R'(\mathbf{Q})}(St_P, A^*).$$

The map $R \longrightarrow R'$ defines a bijection between the set of proper $\mathbf{Q}$–rational parabolic subgroups of L_P with the set of proper $\mathbf{Q}$–rational parabolic subgroups contained in P , see [H–Ch: Lemma 2]. If now $R' \subset P$ we have a restriction map

$$\begin{aligned}\mathrm{res}(R', P) : H^{j-\ell}(P(\mathbf{Q}),\ &\mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V)))\\ &\longrightarrow H^{j-\ell}(R'(\mathbf{Q}),\ \mathrm{Hom}(St_P, C^\infty(G(\mathbf{A}_f), V)))\end{aligned}$$

and one can see

$$H^j_c(S_P,\tilde V)_{cc} = \bigcap_{R'\subset P} \ker(\mathrm{res}^c(R',P))\ ,$$

where R' runs in the set of all proper $\mathbf{Q}$–rational parabolic subgroups contained in P. Hence we see, that $H^j_c(S_P,\tilde V)_{cc}$ is defined in complete analogy to $H^j(S_G,\tilde V)_{cc}$.

3.10. (i) Let N_B be the unipotent radical of the minimal Borel subgroup $B\subset P$. Then $P\cap L_{{}^{w^{-1}}Q}N_B =: P'\subset P$ is a parabolic subgroup of P and since $N_B\subset P$ we have $P'\cap L_{{}^{w^{-1}}Q} = P\cap L_{{}^{w^{-1}}Q}$. Using 3.4., 3.5. and 3.9. (iii) we deduce the equality of

$$\mathrm{cor}(L_{{}^{w^{-1}}Q}, P\cap L_{{}^{w^{-1}}Q})\circ \mathrm{res}^c(P\cap L_{{}^{w^{-1}}Q},P)$$

and

$$\mathrm{cor}(L_{{}^{w^{-1}}Q}, P'\cap L_{{}^{w^{-1}}Q})\circ \mathrm{res}^c(P'\cap L_{{}^{w^{-1}}Q},P')\circ \mathrm{res}^c(P',P).$$

(ii) Let P,Q be standard parbolic subgroups of the same rank, i.e. $\dim A_P = \dim A_Q$. Then it follows from [H–Ch: Lemma 29] that $P\cap L({}^{w^{-1}}Q)$ is a proper parabolic subgroup of $L({}^{w^{-1}}Q)$ unless ${}^{w^{-1}}A_Q = A_P$.

If ${}^{w^{-1}}A_Q = A_P$ then P and Q are called *associate* parabolic subgroups. If $P=Q$ it follows that $w\in W(A_P)$, where $W(A_P)$ is the subgroup of elements of W which can be represented in the $\mathbf{Q}$–rational points $N(A_P)(\mathbf{Q})$ of the normalizer $N(A_P)$ of A_P in G .

3.11. In 3.7 we have defined an isomorphism

$$T(w) : \mathrm{Hom}_{{}^{w^{-1}}P(\mathbf{Q})}(St_G, A^*) \xrightarrow{\sim} \mathrm{Hom}_{P(\mathbf{Q})}(St_G, A^*)$$

Assume now that $w\in W(A_P)$. If $L = L_P$ then ${}^wL = L$ and $St_G \xrightarrow{\sim} \mathbf{Z}[{}^wP(\mathbf{Q})]\otimes_{\mathbf{Z}[L(\mathbf{Q})]} St_L$. Hence

$$\mathrm{Hom}_{{}^{w^{-1}}P(\mathbf{Q})}(St_G, A^*) \cong \mathrm{Hom}_{L(\mathbf{Q})}(St_L, A^*)$$

and we see that $T(w)$ corresponds to an isomorphism $\mathrm{Hom}_{L(\mathbf{Q})}(St_L,A^*)\longrightarrow \mathrm{Hom}_{L(\mathbf{Q})}(St_L,A^*)$, where

$$\varphi\to T(\omega)\varphi = (-1)^{|w|}\, {}^w\varphi\ .$$

Recall that here ${}^w\varphi(s)$ is defined by $n_w\varphi(n_w^{-1}s)$ for $s\in St_L, n_w\in N(A_P)(\mathbf{Q})$ and that $n_w : St_L\to St_L$ maps τ_L to τ_L and if $\ell\in L(\mathbf{Q})$ it maps $\ell\cdot\tau_L$ to $n_w\ell n_w^{-1}\cdot\tau_L$.

Let now $\varphi\in \mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H^j_c(S_L^\natural,\tilde V)$. Then

$$\varphi : G(\mathbf{A}_f)\longrightarrow H^j_c(S_L^\natural,\tilde V)$$

is an $L(\mathbf{A}_f)$–equivariant map. Let $a\in G(\mathbf{A}_f)$. Then $\varphi(a)\in H^j_c(S_L^\natural,\tilde V)$ and $({}^w\varphi)(a) = n_w(\varphi(n_w^{-1}a))$ is well defined, where n_w denotes the map

$$H^j_c(S_L^\natural,\tilde V)\longrightarrow H^j_c(S_L^\natural,\tilde V)$$

given by conjugation with n_w . Hence we see:

3.12. Corollary. *If* $[\varphi] \in H^j_c(S_P, \tilde{V})_{cc}$ *then*

$$\operatorname{res}^c(P,G) \circ \operatorname{cor}(G,P)([\varphi]) = \sum_{w \in W(A_P)} (-1)^{|w|}\, {}^w(\operatorname{res}^c(L_P,P)([\varphi])).$$

We denote by *sign* the 1–dimensional representation of $W(A_P)$, where $w \in W(A_P)$ acts by multiplication with $(-1)^{|w|}$. Then we get:

3.13. Corollary. *The* $G(\mathbf{A}_f)$*–modules*

$$\operatorname{res}^c(P,G) \circ \ \operatorname{cor}(G,P)(H^*_c(S_P,\tilde{V})_{cc})$$

and

$$((\operatorname{res}^c(L_P,P)(H^*_c(S_P,\tilde{V})_{cc})) \otimes \mathit{sign})^{W(A_P)}$$

coincide.

3.14. Remark. The following considerations will make the content of 3.13 more transparent: We investigate the map

$$\begin{aligned}\operatorname{res}^c(L_P,P) : H^j(P(\mathbf{Q}),\ \operatorname{Hom}(St_P, C^\infty(G(\mathbf{A}_f),V))) \\ \to H^j(P(\mathbf{Q}),\ \operatorname{Hom}(St_G, C^\infty(G(\mathbf{A}_f),V))).\end{aligned}$$

Since $C^\infty(G(\mathbf{A}_f),V) = \operatorname{Ind}^{G(\mathbf{A}_f)}_{P(\mathbf{A}_f)} C^\infty(P(\mathbf{A}_f),V)$ and $\operatorname{Ind}^{G(\mathbf{A}_f)}_{P(\mathbf{A}_f)}$ is an exact functor we only have to consider the map

$$\begin{aligned}\operatorname{res}^c(L_P,P)_1 : H^j(P(\mathbf{Q}),\ \operatorname{Hom}(St_P, C^\infty(P(\mathbf{A}_f),V))) \\ \longrightarrow H^j(P(\mathbf{Q}), \operatorname{Hom}(St_G, C^\infty(P(\mathbf{A}_f),V))).\end{aligned}$$

Using the Hochschild–Serre spectral sequence we get for $j = r+s$ maps

$$\begin{aligned}H^r(L(\mathbf{Q}), H^s(N_P(\mathbf{Q}), \operatorname{Hom}(St_P, C^\infty(P(\mathbf{A}_f),V)))) \\ \to H^r(L_P(\mathbf{Q}), H^s(N_P(\mathbf{Q}),\ \operatorname{Hom}(St_G, C^\infty(P(\mathbf{A}_f),V)))).\end{aligned}$$

Since $St_G \cong \mathbf{Z}[P(\mathbf{Q})] \otimes_{\mathbf{Z}[L_P(\mathbf{Q})]} St_P$ is an induced $N_P(\mathbf{Q})$–module

$$H^s(N_P(\mathbf{Q}),\ \operatorname{Hom}(St_G, C^\infty(P(\mathbf{A}_f),V))) = 0$$

for $s > 0$ and

$$H^0(N_P(\mathbf{Q}), \operatorname{Hom}(St_G, W) \cong \operatorname{Hom}(St_L, C^\infty(P(\mathbf{A}_f),V))).$$

Moreover since $N_P(\mathbf{Q})$ acts trivially on St_P

$$H^0(N_P(\mathbf{Q}), \mathrm{Hom}(St_P, C^\infty(P(\mathbf{A}_f), V)))) \cong \mathrm{Hom}(St_L, C^\infty(P(\mathbf{A}_f), V)^{N_P(\mathbf{Q})}).$$

We assume now that $V = \mathbf{C}$ is the trivial representation. Since $N_P(\mathbf{Q})$ is dense in $N_P(\mathbf{A}_f)$ we see that

$$C^\infty(P(\mathbf{A}_f), \mathbf{C})^{N_P(\mathbf{Q})} = C^\infty_{N_P(\mathbf{A}_f)}(P(\mathbf{A}_f), \mathbf{C}) ,$$

which is identified with $C^\infty(L_P(\mathbf{A}_f), \mathbf{C})$, considered as trivial $N_P(\mathbf{Q}) \times N_P(\mathbf{A}_f)$–module. Hence $\mathrm{res}^c(L_P, P)$ is surjective if $V = \mathbf{C}$.

Let now A'_P be as in the proof of 3.4. The natural fibration $S^\natural_{L_P} \longrightarrow S_{L_P}$ is $W(A_P)$ –equivariant with fibre A'_P and $w \in W(A_P)$ acts by multiplication with $(-1)^{|w|}$ on $H_c^{\ell(P)}(A'_P, \tilde{\mathbf{C}})$. Hence we get

3.15. Corollary. *Let $V = \mathbf{C}$ be the trivial representation. Then*

$$\mathrm{res}^c(P, G) \circ \mathrm{cor}(G, P) : H_c^*(S_P, \tilde{\mathbf{C}})_{cc} \to \left(\mathrm{Ind}_{P(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^{*-\ell(P)}(S_{L_P}, \tilde{\mathbf{C}})_{cc}\right)^{W(A_P)}$$

is surjective.

3.16. Remark. We have used throughout the group theoretical description of the maps between the relevant cohomology groups. We now add some remarks on the translation to the topological description of these maps. Full details for this will appear elsewhere.

(i) Let Q be a proper $\mathbf{Q}$–rational θ–stable parabolic subgroup. We assume that K_f is an open and compact subgroup. Then the inclusion $L_Q \subset G$ induces a proper and closed map

$$j : P(\mathbf{Q}) \cap L_Q(\mathbf{Q}) \backslash X_{L_Q} / K_f \cap L_Q(\mathbf{A}_f) \to S_P / K_f ,$$

see [A: 2.7]. Hence there is an induced map

$$j^* : H_c^i(S_P/K_f, \tilde{V}) \to H_c^i(P(\mathbf{Q}) \cap L_Q(\mathbf{Q}) \backslash X_{L_Q} / K_f \cap L_Q(\mathbf{A}_f), \tilde{V})$$

and a map

$$\phi : H_c^i(S_P, \tilde{V})^{K_f} \to \left(\mathrm{Ind}_{L(\mathbf{A}_f)}^{G(\mathbf{A}_f)} H_c^i\left(P(\mathbf{Q}) \cap L_Q(\mathbf{Q}) \backslash X_{L_Q}, \tilde{V}\right)\right)^{K_f}$$

so that

$$\phi([\varphi]) : G(\mathbf{A}_f) \to H_c^i(P(\mathbf{Q}) \cap L_Q(\mathbf{Q}) \backslash X_{L_Q}, \tilde{V}),$$

is defined by $\phi([\varphi])(a) = j^*(R(a)[\varphi])$ where $R(a)$ denotes the map induced by right translation on S_P . It can be shown that this map is the one induced by $\mathrm{res}^c(P \cap L_Q, P)$ as defined in 3.4 (iii).

(ii) Let ψ be the family of supports on S_P which are compact modulo $G(\mathbf{Q})$. Then we have an obvious restriction map

$$H_c^j(S_P,\tilde V)\to H_\psi^j(S_P,\tilde V):=\varinjlim_{K_f} H_\psi^j(S_P/K_f,\tilde V)$$

which can be identified with the map

$$\begin{aligned}\mathrm{res}^c(L_P,P): H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_P,C^\infty(G(\mathbf{A}_f),V)))\\ \to H^{j-\ell}(P(\mathbf{Q}),\mathrm{Hom}(St_G,C^\infty(G(\mathbf{A}_f),V))).\end{aligned}$$

The isomorphism

$$H_\psi^*(S_P,\tilde V)\xrightarrow{\sim} H^{j-\ell}(P(\mathbf{Q}),\ \mathrm{Hom}(St_G,C^\infty(G(\mathbf{A}_f),V)))$$

can be seen as in [Ro 1: 4.4].

3.17. As an application of 3.13 and 3.15 we explain a result of Ash and Borel which roughly says that the fundamental class of a Levi–factor L_P is a nontrivial modular symbol, see [A–B: Thm. 2.5].

We observe $H_c^j(S^\natural_{L_P},\tilde V)=0$ if $j>\dim(X_{L_P})_\infty$. Hence if $d=\dim X_\infty$ then

$$H_c^d(S_G,\tilde V)_{cc}=H_c^d(S_G,\tilde V)\,.$$

Similarly we get

$$H_c^r(S^\natural_{L_P},\tilde{\mathbf{C}})_{cc}=H_c^r(S^\natural_{L_P},\tilde{\mathbf{C}})$$

if $r=\dim(X_{L_P})_\infty$.

Let K_f be such that all Γ_i are torsionfree, $\omega_G(\gamma)=1$ for all $\gamma\in\Gamma_i$ and all $\gamma\in\Gamma_{P,j}$ act orientation preserving on $N_P(\mathbf{R})$ by conjugation, see 1.6 for notation.

Consider $1\neq w\in W(A_P)\subset W/W_P\subset G(\mathbf{Q})/P(\mathbf{Q})$ and let $n_w\in N_{L_P}(\mathbf{Q})$ represent w. Then $n_w\notin P(\mathbf{A}_f)$. Since $P(\mathbf{A}_f)$ is closed in $G(\mathbf{A}_f)$ we choose in addition K_f such that $n_wK_f\notin P(\mathbf{A}_f)$ for all $n_w, w\neq 1$. Then $X_\infty\times wP(\mathbf{Q})K_f\cap X_\infty\times P(\mathbf{Q})K_f=\emptyset$ if $1\neq w\in W(A_P)$. If now $[\psi]\in H_c^r(S_P,\tilde{\mathbf{C}})^{K_f}$ is represented by a $P(\mathbf{Q})-$ invariant r–form $\psi\in\Omega^r(X_\infty)\otimes C^\infty(G(\mathbf{A}_f),\mathbf{C})^{K_f}$ with support in $X_\infty\times P(\mathbf{Q})K_f$ then with notation from 3.16.(i)

$$\phi(\mathrm{res}^c(P,G)\circ\mathrm{cor}(G,P)([\psi]))(1)=\mathrm{res}^c(L_P,P)([\psi]).\qquad(*)$$

Next we construct $[\psi]$. We have $P(\mathbf{Q})\backslash X_\infty\times P(\mathbf{Q})\times K_f\cong\bigcup_{h=1}^h\Gamma_{P,i}\backslash X_\infty$ for suitable arithmetic subgroups $\Gamma_{P,i}\subset P(\mathbf{Q})$. We have fibrations

$$p_i:\Gamma_{P,i}\backslash X_\infty\longrightarrow\Gamma_{P,i}N_P(\mathbf{R})\backslash X_\infty$$

with orientable base and fibres and induced finite coverings

$$\Gamma_{P,i}\cap L_P(\mathbf{Q})\backslash X(L_P)_\infty\longrightarrow\Gamma_{P,i}N_P(\mathbf{R})\backslash X_\infty$$

where the covering group acts orientation preserving. We chose compatible orientations. The fundamental classes $f_i \in H_c^r(\Gamma_{P,i} N_P(\mathbf{R}) \backslash X_\infty, \tilde{\mathbf{C}})$ determine a class $[\psi] = p_1^* f_1 + \ldots + p_h^* f_h$ such that $\mathrm{res}^c(L_P, P)[\psi]$ is a fundamental class for $\bigcup_{i=1}^h \Gamma_{P,i} \cap L_P(\mathbf{Q}) \backslash (X_P)_\infty$. According to [D: 1.15] we can shrink K_f such that we get an injection

$$S_{L_P}^\natural / K_f \cap L_P(\mathbf{A}_f) \subset S_P / K_f \ .$$

Hence $(*)$ shows for sufficiently small K_f that the fundamental class of $S_{L_P}^\natural / K_f \cap L_P(\mathbf{A}_f)$ is a non trivial modular symbol.

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