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Pseudo Eisenstein forms and the cohomology of arithmetic groups III: Residual cohomology classes.

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Abstract. Let $\mathbb{G}$ be a reductive algebraic group defined over $\mathbb{Q}$ and fix a maximal compact subgroup K of $\mathbb{G}(\mathbb{R})$. A compact subgroup $K_A = K\,K_f \subset \mathbb{G}(\mathbf{A})$ defines a locally symmetric space $S(K_f) = \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/K\,K_f A_G$. We consider the "residual Eisenstein cohomology" of $S(K_f)$. Its classes can be represented by harmonic differential forms on $S(K_f)$, which are the residues of Eisenstein forms. Then we construct pseudo Eisenstein forms representing nontrivial cohomology classes with compact support in the Poincaré dual of the "residual Eisenstein cohomology". We use these classes to prove that Sp_4 defines a nontrivial modular symbol in $S(K_f)$ for $\mathbb{G} = \mathrm{GL}_4$. We also sketch the connection with the formula for the volume of a locally symmetric space.

Dedicated to Freydoon Shahidi

I. Introduction

Let $\mathbb{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and let Γ be a torsion free arithmetic subgroup of $G = \mathbb{G}(\mathbb{R})$, K a maximal compact subgroup of G, and $X_\infty = G/K$ the symmetric space. Denote by $\Gamma\backslash X_\infty$ the associated locally symmetric space.

The deRham cohomology $H^*(\Gamma\backslash X_\infty, \mathbb{C})$ is isomorphic to the relative Lie algebra cohomology

$$H^*(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G)),$$

where $\mathcal{A}(\Gamma\backslash G)$ is the space of automorphic forms [**9**] and $\mathfrak{g} = \mathrm{Lie}(G)$. Consider the $(\mathfrak{g}, K)$–module $\mathcal{A}_{cusp}(\Gamma\backslash G)$ of cusp forms. By a result of

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A.Borel [**4**] there is an injection

$$H^*(\mathfrak{g}, K, \mathcal{A}_{cusp}(\Gamma\backslash G)) \hookrightarrow H^*(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G))$$

and its image is called the cuspidal cohomology of Γ.

In this paper we assume that $\Gamma\backslash G$ is not compact (mod center) and and consider the "residual cohomology"

$$H^*_{res}(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G))$$

which is defined as follows: Consider the $(\mathfrak{g}, K)$–module of the residual automorphic functions $\mathcal{A}_{res}(\Gamma\backslash G)$. The inclusion

$$J_{res} : \mathcal{A}_{res}(\Gamma\backslash G) \hookrightarrow \mathcal{A}(\Gamma\backslash G)$$

defines a map

$$J^*_{res} : H^*(\mathfrak{g}, K, \mathcal{A}_{res}(\Gamma\backslash G)) \to H^*(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G)),$$

which is not injective. Its image is denoted by $H^*_{res}(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G))$.

The residual cohomology $H^0_{res}(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G))$ is always nontrivial since the constant function representing the trivial representation is in the residual spectrum and represents a cohomology class in degree 0. Determining $J^*_{res}(H^*(\mathfrak{g}, K, \mathbb{C}))$ is equivalent to determining the nontrivial cohomology classes which are represented by invariant differential forms. This was solved by J.Franke in [**10**].

In general for irreducible unitary representations $A_{\mathfrak{q}}$ in the residual spectrum $J^*_{res}(H^*(\mathfrak{g}, K, A_{\mathfrak{q}}))$ is unknown. In this paper we show

Theorem I.1. *Suppose that $A_{\mathfrak{q}}$ is a representation in the residual spectrum and let $r_{\mathfrak{q}}$ be the lowest degree so that*

$$H^{r_{\mathfrak{q}}}(\mathfrak{g}, K, A_{\mathfrak{q}}) \neq 0.$$

Suppose that $P = MAN$ is a parabolic subgroup of G, π tempered irreducible representation of M and that $I(P, \pi, \nu)$ is a principal series representation with Langlands subquotient $A_{\mathfrak{q}}$. Suppose that we have a residual Eisenstein intertwining operator

$$E_{res} : I(P, \pi, \nu) \to A_{\mathfrak{q}} \subset \mathcal{A}_{res}(\Gamma\backslash G).$$

Then

$$J^{r_{\mathfrak{q}}}_{res}(H^{r_{\mathfrak{q}}}(\mathfrak{g}, K, A_{\mathfrak{q}}))$$

is a nontrivial class in

$$H^{r_{\mathfrak{q}}}_{res}(\mathfrak{g}, K, \mathcal{A}(\Gamma\backslash G)).$$

For all presently known subrepresentations of the residual spectrum the assumptions of the theorem are satisfied, but since not all representations π in the cuspidal spectrum are tempered, these assumptions might not be always satisfied.

Denoting the cohomology with compact support by $H_c^*(\Gamma\backslash X_\infty, \mathbb{C})$ and using Poincaré duality and the ideas and techniques introduced [**25**] we deduce

THEOREM I.2. *Under the assumptions of the theorem*

$$H_c^{dim\ X_\infty - r_{\mathfrak{q}}}(\Gamma\backslash X_\infty, \mathbb{C}) \neq 0.$$

The nontrivial classes in $H_c^{dim\ X_\infty - r_{\mathfrak{q}}}(\Gamma\backslash X_\infty, \mathbb{C})$ *are represented by differential forms* $E(\tilde{\omega}_{\mathfrak{q},\mu_o})$ *with compact support given as pseudo Eisenstein forms in section IV.*

We use these results to describe some residual cohomology classes of $S(K_f) = GL_n(\mathbb{Q})\backslash GL_n(\mathbf{A})/K\, K_f A_G$, (see also 5.6 in [**11**]) . Here $\mathbf{A}$ are the adeles of $\mathbb{Q}$, A_G the connected component of the identity of the scalar matrices and K_f is a open compact subgroup of the finite adeles $\mathbf{A}_f$.

THEOREM I.3. *Suppose that* $\mathbb{G} = GL_n$ *and that* $n = r\,m$. *Then for* K_f *small enough*

$$H^j(S(K_f), \mathbb{C}) \neq 0$$

if

(1) *r and m even and* $j = \frac{r(r+1)m}{4} + \frac{r^2 m(m+2)}{2}$

(2) *r even and m odd and* $j = \frac{r(r+1)(m-1)}{4} + \frac{r^2(m-1)(1+m)}{2}$,

(3) *m=2 and* $j = r(r+1)/2$.

The second part of the paper is concerned with applications of theorem I.2 to modular symbols and to a formula of the volume of a symmetric space.

Suppose that $\mathbb{H} \subset \mathbb{G}$ is a $\mathbb{Q}$–rational reductive subgroup so that $K \cap H$ is maximal compact in $H := \mathbb{H}(\mathbb{R})$. Then the inclusion

$$H \to G$$

induces an inclusion

$$X_{H,\infty} := H/H \cap K \to X_\infty = G/K$$

and hence it induces a proper map

$$j : \Gamma \cap X_{H,\infty} \backslash X_{H,\infty} \to \Gamma \backslash X_\infty.$$

Assume that $\Gamma\backslash X_\infty$ is not compact and that $\Gamma \cap H \backslash X_{H,\infty}$ is an oriented noncompact manifold. If a closed d–form ω represents

$$[\omega] \in H_c^d(\Gamma\backslash X_\infty, \mathbb{C}), \quad d = \dim X_{H,\infty}$$

then

$$\int_{\Gamma \cap H \backslash X_{H,\infty}} j^*\omega$$

is defined. This means that the integral over $\Gamma \cap H \backslash X_{H,\infty}$ determines a map

$$[X_{\Gamma \cap H \backslash H}] : H_c^d(\Gamma\backslash X_\infty, \mathbb{C}) \to \mathbb{C},$$

which is called the *modular symbol* attached to $\mathbb{H}$. Using Poincaré duality we identify the modular symbol $[X_{\Gamma \cap H \backslash H}]$ with an element in $H^*(\Gamma\backslash X_\infty, \mathbb{C})$. If we can find an class $[\omega] \in H_c^*(\Gamma\backslash X_\infty, \mathbb{C})$ such that $[X_{\Gamma \cap H \backslash H}]([\omega]) \neq 0$ then $[X_{\Gamma \cap H \backslash H}]$ is a nontrivial modular symbol.

Ash, Ginzburg and Rallis list 6 families of pairs $(\mathbb{G}, \mathbb{H})$ and show that the restriction of $[X_H]$ to the cuspidal cohomology is zero. One of these pairs is $\mathbb{G} = GL_{2n}$ and $\mathbb{H} = Sp_{2n}$, $n \leq 2$.

We consider generalized modular symbols corresponding to $\mathbb{H} = Sp_4$ in GL_4 in theorem VI.3. and detect a non trivial modular symbol by use of a pseudo Eisenstein series.

Theorem I.4. *Suppose that $\mathbb{G} = GL_4$ and $\mathbb{H}$ is a symplectic group compatible with the choice of the maximal compact subgroup $K \subset GL_4(\mathbb{R})$. For K_f small enough*

$$[\mathbb{H}(\mathbb{Q})\backslash \mathbb{H}(\mathbf{A})/(K \cap \mathbb{H}(\mathbb{R}))(K_f \cap \mathbb{H}(\mathbf{A}_f))]$$

is a nontrivial modular symbol in $H^3(\mathbb{G}(\mathbb{Q})\backslash \mathbb{G}(\mathbf{A})/A_G K K_f, \mathbb{C})$.

In the arithmetic of modular curves and automorphic forms modular symbols provide a link between geometry and arithmetic. "Period integrals" of Eisenstein classes or cuspidal cohomology classes over compact modular symbols have been used by G.Harder to obtain information about special values of L-functions [**15**], [**16**]. We conjecture that the value of the modular symbol on this residual cohomology class is related to special values of Rankin convolutions of cusp forms.

In the last section we sketch the connection of our techniques with the formula for the volume of a locally symmetric space due to Langlands [**21**]. In the proof of theorem I.2 we have used a formula for the

integral over a product of a pseudo Eisenstein form with an Eisenstein form. We show how this formula is related to the computation of the Tamagawa number $\tau(\mathbb{G})$ of $\mathbb{G}$. We sketch this only in the simplest case, i.e. if $\mathbb{G}/\mathbb{Q}$ is split and simply connected.

The results of this article will be used in a sequel to this paper to construct other nontrivial modular symbols.

The outline of the paper is as follows. We first introduce the notation and then we relate the $(\mathfrak{g}, K)$-cohomology of principal series representations to the $(\mathfrak{g}, K)$–cohomology of their unitary Langlands subrepresentations. We exhibit in section III. nontrivial "residual" cohomology classes of $S(K_f)$ and their harmonic representatives. In the next section we discuss the example GL_n. In section V. we construct a family of differential forms with compact support which represent classes in the Poincaré dual of the residual cohomology classes constructed in the previous section. In the section VI. we apply these techniques to show that Sp_4 defines a nontrivial modular symbol in $S(K_f)$ for $\mathbb{G}$=GL_4. We sketch in the last section the connection of our techniques with the well known formula for the volume of $S(K_f)$ for a maximal compact subgroup K_f.

II. Representations with nontrivial $(\mathfrak{g}_0, K)$–cohomology

In this section we first introduce the notation. Then we prove some results relating the $(\mathfrak{g}, K)$-cohomology of principal series representations and their unitary Langlands subrepresentations. The results and techniques of this section are purely representation theoretic. They will be used in the later in the context of representations in the residual spectrum.

2.1 Let $\mathbb{G}$ be a reductive algebraic group over $\mathbb{Q}$, $G = \mathbb{G}(\mathbb{R})$, K a maximal compact subgroup of G, $\mathbb{A}_G$ be a maximal $\mathbb{Q}$–split torus in the center of $\mathbb{G}$ and connected component A_G of $\mathbb{A}_G(\mathbb{R})$. This defines the symmetric space

$$X_\infty = G/K \cdot A_G.$$

For a rational subgroup $\mathbb{H}$ of $\mathbb{G}$ the Lie algebra of H will be denoted by $\mathfrak{h}$ and we put $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes \mathbb{C}$. Let $\mathfrak{g}_0$ be the Lie algebra of the real points of the intersection of the kernels of all rational characters of $\mathbb{G}$. Then $\mathfrak{g}/\mathfrak{a}_G \simeq \mathfrak{g}_0$. The Cartan decomposition is denoted by $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathbf{p}$. Let θ be the corresponding Cartan involution.

We fix a minimal parabolic subgroup P_0 of G. Parabolic subgroups and their decompositions are denoted by $P = LN = MA_PN$. We denote the positive roots of $(\mathfrak{a}_P/\mathfrak{a}_G, \mathfrak{g}_0)$ defined by the parabolic subgroup P by Σ_P^+ and define

$$\rho_P := 1/2 \sum_{\beta \in \Sigma_P^+} \beta.$$

For a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ and positive roots compatible with the choice of P_0 we denote by ρ_G half the sum of the positive roots of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$.

2.2 We consider a standard representation

$$I(P, \pi_L, \mu_0) = \mathrm{ind}_{MA_PN}^G \pi_L \otimes e^{\mu_0 + \rho_p} \otimes 1$$

with nontrivial $(\mathfrak{g}, K)$–cohomology

$$H^*(\mathfrak{g}_0, K, I(P, \pi_L, \mu_0)) \neq 0.$$

Recall that μ_0 is the differential of a character of A and that π_L is tempered. We assume that $P = MA_PN$ is a standard parabolic subgroup and that $\Re(\mu_0) \in \mathfrak{a}_P^*$ is in the interior of the dominant Weyl chamber with respect to Σ_P^+, so that there is a unique maximal nontrivial subrepresentation $U(P, \pi_L, \mu_0)$ and Langlands subquotient

$$L(P, \pi_L, \mu_0) := I(P, \pi_L, \mu_0)/U(P, \pi_L, \mu_0)$$

(section IV in [**7**]). Furthermore we assume in this section that the Langlands subquotient $L(P, \pi_L, \mu_0))$ is unitary and that A_G^0 acts trivially.

This implies that the hermitian dual $I(P, \bar{\pi}_L^*, -\mu_0)$ of the representation $I(P, \pi_L, \mu_0))$ is isomorphic to $I(P, \pi_L^{w_0}, w_0(\mu_0))$ for a element $w_0 \neq 1$ with $w_0^2 = 1$ of the Weyl group $W_P = \mathrm{Norm}(A_P)/\mathrm{Cent}(A_P)$ of the parabolic subgroup P. [**30**] Here π^* is the contragredient representation of π. Note that π_L unitary implies that $\bar{\pi}_L^*$ is isomorphic to π_L.

Since $I(P, \pi_L, \mu_0)$ has nontrivial $(\mathfrak{g}, K)$– cohomology the infinitesimal characters of $I(P, \pi_L, \mu_0)$ and $I(P, \bar{\pi}_L^*, -\mu_0)$ are equal to ρ_G. Using the formulas in Borel-Wallach [3.3 in [**7**]] we conclude that there exists s in the Weyl group W of $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ so that

$$\mu_0 = -(s\,\rho_G)_{|\mathfrak{a}_P}$$

and there exists a finite dimensional irreducible representation E_s of M so that

$$\begin{aligned} & H^{l(s)+q}(\mathfrak{g}_0, K, I(P, \pi_L, \mu_0)) = \\ & = \ (H^*(\mathfrak{m}_0, K_M, \pi_L \otimes E_s) \otimes \wedge^*(\mathfrak{a}_P/\mathfrak{a}_G)^*)^q \\ & = \ \oplus_{r+k=q} H^r(\mathfrak{m}_0, K_M, \pi_L \otimes E_s) \otimes \wedge^k(\mathfrak{a}_P/\mathfrak{a}_G)^* \end{aligned}$$

where $l(s)$ is the length of s and $K_M = K \cap M$. In particular the highest weight of E_s is $(s(\rho_G) - \rho_G)_{|\mathbf{m}}$.

Notation: *For later reference we denote the lowest degree i in which*

$$H^i(\mathfrak{g}_0, K, I(P, \pi_L, \mu_0)) \neq 0$$

by $e(P, \pi_L, \mu_0)$. If no confusion is possible we drop the subscript L.

Poincaré duality implies that the $(\mathfrak{g}, K)$–cohomology of $I(P, \bar{\pi}_L^*, -(\mu_0))$ is non zero and isomorphic as a vector space to

$$H^*(\mathfrak{m}_0, K_M, \bar{\pi}_L^* \otimes \bar{E}_s^*) \otimes \wedge(\mathfrak{a}_P/\mathfrak{a}_G)^*.$$

where E_s^* is the contragredient representation to E_s.

2.3 We obtain a representative of the one dimensional space $H^{e(P,\pi,\mu)}(\mathfrak{g}_0, K, I(P, \pi, \mu))$ as follows:

Recall that $\mathbf{p}$ is the -1 eigenspace of the Cartan involution θ. Let ω_N be a harmonic form representing the lowest weight of the $\mathfrak{m}$-module E_s, let ω_M be the highest weight vector of a representation in $\bigwedge^*(\mathbf{p} \cap \mathfrak{m})$ and v a highest weight vector in the lowest K_M–type of π_L. Then $v \otimes \omega_N \otimes \omega_M^*$ represents a nontrivial map in

$$\mathrm{Hom}_{M \cap K}(\wedge^*(\mathbf{p} \cap \mathfrak{m}), \pi_L \otimes E_s) = H^*(\mathfrak{m}_0, K_M, \pi_L \otimes E_s).$$

Let

$$\psi : \mathfrak{n} \to \mathbf{p} \quad \text{defined by} \quad X \to X - \theta(X).$$

Then $\omega_M^* \wedge \psi(\omega_N)$ is the lowest weight vector of a representation of K. This representation of K is a K-type of $I(P, \pi, \mu)$ and thus it defines a nontrivial element in $\mathrm{Hom}_K(\bigwedge^{e(P,\pi,\mu_0)} \mathbf{p}, I(P, \pi, \mu))$. It represents the nontrivial $(\mathfrak{g}, K)$–cohomology class in degree $e(P, \pi, \mu_0)$.

For more details see [**27**] or [**13**]

2.4 For a connected semisimple Lie group with a maximal compact subgroup K the unitary $(\mathfrak{g}, K)$–modules with nontrivial $(\mathfrak{g}, K)$–cohomology have been constructed and classified by Parthasarathy, Vogan and Zuckerman [**31**]. They are parametrized by equivalence classes

of θ–stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ and are denoted by $A_{\mathfrak{q}}$. For more general description for disconnected groups G see [**20**].

By 2.2 there exists an intertwining operator

$$M(P, \pi_L, \mu_0, w) : I(P, \pi_L, \mu_0) \to I(P, \bar{\pi}_L, -\mu_0).$$

Since all unitary representation with infinitesimal character ρ_G do have non trivial $(\mathfrak{g}, K)$–cohomology there exists a θ–stable parabolic subalgebra $\mathfrak{q}$ and the unique irreducible subrepresentation $L(P, \bar{\pi}_L, \mu_0)$ of $I(P, \bar{\pi}_L, -\mu_0)$ is isomorphic to a representation $A_{\mathfrak{q}}$ for a θ–stable parabolic $\mathfrak{q}$ of $\mathfrak{g}_0 \otimes \mathbb{C}$ [**26**]. Here we extend the $(\mathfrak{g}_0, K)$–module $A_{\mathfrak{q}}$ to a $(\mathfrak{g}, K)$–module by $\mathfrak{a}_G$ acting trivially on $A_{\mathfrak{q}}$.

If $\mathfrak{q} = \mathfrak{l}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}$ is the Levi decomposition of $\mathfrak{q}$ we define $r_{\mathfrak{q}} = \dim \mathfrak{n}_{\mathfrak{q}} \cap \mathbf{p}$. Then

$$\begin{aligned} H^i(\mathfrak{g}_0, K, A_{\mathfrak{q}}) &= 0 \quad \text{if } i < r_{\mathfrak{q}} \\ &= \mathbb{C} \quad \text{if } i = r_{\mathfrak{q}} \end{aligned}$$

Furthermore since $A_{\mathfrak{q}}$ is unitary and irreducible

$$H^*(\mathfrak{g}_0, K, A_{\mathfrak{q}}) = \operatorname{Hom}_K(\wedge^* \mathbf{p}, A_{\mathfrak{q}}).$$

Denote by $\mathfrak{t}_K$ the intersection of a fundamental θ–stable Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}$ with $\mathfrak{k}$.. The minimal K–type $F_{\mathfrak{q}}$ of $A_{\mathfrak{q}}$ has highest weight $\lambda_{\mathfrak{q}}$, the sum of the weights of $\mathfrak{t}_K$ on $\mathfrak{n}_{\mathfrak{q}} \cap \mathbf{p}$. See [**31**].

Proposition II.1. *Suppose that $A_{\mathfrak{q}}$ is the Langlands subrepresentation of $I(P, \bar{\pi}_L, -\mu_0)$. Then*

$$H^{e(P,\pi,-\mu_0)}(\mathfrak{g}_0, K, A_{\mathfrak{q}}) \neq 0.$$

Furthermore the inclusion of $A_{\mathfrak{q}}$ into $I(P, \bar{\pi}_L, -\mu_0)$ defines an injective map of $H^{e(P,\pi,-\mu_0)}(\mathfrak{g}_0, K, A_{\mathfrak{q}})$ into $H^{e(P,\pi,-\mu_0)}(\mathfrak{g}_0, K, I(P, \bar{\pi}_L, -\mu_0))$.

Proof: The formulas on page 82 of [**31**] show that a K–type of $\wedge^{e(P,\pi,-\mu_0)}\mathbf{p}$ which represents the nontrivial class in

$$\operatorname{Hom}_K(\wedge^{e(P,\pi,-\mu_0)}\mathbf{p}, I(P, \bar{\pi}_L, -\mu_0))$$

has extremal weight $\lambda_{\mathfrak{q}}$ and hence is equal to the K–type $F_{\mathfrak{q}}$, which also occurs in in the subrepresentation $A_{\mathfrak{q}}$. This K–type has multiplicity one in $I(P, \pi_L, -\mu_0)$ and hence the map $\operatorname{Hom}_K(F_{\mathfrak{q}}, I(P, \bar{\pi}_L, -\mu_0)))$ factors over $A_{\mathfrak{q}}$. $\square$

To prove that

$$\begin{aligned} H^i(\mathfrak{g}_0, K, A_{\mathfrak{q}}) &= 0 \quad \text{if } i < e(P, \pi, -\mu_0) \\ &= \mathbb{C} \quad \text{if } i = e(P, \pi, -\mu_0) \end{aligned}$$

i.e that $r_{\mathfrak{q}} = e(P, \pi, -\mu_0)$ we review some results from exterior algebra. See chapter I in [**8**].

Let V be a finite dimensional vector space. A subring I of $\wedge^*(V^*)$ is an ideal if
a.) $\alpha \in I$ implies $\alpha \wedge \beta \in I$ for all $\beta \in \wedge^*(V^*)$
b.) $\alpha \in I$ implies that all its homogenous components in $\wedge^*(V^*)$ are contained in I.

Given an ideal I of $\wedge^*(V^*)$ its retracting space is the smallest subspace $W^* \subset V^*$ so that I is generated as an ideal by a set S of elements of $\wedge^*(W^*)$, i.e an element of I is a sum of elements of the form $\sigma \wedge \alpha$ with $\sigma \in S$ and $\alpha \in \wedge^*(V^*)$.

Let $W^* \subset V^*$ be a subspace of dimension r and let I_W be the ideal generated by $\wedge^*(W^*)$. Then by 1.3 of [**8**] W^* is also the retracting space of I_W.

THEOREM II.2. *Suppose that $A_{\mathfrak{q}}$ is the Langlands subrepresentation of the standard representation $I(P, \bar{\pi}_L, -\mu_0)$. Then*

$$r_{\mathfrak{q}} = e(P, \pi, -\mu_0).$$

Proof: Let P^0 be the kernel of the absolute values of the rational characters of P. Since the symmetric space X_∞ has a covering $A_P \times P^0/K_M$ we have a vector space decomposition

$$\mathbf{p} = \mathfrak{a}_P \oplus \mathbf{p}_{\mathfrak{m}} \oplus \psi(\mathfrak{n}),$$

where $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{m} \oplus \mathbf{p}_{\mathfrak{m}}$ is the corresponding Cartan decomposition of $\mathfrak{m}$. Consider the subspace $\mathfrak{a}_P \subset \mathbf{p}$ and the ideal $I(\mathfrak{a}_P)$ of $\bigwedge^*(\mathbf{p})$ generated by $\mathfrak{a}_P$. Note that the ideal $I(\mathfrak{a}_P)$ is a $U(\mathfrak{m})$-invariant subspace of $\wedge^*(\mathbf{p})$.

The differential forms representing the nontrivial $(\mathfrak{g}_0, K)$–cohomology classes of $I(P, \bar{\pi}_L, -\mu_0)$ have representatives in $\mathrm{Hom}_K(\bigwedge^* \mathbf{p}, I(P, \bar{\pi}_L, -\mu_0))$ and are determined by their values on the K–isotypic components of $\wedge^*(\mathbf{p})$. Consider the subrepresentation $F_{\mathfrak{q}}$ of $\wedge^*(\mathbf{p})$ in degree $e(P, \pi, -\mu_0)$. We showed that this representation determines an nontrivial $(\mathfrak{g}_0, K)$–cohomology class of $I(P, \bar{\pi}_L, -\mu_0)$ in degree $e(P, \pi, -\mu_0)$.

Considering a realization of $I(P, \bar{\pi}_L, -\mu_0)$ in the functions on G we consider $I(P, \bar{\pi}_L, -\mu_0) \otimes \wedge^* \mathbf{p}^*$ as differential forms on X. The formula in 2.3 shows together with the results in [**13**], [**27**] that we can find a harmonic representative of the nontrivial cohomology class of $I(P, \bar{\pi}_L, -\mu_0)$ in degree $e(P, \pi, -\mu_0)$. This representative is not in $I(P, \bar{\pi}_L, -\mu_0) \otimes I(\mathfrak{a}_P)^*$ and none of its homogeneous components are in $I(P, \bar{\pi}_L, -\mu_0) \otimes I(\mathfrak{a}_P)^*$.

Since $A_{\mathfrak{q}}$ is unitary and irreducible

$$H^*(\mathfrak{g}_0, K, A_{\mathfrak{q}}) = \mathrm{Hom}_K(\wedge^*\mathbf{p}, A_{\mathfrak{q}}).$$

Now $F_{\mathfrak{q}}$ is the only subrepresentation of $\wedge^*\mathbf{p}$ which is also a K–type of $A_{\mathfrak{q}}$ [**31**]. Write $\mathfrak{q} = \mathfrak{l}_{\mathfrak{q}} \oplus \mathfrak{n}_{\mathfrak{q}}$ for the Levi decomposition. D.Vogan and G.Zuckerman show in [**31**] that the highest weight vectors of the subrepresentations $F_{\mathfrak{q}} \subset \wedge^*(\mathbf{p})$ are of the form $\omega_L \wedge V_{r_{\mathfrak{q}}}$ where ω_L is an L–invariant differential form in $\wedge^*(\mathbf{p} \cap \mathfrak{l}_{\mathfrak{q}})$ and $V_{r_{\mathfrak{q}}}$ is representation with highest weight $2\rho(\mathfrak{n}_{\mathfrak{q}} \cap \mathbf{p})$. These forms are harmonic representatives of the cohomology classes.

Using that $A_{\mathfrak{q}} \hookrightarrow I(P, \bar{\pi}_L, -\mu_0)$ we can conclude that these forms are also harmonic forms on X and in degree $e(P, \pi, -\mu_0)$ there is also a harmonic form representing a nontrivial class of $I(P, \bar{\pi}_L, -\mu_0)$.

Now A_P is the maximal $\mathbb{R}$–split torus in $L_{\mathfrak{q}}$ [**31**], and so the only harmonic representative for the nontrivial $(\mathfrak{g}_0, K)$–cohomology classes of $A_{\mathfrak{q}}$ which may not have a homogenous component in $I(\mathfrak{a}_P^*)$ is in degree $r_{\mathfrak{q}}$.

Thus the previous proposition implies that $r_{\mathfrak{q}} = e(P, \pi, -\mu_0)$. $\square$

III. Residual Eisenstein classes

In this section residual Eisenstein forms and classes are introduced. The main reference for Eisenstein series and their residues is the book by C.Moeglin and Waldspurger [23]. We also freely use their terminology.

3.1 Let $\mathbf{A}_f \subset \mathbf{A}$ be the finite adeles of the adele ring $\mathbf{A}$ over $\mathbb{Q}$. We give $\mathbb{G}(\mathbf{A}_f)$ the topology induced by the topology of $\mathbf{A}_f$ and define the global symmetric space $X = X_\infty \times \mathbb{G}(A_f)$. $\mathbb{G}(\mathbb{Q})$ acts on X and we obtain the global local symmetric space $S = \mathbb{G}(\mathbb{Q})\backslash X$. For a compact open subgroup K_f of $\mathbb{G}(\mathbf{A}_f)$ we consider the locally symmetric space

$$S(K_f) = \mathbb{G}(\mathbb{Q})\backslash X/K_f.$$

We fix a minimal parabolic subgroup $\mathbb{P}_0$ of $\mathbb{G}$ defined over $\mathbb{Q}$ and a Levi subgroup $\mathbb{L}_0$ of $\mathbb{P}_0$ also defined over $\mathbb{Q}$.

3.2 Let $\mathbb{P}$ be a standard parabolic subgroup of $\mathbb{G}$ defined over $\mathbb{Q}$ with Levi decomposition $\mathbb{P} = \mathbb{U}_P\mathbb{L}_P$ and let A_P be the connected component of the maximally split torus in the center of $L_P := \mathbb{L}_P(\mathbb{R})$. Following Arthur we define the height function

$$H_P : \mathbb{L}_P(\mathbf{A}) \to A_P.$$

The kernel of H_P is denoted by $\mathbb{L}^1(\mathbf{A})$. We write $\mathbb{P}^1(\mathbf{A}) = \mathbb{U}_P(\mathbf{A})\mathbb{L}^1_P(\mathbf{A})$.

A parameter $\mu \in \mathfrak{a}^*_P$ defines character χ_μ of $\mathbb{L}_P(\mathbf{A})$ by

$$\chi_\mu(l_{\mathbf{A}}) = e^{\mu(log\ H_P(l_{\mathbf{A}}))}, \qquad l_{\mathbf{A}} \in \mathbb{L}_P(\mathbf{A}).$$

Let Σ^+_P be the roots of A_P on $U_P = \mathbb{U}_P(\mathbb{R})$ and ρ_P half the sum of the positive roots. If no confusion is possible we omit the subscript P.

Suppose that $\mathbb{G}^\flat$ is a reductive subgroup of $\mathbb{G}$. Recall that an irreducible unitary representation $\pi_{\mathbf{A}}$ of a reductive group $\mathbb{G}^\flat(\mathbf{A})$ is called automorphic of it occurs discretely in $L^2(\mathbb{G}^\flat(\mathbf{A}), \xi)$ for character ξ of which is trivial on $\mathbb{A}_{G^\flat}(\mathbb{Q})$ and $A_{G^\flat}$.

We now fix a unitary character ξ of $\mathbb{A}_G(\mathbf{A})$ which is trivial on $\mathbb{A}_G(\mathbb{Q})$ and A_G.

Let $\pi_{\mathbb{L}(\mathbf{A})} = \prod_v \pi_v$ be an irreducible unitary automorphic representation of $\mathbb{L}(\mathbf{A})$ on V_π which transforms under $\mathbb{A}_G(\mathbf{A})$ by ξ. We define using normalized induction a representation

$$\begin{aligned} I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A})}, \mu) &= \mathrm{ind}_{\mathbb{P}(\mathbf{A})}^{\mathbb{G}(\mathbf{A})} \chi_{\mu+\rho_P} \otimes \pi_{\mathbb{L}(\mathbf{A})} = \\ &= \prod_\nu \mathrm{ind}_{\mathbb{P}(\mathbb{Q}_\nu)}^{\mathbb{G}(\mathbb{Q}_\nu)} (\chi_{\mu+\rho_P})_\nu \otimes \pi_{\mathbb{L}(\mathbb{Q}_\nu)} \end{aligned}$$

This representation acts on the space of functions f on $\mathbb{G}(\mathbf{A})$ with values in V_π which satisfy

$$f(z_{\mathbf{A}} u_{\mathbf{A}} l_{\mathbf{A}} g_{\mathbf{A}}) = \xi(z_{\mathbf{A}}) \pi_{\mathbb{L}(\mathbf{A})}(l_{\mathbf{A}}) \chi_{\mu+\rho_P}(l_{\mathbf{A}}) f(g_{\mathbf{A}}).$$

Here $l_{\mathbf{A}} \in \mathbb{L}(\mathbf{A}),\ u_{\mathbf{A}} \in \mathbb{U}(\mathbf{A}),\ g_{\mathbf{A}} \in \mathbb{G}(\mathbf{A}), z_{\mathbf{A}} \in \mathbb{A}_G(\mathbf{A})$.

The automorphic representation

$$\pi_{\mathbb{L}(\mathbf{A})} = \prod_\nu \pi_{\mathbb{L}(\mathbb{Q}_\nu)} = \pi_L \prod_p \pi_{\mathbb{L}(\mathbb{Q}_p)}$$

is in the cuspidal spectrum of

$$\mathbb{L}(\mathbb{Q}) A_P \backslash \mathbb{L}(\mathbf{A}) / K K_f \cap \mathbb{L}(\mathbf{A})$$

if its factor π_L at the infinite places is tempered [**32**]. We will therefore assume this for the rest of the article.

3.3 Assume now as in the previous section that $\Re(\mu)$ is in the dominant Weyl chamber with respect to Σ^+_P. Let $W(L)$ be the set of elements w in the Weyl group of $\mathbb{G}$ of minimal length modulo the Weyl group of $\mathbb{L}$, and such that $w\mathbb{L}w^{-1}$ is also a standard Levi of $\mathbb{G}$ [23]. For every w in $W(L)$ there exists an intertwining operator

$$\mathbf{M}(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w) : I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A})}, \mu) \to I(\mathbb{P}, \pi^w_{\mathbb{L}(\mathbf{A})}, w(\mu)).$$

Choosing an isomorphism

$$A_w : V_\pi \to V_{\pi^w} = V_\pi$$

we identify it with an intertwining operator $M_{\mathbf{A}}(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w)$, which has for $\Re(\mu)$ large an Euler product

$$M_{\mathbf{A}}(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w) = \prod_\nu M(\pi_{\mathbb{L}(\mathbf{Q}_\nu)}, \mu, w).$$

(see II.1.9 in [**23**] for details).

There exists a meromorphic function $m(\pi_{\mathbb{L}(\mathbb{Q}_\nu)}, \mu, w)$ so that the local intertwining operator

$$\mathcal{M}(\pi_{\mathbb{L}(\mathbb{Q}_\nu)}, \mu, w) = m(\pi_{\mathbb{L}(\mathbb{Q}_\nu)}, \mu, w) M(\pi_{\mathbb{L}(\mathbb{Q}_\nu)}, \mu, w)$$

is holomorphic and nonzero for μ in the dominant Weyl chamber. See [**1**]. There is also a meromorphic function $m(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w)$, which for large dominant μ is the product of the local factors, so that

$$\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w) = m(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w) M_{\mathbf{A}}(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w)$$

is holomorphic and nonzero for μ in the dominant Weyl chamber.

We say that the operator $M(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w)$ has a pole of order r at μ_0 if the function $m(\pi_{\mathbb{L}(\mathbf{A})}, \mu, w_0)$ has a zero at $\mu = \mu_0$ of order r.

We fix the level K_f and are interested in automorphic forms on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/A_G K_f$. Thus from now on we consider representations $I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A}_f)}, \mu)$ with a K_f-invariant vector, i.e. $I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A}_f)}, \mu)^{K_f} \neq 0$.

3.4 By our our assumptions the representation $\pi_{\mathbb{L}(\mathbf{A})}$ is a subrepresentation of the cuspidal spectrum of $\mathbb{L}(\mathbb{Q})A_L\backslash\mathbb{L}(\mathbf{A})$; hence we consider $I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A})}, \mu)$ as realized in the functions on $\mathbb{P}(\mathbb{Q})\mathbb{U}(\mathbf{A})\backslash\mathbb{G}(\mathbf{A})$. A form

$$\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu} \in \wedge^i \mathbf{p}^* \otimes_K I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A})}, \mu)^{K_f} \cong \mathrm{Hom}_K(\wedge^i \mathbf{p}, I(\mathbb{P}, \pi_{\mathbb{L}(\mathbf{A})}, \mu))^{K_f}$$

defines a differential form on $\mathbb{P}(\mathbb{Q})\mathbb{U}(\mathbf{A})\backslash\mathbb{G}(\mathbf{A})/KK_f$. For $\Re(\mu)$ large and dominant we define following [**13**] the Eisenstein differential form

$$E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu}) = \sum_{g^{-1}\in\mathbb{P}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{Q})} g^* \eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu}$$

on $S(K_f)$.

The constant term $E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu})^P$ of the form $E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu})$ with respect to the parabolic subgroup $\mathbb{P} = \mathbb{L}\mathbb{U}$ is equal to

$$\begin{aligned} E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu})^P &= \sum_{w\in W(L)} M(\pi_{\mathbb{L}(\mathbf{A})},\mu,w)\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu} \\ &= \sum_{w\in W(L)} m(\pi_{\mathbb{L}(\mathbf{A})},\mu,w)^{-1}\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu,w)\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu}. \end{aligned}$$

Recall that $W(L)$ is the set of elements w in the Weyl group of G of minimal length modulo the Weyl group of L, and such that wLw^{-1} is also a standard Levi of G [23]. Furthermore

$$M(\pi_{\mathbb{L}(\mathbf{A})},\mu,w)\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu} \in \wedge^i\mathbf{p}^* \otimes_K M(\pi_{\mathbb{L}(\mathbf{A})},\mu,w)I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},\mu)^{K_f}.$$

If the constant term $E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu})^P$ has a pole of order dim $(\mathfrak{a}_P)$ for $\mu = \mu_0$ dominant, then so does the Eisenstein form $E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu})$ [**23**]. We write $E(\eta^{res}_{\pi_{\mathbb{L}(\mathbf{A})},\mu_0})$ for the residue at $\mu = \mu_0$ and call this a residual Eisenstein form. The form

$$E(\eta^{res}_{\pi_{\mathbb{L}(\mathbf{A})},\mu_0})$$

is square integrable.

Let w_0 be the Weyl group element considered in 2.2. If the constant term $E(\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu})^P$ has a pole of order dim $(\mathfrak{a}_P)$ for $\mu = \mu_0$ dominant, then $M(\pi_{L(\mathbf{A})},\mu,w_0)$ has a pole at $\mu = \mu_0$ of order dim $\mathfrak{a}_P$ [**23**]. The image of $\mathcal{M}(\pi_{L(\mathbf{Q_A})},\mu_0,w_0)$ is a unitary representation which is isomorphic to a representation in the residual spectrum.

The constant term of the residual Eisenstein form $E(\eta^{res}_{\pi_{\mathbb{L}(\mathbf{A})},\mu_0})$ determines its growth at infinity. Since a residual Eisenstein form is square integrable, it is decaying at infinity and its constant term $E(\eta^{res}_{\pi_{\mathbb{L}(\mathbf{A})},\mu_0})^P$ has only the term

$$m(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,w_0)^{-1}\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,w_0)\eta_{\pi_{\mathbb{L}(\mathbf{A})},\mu_0}$$

for $w_0 \in W(L)$ as introduced in 2.2.

3.5 If in addition

$$H^*(\mathfrak{g}_0, K, I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},\mu_0)) \neq 0$$

then the factor $I(P,\pi_L,\mu_0)$ of $I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},\mu_0)$ at the infinite place satisfies the assumptions of the previous section and we conclude that in degree $r_{\mathfrak{q}} = e(P,\pi,-\mu_0)$ the form

$$E(\eta^{res}_{\pi_{\mathbb{L}(\mathbf{A})},\mu_0}) \in \mathrm{Hom}_K(\wedge^{r_{\mathfrak{q}}}\mathbf{p}, L^2_{res}(\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbf{A})))_{A_{\mathfrak{q}}}$$

where $L^2_{res}(\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbf{A}))_{A_\mathfrak{q}}$ is the isotypic component of type $A_\mathfrak{q}$, i.e

$$L^2_{res}(\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbf{A}))_{A_\mathfrak{q}} = \mathrm{Hom}_{(\mathfrak{g},K)}(A_\mathfrak{q}, L^2_{res}(\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbf{A}))) \otimes A_\mathfrak{q}).$$

The standard principal series representations $I(P,\pi_L,\mu_0)$ and $I(P,\bar{\pi}_L,-\mu_0)$ have $A_\mathfrak{q}$ as Langlands subquotient, respectively subrepresentation, and they have both the same K–types. Thus we have a nonzero closed form

$$\eta_\mathfrak{q} \in \wedge^{r_\mathfrak{q}}\mathbf{p} \otimes_K I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},\mu_0)$$

defined by the subrepresentation of $\wedge^{r_\mathfrak{q}}\mathbf{p}$ with highest weight

$$2\rho(\mathfrak{n}_\mathfrak{q}\cap\mathbf{p}_\mathbb{C}) = \sum_{\beta\in\Sigma(\mathfrak{n}_\mathfrak{q}\cap\mathbf{p}_\mathbb{C})}\beta,$$

where $\mathfrak{n}_\mathfrak{q}$ is the nilradical of the θ–stable parabolic subalgebra $\mathfrak{q}$.

The constant Fourier coefficient $E(\eta^{res}_\mathfrak{q})^P$is a nonzero multiple of $\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,w_0)\eta_\mathfrak{q}$. We have the exact sequence

$$\begin{aligned} 0 \to Im(\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,w_0)) &\to I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},-\mu_0) \\ &\to \ I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},-\mu_0)/Im(\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,w_0)) \to 0 \end{aligned}$$

and by 2.2 it induces in degree $r_\mathfrak{q}$ the isomorphism

$$H^{r_\mathfrak{q}}(\mathfrak{g},K,Im(\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,w_0))) \ = \ H^{r_\mathfrak{q}}(\mathfrak{g},K,I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},-\mu_0)).$$

Hence

$$[\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu,w_0)\eta_\mathfrak{q}]$$

defines a nontrivial class in $H^{e(P,\pi,-\mu_0)}(\mathfrak{g}_0,K,I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},-\mu_0))$ and so does $[E(\eta^{res}_\mathfrak{q})^P]$.

3.6 To prove that there are nontrivial residual Eisenstein classes $[E(\eta^{res}_{\Pi_{L(\mathbf{A})},\mu_0})]$ in degree $r_\mathfrak{q}$ we follow the ideas of [**13**] and [**27**] and consider the restriction of a form to a face

$$e(\mathbb{P},K_f) = \mathbb{P}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/K\ K_f A_P$$

of the boundary of the Borel Serre compactification. See [**6**] for details on the Borel Serre compactification.

The cohomology of the face $e(\mathbb{P},K_f)$ is isomorphic to

$$\mathrm{ind}^{\mathbb{G}(\mathbf{A}_f)}_{\mathbb{P}(\mathbf{A}_f)} H^*(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/K_{\mathbb{L}(\mathbf{A})}A_P, H^*(\mathbf{n},\mathbb{C}))$$

where $K_{\mathbb{L}(\mathbf{A})} = (K\cap L)(K_{\mathbf{A}_f}\cap\mathbb{L}(\mathbf{A}_f)) = K_L\, K_{\mathbb{L}(\mathbf{A}_f)}$. We have

$$\begin{aligned} &H^*_{\mathrm{cusp}}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/K_{\mathbb{L}(\mathbf{A})}A_P, H^*(\mathbf{n},\mathbb{C})) \\ &\qquad\qquad \subset H^*(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/K_{\mathbb{L}(\mathbf{A})}A_P, H^*(\mathbf{n},\mathbb{C})) \end{aligned}$$

and

$$H^*_{\text{cusp}}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/K_{\mathbb{L}(\mathbf{A})}A_P, H^*(\mathbf{n},\mathbb{C}))$$

is equal to the direct sum of

$$\text{Hom}_{\mathbb{L}(\mathbf{A})}(\Pi_{\mathbb{L}(\mathbf{A})}, L^2_{\text{cusp}}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/K_{\mathbb{L}(\mathbf{A})}A_P)\otimes\text{Hom}_{K_L}(\wedge^*\mathbf{p}_{\mathfrak{m}}, \Pi_{\mathbb{L}(\mathbf{A})}\otimes H^*(\mathfrak{n},\mathbb{C}))),$$

where $\mathbf{p}_{\mathfrak{m}} = \mathbf{p}\cap\mathfrak{m}$.

Since $\pi_{\mathbb{L}(\mathbf{A})}$ is a cuspidal representation, $[E(\eta^{res}_{\mathfrak{q}})^P]$ defines by section II a nontrivial class in

$$\text{ind}^{\mathbb{G}(\mathbf{A}_f)}_{\mathbb{P}(\mathbf{A}_f)} H^*(\mathfrak{m}, K_M, \pi_{\mathbb{L}(\mathbf{A})}\otimes H^*(\mathfrak{n},\mathbb{C}))\otimes\wedge^0\mathfrak{a}_P.$$

But $H^*(\mathfrak{m}, K_M, \pi_{\mathbb{L}(\mathbf{A})}\otimes H^*(\mathfrak{n},\mathbb{C}))\otimes\wedge^0\mathfrak{a}_P$ can be considered as a subspace of

$$H^*_{\text{cusp}}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/K_{\mathbb{L}(\mathbf{A})}A_P, H^*(\mathbf{n},\mathbb{C})).$$

Hence $[E(\eta^{res}_{\mathfrak{q}})^P]$ can be considered as a class in $H^*_{cusp}(e(\mathbb{P},K_f))$.

In [**13**] it is proved that computing the restriction of an Eisenstein form to the face $e(\mathbb{P},K_f)$ corresponds to taking the constant Fourier coefficient. Thus we proved

Theorem III.1. *Under our assumptions* $[E(\eta^{res}_{\mathfrak{q}})]$ *is a nontrivial cohomology class of* $S(K_f)$ *in degree* $e(P,\pi,-\mu_0) = r_{\mathfrak{q}}$.

IV. An example: Residual cohomology classes for $\mathbf{GL}_n$

Let $\mathbf{A}$ be the adeles of $\mathbb{Q}$. We use the description of the residual spectrum of the general linear group by C.Moeglin and J.L.Waldspurger and the nonvanishing results of the cuspidal cohomology of [**6**] to prove nonvanishing results for the residual cohomology of the general linear group.

4.1 In [**3**] it is proved that there are cuspidal representations $\pi_{\mathbf{A}}$ of $GL_n(\mathbf{A})$ which are invariant under the Cartan involution, i.e are self dual and which satisfy

$$H^*(\mathfrak{g}_0, K, \pi_{\mathbf{A}}\otimes F_\lambda)\neq 0$$

for a finite dimensional representation F_λ with highest weight λ provided that $<\lambda,\alpha>\in 2\mathbb{N}^+$. If $n=2$ it is know from the classical theory of automorphic forms that for all finite dimensional representations F_λ of $GL_2(\mathbb{R})$ there exist cuspidal automorphic representations with

$$H^*(\mathfrak{g}_0, K, \pi_{\mathbf{A}}\otimes F_\lambda)\neq 0.$$

C. Moeglin and J.L. Waldspurger proved that the representations in the residual spectrum of $GL_n(\mathbb{Q})A_G\backslash GL_n(\mathbf{A})$ are obtained as follows as follows [**24**]: A partition $(m, m, \dots m)$ of $n = r\,m$ defines a parabolic subgroup $\mathbb{P}_m$. Then

$$\mathbb{L}_m(\mathbf{A}) = \prod GL_m(\mathbf{A}).$$

Let $\pi_{\mathbf{A},m}$ be a representations in the cuspidal spectrum of $GL_m(\mathbb{Q})A_{G_m}\backslash GL_m(\mathbf{A})$. Consider the standard representation

$$I(\mathbb{P}_m, \prod \pi_{\mathbf{A},m}, \mu) = \mathrm{ind}_{\mathbb{P}_m(\mathbf{A})}^{GL_n(\mathbf{A})} \chi_{\mu+\rho_P} \otimes \prod \pi_{\mathbf{A},m}$$

for μ in the dominant with respect to Σ_{P_m}. If $< \alpha, \mu_0 >= 1$ for all simple roots of in Σ_{P_0} the representation $I(\mathbb{P}_m, \prod \pi_{\mathbf{A},m}, \mu_0)$ is reducible and its Langlands quotient $L(\mathbb{P}_m, \prod \pi_{\mathbf{A},m}, \mu_0)$ is in the residual spectrum. Then $L(\mathbb{P}_1, \prod \pi_{\mathbf{A},1}, \mu_0)$ is one dimensional and $L(\mathbb{P}_2, \prod \pi_{\mathbf{A},2}, \mu_0)$ is a Speh representation discussed in [**28**].

4.2 Now suppose that $I(\mathbb{P}_m, \prod \pi_{\infty,m}, \mu_0)$ has nontrivial $(\mathfrak{g}, K)$–cohomology. Then $I(\mathbb{P}_m, \prod \pi_{\infty,m}, \mu_0)$ has infinitesimal character ρ. This determines the infinitesimal character of $\pi_{\infty,m}$ [**7**]. Since $\pi_{\infty,m}$ is tempered there exists exactly one representation of $GL(m, \mathbb{R})$ with this property. A computation shows that if r is even then the infinitesimal character of $\pi_{\infty,m}$ satisfies the condition of [**3**] and hence there exists a cuspidal representation $\pi_{\mathbf{A},m}$ with $\pi_{\infty,m}$ as factor at the infinite place. If m= 2 then such representation exist for all r.

Theorem IV.1. *Suppose that r or m are even. Then there are representations $L(\mathbb{P}_m, \prod \pi_{\mathbf{A},m}, \mu_0)$ with nontrivial $(\mathfrak{g}, K)$-cohomology in the residual spectrum.*

4.3 Since representations are tempered and hence induced, an straight forward computation shows

Lemma IV.2. *Suppose that*

$$L(\mathbb{P}_m, \prod \pi_{\mathbf{A},m}, \mu_0) = L(\mathbb{P}_m, \prod \pi_{\infty,m}, \mu_0) L(\mathbb{P}_m, \prod \pi_{\mathbf{A}_f,m}, \mu_0))$$

is in the residual spectrum and that

$$H^*(\mathfrak{g}_0, K, L(\mathbb{P}_m, \prod \pi_{\infty,m}, \mu_0)) \neq 0.$$

If m is even, the Langlands subquotient $L(\mathbb{P}_m, \prod \pi_m, \mu_0)$ is unitarily induced from a parabolic subgroup $\mathbb{P}_{2r}$ and Speh representations on each factor of $\mathbb{L}(\mathbf{A})$. If m is odd then it is unitarily induced from a parabolic

for a partition $(2r, 2r, \ldots, 2r, r)$ *On each of the factors of the Levi subgroup isomorphic to* GL_{2r} *we induce from a Speh representation and from the trivial representation on the factor isomorphic to* GL_r.

Using the formula 4.2.1 in [**28**] we conclude

Lemma IV.3. *We keep the assumptions of the previous lemma. If* m *is even then*

$$\begin{aligned} H^i(\mathfrak{g}_0, K, L(\mathbb{P}_m, \prod \pi_{\infty,m}, \mu_0)) &= \mathbb{C} \quad \text{if } i = \frac{r(r-1)m}{4} + \frac{r^2 m(m+1)}{2} \\ &= 0 \quad \text{if } i < \frac{r(r-1)m}{4} + \frac{r^2 m(m+1)}{2}. \end{aligned}$$

If m *is odd then*

$$\begin{aligned} &H^i(\mathfrak{g}_0, K, L(\mathbb{P}_m, \prod \pi_{\infty,m}, \mu_0)) = \\ &\quad = \mathbb{C} \qquad \text{if } i = \frac{r(r+1)(m-1)}{4} + \frac{r^2(m-1)(1+m)}{2} \\ &\quad = 0 \qquad \text{if } i < \frac{r(r+1)(m-1)}{4} + \frac{r^2(m-1)(1+m)}{2}. \end{aligned}$$

Now II.2 implies

Theorem IV.4. *Suppose that* $\mathbb{G} = Gl_n$ *and that* $n = r\,m$. *Then for* K_f *small enough*

$$H^j(S(K_f), \mathbb{C}) \neq 0$$

if

(1) r *and* m *even and* $j = \frac{r(r+1)m}{4} + \frac{r^2 m(m+2)}{2}$

(2) r *even and* m *odd and* $j = \frac{r(r+1)(m-1)}{4} + \frac{r^2(m-1)(1+m)}{2}$,

(3) *m=2 and* $j = r(r+1)/2$.

Remark: All the residual cohomology classes in this theorem are in degrees below 1/2 dim X_∞.

V. Some classes with compact support

If K_f is small enough $S(K_f)$ is orientable. We fix an orientation and have Poincare duality; i.e. a nondegenerate pairing of $H^*(S(K_f),\mathbb{C})$ and $H^*_c(S(K_f),\mathbb{C})$. In this section we first recall the construction of closed pseudo Eisenstein forms with compact support [**25**]. Then we compute the integral of their cup product with residual Eisenstein forms to show, that the residual Eisenstein forms represent nontrivial classes in $H^*_c(S(K_f),\mathbb{C})$.

5.1 Suppose that $\rho : G \to End(V)$ is a finite dimensional representation of G. Choose an admissible scalar product on $\mathfrak{g}$. The corresponding volume form is denoted by

$$dx \in \Omega^d(X,\mathbb{C}), \quad d = \text{ dim } X_\infty.$$

Fix an admissible inner product on V (II.2 in [**7**].) Then there is a hermitian scalar product $< \ >$ on $\bigwedge^i(\mathfrak{g}_0/\mathfrak{k})^* \otimes V$ where the superscript indicates the dual space, There is an induced pointwise scalar product on $\Omega^i(X,\tilde{V})$ and on $\Omega^i(S(K_f),\tilde{V})$ and we obtain a map

$$A : \Omega^i(X,\tilde{V}) \times \Omega^{d-i}(X,\tilde{V}^*) \to \Omega^d(X,\mathbb{C})$$

where V^* is the contragredient representation of V. Let $\bar{V}^*$ be the complex conjugate. Then there is map

$$* : \Omega^i(X,\tilde{V}) \to \Omega^i(X,\tilde{\bar{V}}^*)$$

characterized by the equation

$$A(\omega_1, *\omega_2) =< \omega_1, \omega_2 > dx$$

for $\omega_1, \omega_2 \in \Omega^i(X,\tilde{V})$ and similarly for forms on $S(K_f)$ and on $\mathbb{P}(\mathbb{Q})\backslash X/KK_f$.

5.2 Suppose that $\pi_{\mathbb{L}(\mathbf{A})}$ is a cuspidal representation with $\pi_{\mathbb{L}(\mathbf{A})}^{K_f\cap\mathbb{L}(\mathbf{A}_f)} \neq 0$ and that

$$H^*(\mathfrak{m}_0, K\cap L, \pi_{\mathbb{L}(\mathbf{A})} \otimes F_L) \neq 0$$

for an irreducible finite dimensional representation F_L. Let r_π be the lowest degree in which the $(\mathfrak{m}, K\cap L)$–cohomology of $\pi_{\mathbb{L}(\mathbf{A})}$ is nontrivial. Consider a nontrivial cohomology class $[\omega_\pi]$ in degree r_{π_L}. Since $\pi_{\mathbb{L}(\mathbf{A})}$ is a cuspidal representation with a $K_f\cap\mathbb{L}(\mathbf{A}_f)$–fixed vector we consider ω_π as a differential form representing a nontrivial class in

$$\text{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H^{r_\pi}_{cusp}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/A_L\,(K\,K_f\cap\mathbb{L}(\mathbf{A})),\ F_L).$$

We fix an admissible inner product on F_L and a volume form dx_L. Then $*\omega_\pi$ is a differential form with compact support representing a class in

$$\mathrm{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H_{cusp}^{d_L-r_\pi}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/A_L\,(K\,K_f\cap\mathbb{L}(\mathbf{A})),\bar{F_L}^*)$$

dual to it with respect to Poincaré duality. Here d_L is the dimension of the symmetric space of L.

On the other hand following I.1.4 and I.7.1 in [**7**] we can use representation theory to defined the Poincaré dual

$$*_{rep}\omega_\pi \in \mathrm{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H^{d_L-r_\pi}(\mathfrak{m}_0, K\cap L, \bar{\pi}^*_{L(\mathbf{A})}\otimes \bar{F}^*_L)$$

of the form

$$\omega_\pi \in \mathrm{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H^{r_\pi}(\mathfrak{m}_0, K_\infty\cap L, \pi_{L(\mathbf{A})}\otimes F_L).$$

Since $\pi_{L(\mathbf{A})}$ is a unitary representation $\bar{\pi}^*_{L(\mathbf{A})}\cong\pi_{L(\mathbf{A})}$ and thus we have $\bar{F_L}^*\cong F_L$. The Poincaré dual of a cuspidal class is again a cuspidal class and the results in section II of [**7**] show that $[*_{rep}\omega_\pi]$ and $[*\omega_\pi]$ represent the same cohomology class in

$$\mathrm{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H_{cusp}^{d_L-r_\pi}(\mathbb{L}(\mathbb{Q})\backslash\mathbb{L}(\mathbf{A})/A_L(K\,K_f\cap\mathbb{L}(\mathbf{A})),F_L).$$

5.3 We write $A_P = A_G A_L$ where $\mathfrak{a}_L\subset\mathfrak{g}_0$. Let ω_A be a differential form of compact support on A_L in the top degree so that

$$0\neq[\omega_A]\in H_c^{dim\ A_L}(A_L)$$

and normalized so that $\int_{A_L}\omega_A = 1$.

We use the notation and assumptions of section III. Suppose that $[E(\eta_{\mathfrak{q}}^{res})]$ is a nontrivial cohomology class of $S(K_f)$ in degree

$$e(P,\pi,-\mu_0) = r_{\mathfrak{q}}$$

. Then $r_{\mathfrak{q}} = r_\pi + r_N$ and there exists an irreducible cuspidal representation $\pi_{\mathbb{L}(\mathbf{A})}$, a finite dimensional subrepresentation $F_L\subset H^{r_N}(\mathfrak{n},\mathbb{C})$ of L so that

$$H^{r_\pi}(\mathfrak{m}_0, K\cap L, \pi_{\mathbb{L}(\mathbf{A})}\otimes F_L)\neq 0.$$

We write ω_π for a closed and coclosed form so that

$$[\omega_\pi]\in\mathrm{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H^{r_\pi}(\mathfrak{m}_0, K_\infty\cap L, \pi_{\mathbb{L}(\mathbf{A})}\otimes F_L),$$

represents the same class as $E(\eta_{\mathfrak{q}}^{res})^P$.

Note that we have Poincaré duality on $H^*(\mathfrak{n},\mathbb{C})$. So we consider $F\xrightarrow{\sim}\bar{F}^*\subset H^{dim\ \mathfrak{n}-r_N}(\mathfrak{n},\mathbb{C})$. Then

$$[*_{rep}\omega_\pi]\in\mathrm{ind}_{\mathbb{P}(\mathbf{A}_f)}^{\mathbb{G}(\mathbf{A}_f)} H^{d_L-r_\pi}(\mathfrak{m}_0, K\cap L, \pi_{\mathbb{L}(\mathbf{A})}\otimes H^{dim\ \mathfrak{n}-r_N}(\mathfrak{n},\mathbb{C}))$$

can be considered as a form in degree $d_L - r_\pi + \dim \mathfrak{n} - r_N$.

We consider $*\omega_\pi$ as form on the face $e(\mathbb{P}, K_f)$ of the Borel–Serre boundary on $S(K_f)$ and use the natural identification

$$\mathbb{P}(\mathbb{Q})\backslash X/KK_f \xrightarrow{\sim} e(\mathbb{P}, K_f) \times A_P \,.$$

Then

$$\tilde{\omega}_{\pi,\mu} := *\omega_\pi \wedge \chi_{\mu-\rho_P}\omega_A$$

is a closed form with compact support on $\mathbb{P}(\mathbb{Q})\backslash X/KK_f$ and we get a compactly supported Eisenstein form

$$E(\tilde{\omega}_{\pi,\mu}) = \sum_{\gamma\in G(\mathbb{Q})/\mathbb{P}(\mathbb{Q})} \gamma^*\tilde{\omega}_{\pi,\mu}.$$

For more detail of this construction, see [25].

Theorem V.1. *The pseudo Eisenstein $E(\tilde{\omega}_{\pi,\mu_0})$ series represents a non trivial cohomology class with compact support such that*

$$\int_{\mathbb{G}(\mathbb{Q})\backslash(\mathbb{G}(\mathbf{A})} E(\tilde{\omega}_{\pi,\mu_0}) \wedge res_{\mu=\mu_0} E(\eta_q, \mu) \neq 0 \,.$$

Proof: We have

$$\begin{aligned}
&\int_{\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbf{A})} E(\tilde{\omega}_{\pi,\mu_0}) \wedge res_{\mu=\mu_0} E(\eta_q, \mu) \\
&= \int_{\mathbb{P}(\mathbb{Q})\mathbf{A}_G\backslash G(\mathbf{A})} \tilde{\omega}_{\pi,\mu_0} \wedge res_{\mu=\mu_0} E(\eta_q, \mu) \\
&= \int_{\mathbb{P}(\mathbb{Q})\mathbb{N}(\mathbf{A})A_G\backslash G(\mathbf{A})} \tilde{\omega}_{\pi,\mu_0} \wedge res_{\mu=\mu_0} E(\eta_\rho, \mu_0)^P \\
&= \mathrm{vol}(KK_f) \int_{A_P/A_G} \int_{\mathbb{L}(\mathbb{Q})\mathbb{L}(\mathbf{A})/A_P} *\omega_\pi \wedge \omega_\pi \chi_{\mu_0-\rho_P}\chi_{\rho_P-\mu_0}\omega_A \\
&= vol(KK_f) \int_{\mathbb{L}(\mathbf{A})\cap KK_f} ||\omega_\pi||^2(x)dx \neq 0 \,.
\end{aligned}$$

Hence the second claim holds and by Stoke's theorem $E(\tilde{\omega}_{\pi,\mu_0})$ cannot be the boundary of a compactly supported form. q.e.d.

Corollary V.2. *$E(\tilde{\omega}_{\pi,\mu_0})$ represents a nontrivial cohomology class in the L^2-cohomology of $S(K_f)$.*

Proof: The form $E(\tilde{\omega}_{\mathfrak{q},\mu_0})$ is square integrable. Therefore we can interpret our formulas in L^2-cohomology. $\square$

The pseudo Eisenstein form $E(\omega_{\mathfrak{q},\mu_0})$ is a differential form with compact support. So we can use the methods of Langlands (as written up in the Moeglin/Waldspurger) to compute its "Fourier " expansion in the L^2– spectrum and in particular its projection on the residual cohomology $\mathrm{Hom}_K(\wedge^{n-r_{\mathfrak{q}}}\mathbf{p}, L^2_{\mathrm{res}}(\mathbb{G}(\mathbb{Q})A_G\backslash\mathbb{G}(\mathbf{A})))$. This idea will be pursued in the following sections.

VI. A symplectic modular symbol

Suppose that $\mathbb{G} = GL_4$. We prove in the section that for K_f small enough the symplectic group defines an nontrivial modular symbol for $S(K_f)$.

6.1 Let $\mathbb{G} = GL_4$ over $\mathbb{Q}$ and let $\theta : g \to (g^t)^{-1}$. Define the skew symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and let $\mathbb{H}$ be the symplectic subgroup of $\mathbb{G}$ defined as the fixed points of the involution

$$\sigma : g \to -J\,(g^t)^{-1}\,J.$$

Since $\sigma\theta = \theta\sigma$ the involution θ defines also a Cartan involution on H for the maximal compact subgroup $H \cap K$.

We fix a Borel subgroup $\mathbb{B}$ consists of the upper triangular matrices.

6.2 Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the θ-stable parabolic subalgebra in $gl_4(\mathbb{C})$ defined by J. Then the centralizer L of J in $GL_4(\mathbb{R})$ is isomorphic to $GL(2,\mathbb{C})$. Furthermore $\mathbf{p}_{\mathbb{C}} = (\mathfrak{n} \cap \mathbf{p}_{\mathbb{C}}) \oplus (\mathbf{p}_{\mathbb{C}} \cap \mathfrak{h}_{\mathbb{C}})$. [**31**]

We denote by $A_{\mathfrak{q}}$ also the unitary representation of the cohomologically induced Harish Chandra module $A_{\mathfrak{q}}$. It is a subrepresentation of $L^2(H\backslash G)$. [**29**]

The abelian subgroup

$$A_P = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & a_2^{-1} \end{pmatrix}, a_1, a_2 \in \mathbb{R}^* \right\}$$

is a maximally split torus in H. Let $\mathbb{P}$ be a rational standard maximal parabolic subgroup of $\mathbb{G}$ with Levi factor $\mathbb{L}$ of $GL_2 \times GL_2$. For

K_f small enough there exists a representation $\Pi_{\mathbb{L}(\mathbf{A})}$ in the cuspidal spectrum of $L^2(\mathbb{L}(\mathbb{Q})A_P\backslash\mathbb{L}(\mathbf{A})/\mathbb{L}(\mathbf{A}_f)\cap K_f)$ which has a discrete series representation at the infinite place with infinitesimal character 2ρ. The Eisenstein intertwining operator

$$E(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},\mu) : I(\mathbb{P},\pi_{\mathbb{L}(\mathbf{A})},\mu) \to C^\infty(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/A_G\,(K\,K_f))$$

has pole for $\mu_0 = \frac{1}{2}\rho_P$. The image of the residual intertwining operator for $\mu_0 = \frac{1}{2}\rho_P$ is a representation

$$\Pi_{\mathbf{A}} = \Pi_{\mathbf{A}}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0) = \prod_v \Pi_v = \Pi_\infty \Pi_{\mathbf{A}_f}$$

with $\Pi_\infty = A_{\mathfrak{q}}$ with $r_{\mathfrak{q}} = 3$. The representation $\Pi_{\mathbf{A}}$ is isomorphic to a subrepresentation of the residual spectrum of $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/A_G K_f)$. For details see [**28**].

By theorem III.1 the representation $\Pi_{\mathbf{A}}$ defines for K_f small enough a residual cohomology class $[E(\eta_{\mathfrak{q}}^{res})]$ in degree $r_{\mathfrak{q}}= 3$. We have (see V.2)

$$0 \neq [E(\tilde{\omega}_{\mathfrak{q},\mu_0})] \in H_c^6(S(K_f),\mathbb{C}).$$

Following [**29**] we let $0 \neq \omega_H \in \wedge^{\dim \mathfrak{h}\cap\mathfrak{p}}(\mathfrak{h}\cap\mathbf{p})^*$. Then $*\omega_H \in \wedge^{\dim \mathfrak{n}\cap\mathfrak{p}}\mathbf{p}^*$ defines a differential form on $S(K_f)$. Assuming that K_f is small enough and we have a compatible orientation on $S_H(K_f\cap\mathbb{H}(\mathbf{A}_f)$ we identify

$$E(\tilde{\omega}_{\mathfrak{q},\mu_0})\wedge *\omega_H$$

with a function $\mathcal{E}_{\mathfrak{q}} \neq 0$ on $S(K_f)$. There exists a measure $dh_{\mathbf{A}}$ on $S_H(K_f\cap\mathbb{H}(\mathbf{A}_f))$ induced by a left invariant measure on $\mathbb{H}(\mathbf{A})/(K\,K_f\cap\mathbb{H}(\mathbf{A}))$ so that

Lemma VI.1. *Under our assumptions*

$$\int_{S_H(K_f\cap\mathbb{H}(\mathbf{A}_f))} *E(\eta_{\mathfrak{q}}^{res}) = \int_{S_H(K_f\cap\mathbb{H}(\mathbf{A}_f))} \mathcal{E}_{\mathfrak{q}}(h_{\mathbf{A}})dh_{\mathbf{A}}.$$

Proof: This follows from the arguments on page 5.6 in [**29**]. □

6.3 The representations in the discrete spectrum of $L^2(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/A_G)$ with nontrivial $(\mathfrak{g},K)$–cohomology are [**28**] :

(1) tempered and in the cuspidal spectrum; they have nontrivial $(\mathfrak{g}_0,K)$–cohomology in degree 4 and 5,
(2) the trivial representation **1** in the residual spectrum; it has non trivial $(\mathfrak{g},K)$–cohomology in degree 1, 4, 5 and 9,

(3) the representations $\Pi_{\mathbf{A}}$ in the residual spectrum described in 6.2; they have nontrivial $(\mathfrak{g}, K)$–cohomology in degree 3 and 6.

The L^2–cohomology $H^*_{L^2}(S(K_F), \mathbb{C})$ is infinite in degree 4 and 5 [**5**], but finite in degree 1, 3, 6 and 9. In degree 3 it is represented by residual harmonic forms $E(\eta^{res}_{\mathfrak{q}}) \in \mathrm{Hom}_K(\wedge^3 \mathbf{p}, \Pi_{\mathbf{A}})$ for representations $\Pi_{\mathbf{A}}$.

By the previous lemma and the results of Jacquet and Rallis for K_f small enough [**18**]

$$I_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} : \omega \to \int_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} \omega$$

defines a map

$$I_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} : \mathrm{Hom}_K(\wedge^6 \mathbf{p}, L^2_{dis}(\mathbb{G}(\mathbb{Q}) \backslash \mathbb{G}(\mathbf{A}) / A_G K_f)) \to \mathbb{C}.$$

Let $\Omega_{\pi_{\mathbf{A}}}$ be a smooth form with compact support representing the same L^2–cohomology class as $*E(\eta^{res}_{\mathfrak{q}})$. Then

$$*E(\eta^{res}_{\mathfrak{q}}) - \Omega_{\pi_{\mathbf{A}}} = d\eta.$$

The integral $\int_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} d\eta$ is finite.

Let d_H be the differential on $S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))$. An easy check shows that for $h \in S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))$

$$d\eta(h) \wedge *\omega_H(h) = d_H \eta(h) \wedge *\omega_H(h).$$

Thus by the L^2–Stokes theorem for complete manifolds [**12**]

$$\int_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} d\omega = 0.$$

Thus we proved

Proposition VI.2. *For K_f small enough the map*

$$I_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} : \omega \to \int_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} \omega$$

defines a modular symbol

$$[I_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))}] \in H^3(S(K_f), \mathbb{C}).$$

Theorem VI.3. *Suppose that $\mathbb{G} = GL(4)$ and $\mathbb{H}$ is a symplectic group compatible with the choice of the maximal compact subgroup $K \subset GL(4, \mathbb{R})$. There exists a K_f so that*

$$[I_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))}]$$

is not zero.

Proof: The linear functional

$$f \to \int_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} f(h_{\mathbf{A}}) dh_{\mathbf{A}}$$

is not zero on the K–finite functions f by the work of Jacquet and Rallis in the residual spectrum [**18**]. Let $\mathcal{E}_{\mathfrak{q}}$ be the function defined in VI.1. By the previous considerations we may assume that this function is in $L^2_{res}(\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A}/A_G K_f)$. There exists a $g_0 \in \mathbb{G}(\mathbb{Q})$ near the identity so that

$$\int_{S_H(K_f \cap \mathbb{H}(\mathbf{A}_f))} \mathcal{E}_{\mathfrak{q}}(g_0 h_{\mathbf{A}}) dh_{\mathbf{A}} \neq 0.$$

Here we integrate over the orbit of g_0 under $\mathbb{H}(\mathbf{A})$. Since the rational elements are dense in G we may assume that g_0 is rational. There is a subgroup $K_{g_0,f}$ of finite index in both K_f and its g_0 conjugate so that $[S_H(K_{g_0,f} \cap \mathbb{H}(\mathbf{A}_f)))](\mathcal{E}_{\mathfrak{q}}) \neq 0$ $\square$

VII. On the volume of $S(K_f)$

In the proof of theorem V.1 we have used a formula for the integral of a wedge product of a pseudo Eisenstein form with an Eisenstein form. We show how this formula is related to the computation of the Tamagawa number $\tau(\mathbb{G})$ of $\mathbb{G}$. We sketch this only in the simplest case, i.e. if $\mathbb{G}/\mathbb{Q}$ is split and simply connected.

7.1. Preliminaries. Let $\mathbb{G}/\mathbb{Q}$ be a split semi simple connected and simply connected algebraic group with $\mathbb{Q}$–Lie algebra $\mathfrak{g}$. We fix a system R of roots, a system $R^+ \subset R$ of positive roots and a Chevalley basis $\{X_\alpha, H_i\}, i = 1, \ldots, l, \alpha \in R$, of $\mathfrak{g}$ is standard notation. Then the H_i are a $\mathbb{Q}$–basis of the $\mathbb{Q}$–Lie algebra of a split torus $\mathbb{T}$ of $\mathbb{G}$ which is contained in a Borel group $\mathbb{B}$ of $\mathbb{G}$ and $\mathbb{B}$ has unipotent radical $\mathbb{N}$, where the $\mathbb{Q}$–Lie algebra of $\mathbb{N}$ has basis $\{X_\alpha\}_{\alpha \in R^+}$. The Chevalley basis determines an invariant differential form of highest degree and hence invariant measures ω^G_ν on $\mathbb{G}(\mathbb{Q}_\nu)$ for all places v of $\mathbb{Q}$. Then the restricted tensor product

$$\omega^G := \hat{\bigotimes}_v \omega_v$$

is an invariant measure on $\mathbb{G}(\mathbf{A})$. The measure ω^G induces a measure dg on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})$ and by definition the Tamagawa number $\tau(\mathbb{G})$ is defined as

$$\tau(\mathbb{G}) = \int_{G(\mathbb{Q})\backslash G(\mathbf{A})} dg.$$

For all this see [21].

Let p be a prime. Since $\mathbb{G}/\mathbb{Q}$ is a Chevalley group $\mathbb{G}(\mathbb{Z}_p) =: K_p$ makes sense. Put $K_\infty = K$ and let $K \subset \mathbb{G}(\mathbb{R})$ be a maximal compact subgroup of $G(\mathbb{R})$. For all places v of $\mathbb{Q}$ put $N_v := \mathbb{N}(\mathbb{Q}_v), T_v = \mathbb{T}(\mathbb{Q}_v)$, and $G_v = \mathbb{G}(\mathbb{Q}_v)$. Then we have the Iwasawa decomposition $G_v = N_v T_v K_v$.

Put

$$\rho_v(t_v) = |\det Ad(t_v)_{|\mathfrak{n}_v}|_v^{1/2}.$$

Here $|\ |_v$ is the normalized absolute value on $\mathbb{Q}_v$, $\mathfrak{n}_v$ the $\mathbb{Q}_v$–Liealgebra of N_v and $Ad(t_v)$ is the adjoint action of $t_v \in T_v$ on $\mathfrak{n}_v$. Put $|\rho|(t) = \prod_v \rho_v(t_v)$ for $t \in \mathbb{T}(\mathbf{A})$. For $v \neq \infty$ let ω_v^T be the measure on T_v given by the $\{H_i\}$ with normalization $\mathrm{vol}_{\omega_v}(\mathbb{T}(\mathbb{Z}_v)) = 1$. If $v = \infty$ let ω_v^T be the measure determined by the $\{H_i\}$. Then the restricted product

$$\omega^T = \hat{\bigotimes}_v \omega_v^T$$

is a measure on $\mathbb{T}(\mathbf{A})$. We choose on $\mathbb{N}(\mathbf{A})$ the measure $\omega^N = \otimes_v \omega_v^N$, where the measures ω_v^N are determined by the $\{X_\alpha\}_{\alpha \in \mathbb{R}^+}$. Then we have for the finite places

$$\omega_v^G = \rho_v^{-2} \omega_v^N \otimes \omega_v^T \otimes \omega_v^K,$$

where ω_v^K is the measure on K_v given by restriction of ω_v^G . We choose a measure ω_∞^K on K_∞ such that this formula also hold for $v = \infty$. To simplify the notation we write for $\varphi \in C_c(\mathbb{G}(\mathbf{A})$ as usual

$$\begin{aligned} \int_{\mathbb{G}(\mathbf{A})} \varphi(g) d\omega^G(g) &=: \int_{\mathbb{G}(\mathbf{A})} \varphi(g) dg \\ &= \int_{\mathbb{N}(\mathbf{A})} \int_{\mathbb{T}(\mathbf{A})} \int_{KK_f} \varphi(ntk) |\rho|^{-2}(t) dn dt dk. \end{aligned}$$

The measure dn induces a measure of mass 1 also denoted by dn on the quotient $\mathbb{N}(\mathbb{Q})\backslash\mathbb{N}(\mathbf{A})$. The measure dt induces a measure again denoted by dt on the quotient $\mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbf{A})$.

7.2. Let $\varphi : \mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbf{A}) \longrightarrow \mathbb{R}$ be a continuous compactly supported function which is right $(KK_f) \cap \mathbb{T}(\mathbf{A})$–invariant such that

$$\int_{\mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbf{A})} \varphi(t) |\rho|^{-2}(t) dt = 1.$$

Since

$$\mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbf{A})/(KK_f) \cap \mathbb{T}(\mathbf{A}) \xrightarrow{\sim} \mathbb{N}(\mathbf{A})\mathbb{B}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})/KK_f$$

we view φ as left $\mathbb{N}(\mathbf{A})\mathbb{B}(\mathbb{Q})$–invariant and right KK_f–invariant function on $\mathbb{G}(\mathbf{A})$. Then the pseudo-Eisenstein series

$$E(\varphi)(g) := \sum_{\gamma\in\mathbb{B}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{Q})} \varphi(\gamma g), \quad g \in \mathbb{G}(\mathbf{A}) \,,$$

is a compactly supported function on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})$. We use 7.1 and get

$$\int_{\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})} E(\varphi)(g)dg = \mathrm{vol}(KK_f).$$

where dg also denotes the induced measure on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})$. So if $\mathbf{1}$ denotes the constant function,

$$\mathrm{vol}(KK_f)/\tau(\mathbb{G}) \cdot \mathbf{1}$$

is the projection of $E(\varphi)$ on the constant function in the L^2-spectral decomposition.

We can also interpret this integral as the integral of a wedge product of a residual Eisenstein form in degree 0 with a pseudo Eisenstein form in degree $d = \dim X_\infty$.

7.3 To determine another formula for projection of the function $E(\varphi)$ onto the trivial representation we use Fourier analysis on the torus. Let $\lambda : \mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbf{A}) \longrightarrow \mathbb{C}^*$ be a continuous character and put

$$\hat{\varphi}_\lambda(g) = \int_{\mathbb{T}(\mathbb{Q})\backslash\mathbb{T}(\mathbf{A})} |\rho|^{-1}(t)\lambda^{-1}(t)\varphi(tg)dt \,.$$

The function $\hat{\varphi}_\lambda$ is $\mathbb{N}(\mathbf{A})\mathbb{B}(\mathbb{Q})$–left invariant and KK_f–right invariant. Moreover

$$\hat{\varphi}_\lambda(t) = \lambda(t)|\rho|(t)\hat{\varphi}_\lambda(1)$$

and

$$\hat{\varphi}_{|\rho|}(1) = 1$$

by our choice of φ.

As usual we put

$$E(\varphi,\lambda)(g) := \sum_{\gamma\in\mathbb{B}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{Q})} \hat{\varphi}_\lambda(\gamma g).$$

It is well known that this Eisenstein series is convergent if $Re(\lambda)$ is sufficiently big. This holds in particular for $\lambda = |\rho|^s$ and $1 < s \in \mathbb{R}$. We have

$$\int_{\mathbb{N}(\mathbb{Q})\backslash\mathbb{N}(\mathbf{A})} E(\varphi,\lambda)(ng)dn = \sum_{w\in W} M(w,\lambda)\hat{\varphi}_\lambda(g),$$

where

$$M(\omega,\lambda)\hat{\varphi}_\lambda(g)=\int_{\mathbb{N}(\mathbb{Q})\backslash\mathbb{N}(\mathbf{A})}\hat{\varphi}_\lambda(wng)dn.$$

It is well known that $E(\varphi,\lambda)$ has a meromorphic continuation in λ with a residue of order ℓ at $\lambda=|\rho|$ which is denoted by $\mathrm{res}_{\lambda=|\rho|}E(\varphi,\lambda)$. Moreover this residue is a multiple $\mathbf{c}$ of the constant function $\mathbf{1}$ on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})$. To obtain an exact formula for $\mathbf{c}$ and to calculate residue it suffices to consider the one dimensional problem $\lambda_s=|\rho|^s, 1<s\in\mathbb{R}$. We use

$$(M(w,|\rho|^s)\hat{\varphi}_{|\rho|^s})(t)=|\rho|(t)|w\rho|^s(t)(M(w,|\rho|^s)\hat{\varphi}_{|\rho|^s})(1),\quad t\in\mathbb{T}(\mathbf{A}).$$

It is well known that that for $s>1,(M(w,|\rho|^s)\hat{\varphi}_{|\rho|^s})(1)$ can be calculated as a product of local contributions using the results of Gindikin-Karpelevic for all places v of $\mathbb{Q}$. We get

$$\mathrm{res}_{s=1}(M(w_0,|\rho|)^s)\hat{\varphi}_{|\rho|^s})(1)=\mathrm{vol}(KK_f).$$

Hence

$$\mathrm{res}_{s=1}E(\varphi,|\rho|^s)=\mathrm{vol}(KK_f)\,\mathbf{1},$$

where $\mathbf{1}$ is the constant function on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbf{A})$.

We use the notation given in the proof of V.1 and deduce that

$$\mathcal{M}(\pi_{\mathbb{L}(\mathbf{A})},\mu_0,\omega_0)\eta_q=\ \mathrm{vol}(KK_f)\mathbf{1}.$$

So integrating the projection of $E(\varphi)$ on the constant function we get $\mathrm{vol}(KK_f)\tau(\mathbb{G})$. On the other hand in 7.2 we showed that this integral is $\mathrm{vol}(KK_f)$. Hence

$$\tau(\mathbb{G})=1.$$

This is due to Langlands [21].

In a sequel to this paper we will use similar techniques to compute the integrals defined by other residual modular symbols.

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