ON THE RESTRICTION OF REPRESENTATIONS OF SL(2, $\mathbb{C}$) TO SL(2,$\mathbb{R}$)

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ABSTRACT. We prove that for certain range of the the continuous parameter the complementary series representation of SL(2,$\mathbb{R}$) is a direct summand of the complementary series representations of SL(2,$\mathbb{C}$). For this we construct a continuous " geometric restriction map" from the complementary series representations of SL(2,$\mathbb{C}$) to the complementary series representations SL(2,$\mathbb{R}$). In the second part we prove that the Steinberg representation $\sigma$of SL(2,$\mathbb{R}$) is a direct summand of the restriction of the Steinberg representation π of SL(2,$\mathbb{C}$). We show that σ does not contain any smooth vectors of π.

1. **Introduction**

Let $G = SL(2, \mathbb{C})$ and denote by $B(\mathbb{C})$ the (Borel-)subgroup of upper triangular matrices in G, by $N(\mathbb{C})$ the subgroup of unipotent upper triangular matrices in G. Given an element $b = \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}$ of $B(\mathbb{C})$, denote by $\rho(b) =\mid a \mid^2$. The group $K = SU(2)$ is a maximal compact subgroup of G. Given a complex number u, denote by π_u the space of functions on G which satisfy, for all $b \in B(\mathbb{C})$ and all $g \in G(\mathbb{C})$ the formula

$$f(bg) = \rho(b)^{1+u} f(g)$$

and in addition are K-finite under the action of K by right translations. If $Re(u) > 0$ define the map $I_G(u) : \pi_u \to \pi_{-u}$ by the formula (for $x \in G$),

$$(I_G(u)f)(x) = \int_{N(\mathbb{C})} dn f(w_0 nx).$$

The integral converges (for $Re(u) > 0$). If u is real and $0 < u < 1$, then the pairing

$$< f, f >_{\pi_u} = \int_K \overline{f}(k)(I_G(u)f)(k))dk$$

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defines a positive definite G-invariant inner product on K-finite functions in π_u. The completion $\widehat{\pi_u}$ with respect to this inner product is complementary series representation with continuous parameter u.

Given a complex number $u' \in \mathbb{C}$, denote by $\sigma_{u'}$ the representation of $(\mathfrak{h}, K_H)$, where $\mathfrak{h}$ is the Lie algebra of $H = SL(2,\mathbb{R})$, and $K_H = SO(2)$ is the maximal compact subgroup of H, defined as the space of complex valued right K_H-finite functions on H such that for all upper triangular matrices $b = \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}$ in H and all $h \in H$, we have $f(bh) =\mid a \mid^{1+u'} f(h)$. The character $\mid a \mid^2$ is the character $\rho(b)^2$ (the "sum" of positive roots).

Denote by N_H the group of unipotent upper triangular matrices in H. If $Re(u') > 0$, we define the intertwining operator $I_H(u') : \sigma_{u'} \to \sigma_{-u'}$ as follows: for all $g \in H$, set

$$(I_H(u')f)(g) = \int_{N_H(\mathbb{R})} dn f(w_0 n g).$$

The integral is convergent if $Re(u') > 0$. Now for $f_H, g_H \in \sigma_{u'}$ and u' is real and $0 < u' < 1$, the pairing

$$< f_H, g_H >_{\sigma_{u'}} = \int_{K_H} \overline{f}_H(k_H)(I_H(u')g_H)(k_H)dk_H.$$

defines a positive definite H–invariant inner product on σ'_u. The completion is the complementary series representation $\widehat{\sigma'_u}$.

Theorem 1.1. *Let $\frac{1}{2} < u < 1$ and $\widehat{\pi}_u$ denote the completion of the complementary series representation of $SL(2,\mathbb{C})$. Define similarly, the completion $\widehat{\sigma}_{u'}$ of the complementary series $\sigma_{u'}$for $SL(2,\mathbb{R})$. If $u' = 2u-1$, then $\widehat{\sigma}_{u'}$ is a direct summand of $\widehat{\pi}_u$ restricted to $SL(2,\mathbb{R})$.*

This theorem is proved in [5] . In the proof in the present paper we realize the "abstract" projection map from $\widehat{\pi}_u$ to $\widehat{\sigma}_{2u-1}$ as a simple geometric map of sections of a line bundle on the flag varieties of $G = SL(2,\mathbb{C})$ and $H = SL(2,\mathbb{R})$. In a sequel we will use this idea to analyze the restriction of the complementary series representations of SO(n,1)

Consider the **Steinberg representation** $\widehat{\pi} = Ind_B^G(\chi)$. Here, Ind refers to **unitary** induction from a unitary character χ of B. Given two functions $f, f' \in \pi$, the product $\phi = f\overline{f'}$ ($\overline{f'}$ is the complex conjugate of f') lies in π_0. The G-invariant inner product on π is defined by

$$< f, f' >= \int_G (f\overline{f'})dg$$

We have the exact sequence of $(\mathfrak{g}, K)$–modules

$$0 \to \pi \to \pi_1 \to 1 \to 0$$

The $(\mathfrak{h}, K_H)$– module of the Steinberg representation σ of SL(2, $\mathbb{R}$) is defined by the exact sequence

$$0 \to \sigma \to \sigma_1 \to 1 \to 0$$

and the completion $\widehat{\sigma}$ is a direct sum of 2 discrete series representations.

Theorem 1.2. *The restriction to H of $\widehat{\pi}$ contains the Steinberg Representation $\widehat{\sigma}$ of H as a direct summand.*

More precisely, the restriction is a sum of the Steinberg representation σ of H, its complex conjugate $\overline{\sigma}$, and a sum of two copies of $L^2(H/K \cap H)$ where $K \cap H$ is a maximal compact subgroup of H. By a theorem of T. Kobayashi (theorem 4.2.6 in citeko) this implies that $\widehat{\sigma}$ does not contain any nonzero K–finite vectors in $\widehat{\pi}$. Using an explicit description of the functions in the subspace $\widehat{\sigma}$ we prove a stronger result

Theorem 1.3. *The intersection*

$$\widehat{\sigma} \cap \widehat{\pi}^{\infty} = 0.$$

That is $\widehat{\sigma}$ does not contain any nonzero smooth vectors in $\widehat{\pi}$.

It is very important in the above theorems to consider a unitary representation of G respectively H, and not only the unitary $(\mathfrak{g}, K)$ respectively $(\mathfrak{h}, K_H)$-modules as the following example shows.

Fix a semi-simple non-compact real algebraic group G and let $\mathcal{C}_c(G)$ denote the space of continuous complex valued functions on G with compact support. Let π denote an irreducible representation on a Hilbert space (which, we denote again by π) of G of the complementary series, which is unramified (i.e. fixed under a maximal compact subgroup K of G). Fix a non-zero K invariant vector v in π.

Denote by $\| w \|_\pi$ the metric on the space π. Given $\phi \in \mathcal{C}_c(G)$, we get a bounded operator $\pi(\phi)$ on π . Define a metric on $\mathcal{C}_c(G)$ by setting

$$\| \phi \|^2 = \| \pi(\phi)(v) \|_\pi^2 + \| \phi \|_{L^2}^2,$$

where the latter is the L^2-norm of ϕ. The group G acts by left translations on $\mathcal{C}_c(G)$ and preserves the above metric. Hence it operates by unitary transfomations on the completion (the latter is a Hilbert space) of this metric.

Proposition 1.4. *Under the foregoing metric, the completion of* $\mathcal{C}_c(G)$ *is the direct sum of the Hilbert spaces*

$$\pi \oplus L^2(G).$$

The action of the group G on the direct sum, restricted to the subspace $\mathcal{C}_c(G)$, is by left translations.

Note that the direct sum $\pi \oplus L^2(G)$ and $L^2(G)$ both share the same dense subspace $\mathcal{C}_c(G)$ on which the G action is identical, namely by left translations, and yet the completions are different: $\pi \oplus L^2$ is the completion with respect to the new metric and $L^2(G)$ is the completion under the L^2-metric. We have therefore an example of two non-isomorphic unitary G-representations with an isomorphic dense subspace. This is not posible in the case of **irreducible** unitary representations, as can be easily seen.

Proof. The kernel to the map $\phi \mapsto \pi(\phi)v$ on $\mathcal{C}_c(G)$ is just those functions, whose Fourier transform vanishes at a point on $\mathbb{C}$ (the latter is the space of not-necessarily unitary characters of $\mathbb{R}$). This is clearly dense in $\mathcal{C}_c(G)$ and hence dense in $L^2(G)$. The restriction of the new metric to the kernel is simply the L^2-metric, and the kernel is dense in L^2. Therefore, the completion of the kernel gives all of L^2.

Since the map from $\mathcal{C}_c(G)$ to the direct sum is non-zero, it follows that the completion of $\mathcal{C}_c(G)$ can not be only L^2. The irreducibility of π now implies that the completion must be $\pi \oplus L^2(G)$.

2. Complementary Series for $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$

2.1. **Complementary Series for** $SL(2,\mathbb{R})$ **.**

Given a complex number $u' \in \mathbb{C}$, denote by $\sigma_{u'}$ the representation of $(\mathfrak{h}, K_H)$, where $\mathfrak{h}$ is the Lie algebra of $H = SL(2,\mathbb{R})$, and $K_H = SO(2)$ is the maximal compact subgroup of H, defined as the space of complex valued right K_H-finite functions on H such that for all upper triangular matrices $b = \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}$ in H and all $h \in H$, we have $f(bh) = \mid a \mid^{1+u'} f(h)$. The character $\mid a \mid^2$ is the character $\rho(b)^2$ (the "sum" of positive roots). Define $\sigma_{-u'}$ similarly, replacing u' by $-u'$.

Denote by N_H the group of unipotent upper triangular matrices in H. If $Re(u') > 0$, we define the intertwining operator $I_H(u') : \sigma_{u'} \to \sigma_{-u'}$ as follows: for all $g \in H$, set

$$(I_H(u')(f))(g) = \int_{N_H(\mathbb{R})} dnf(w_0ng).$$

The convergence of the integral follows under the assumption that $Re(u') > 0$ and by using the Iwasawa decomposition of the element w_0n. It is easy to check that I_H takes $\sigma_{u'}$ into $\sigma_{-u'}$. We will restrict $f \in \sigma_{u'}$ and $g \in \sigma_{-u'}$ and consider the pairing $(f, g) = \int_{K_H} dkf(k)g(k)$. This is easily seen to be an H-invariant pairing. Now, given $f, g \in \sigma_{u'}$, define the pairing

$$< f, g >_{\sigma_{u'}} = (f, I_H(u')(g)).$$

The fact that I_H is an intertwining operator says that this pairing $< f, g >$ between elements of $\sigma_{u'}$ is also an H-invariant pairing. If u' is real and $0 < u' < 1$, then it may be shown that the pairing between f and $\overline{f}$ is positive unless $f = 0$. Via this isomorphism I_H, we then get an inner product on $\sigma_{-u'}$ for all u' with $0 < u' < 1$.

The space $\sigma_{u'}$ consists, by construction, of K_H-finite vectors and the restriction of $\sigma_{u'}$ to K_H is an injection; under this map, $\sigma_{u'}$ may be identified with trigonometric aolynomials on K_H which are *even*. The space of even trigonometric polynomials is spanned by the characters $\chi_l = \theta \mapsto e^{4\pi l\theta}$, l going through all integers. Each χ_l-eigenspace in $\pi_{u'}$ is one dimensional and has a unique vector , denoted $\chi_l(u, h)$ such that for all $k \in K_H$ we have $\chi_l(u', k) = \chi_l(k)$. Being an intertwining map, the map $I_h(u')$ maps $\chi_l(u', h)$ into a multiple of $\chi_l(-u', h)$. After replacing $I_h(u')$ by a scalar multiple of itself, we may assume that $I_h(u')$ maps $\chi_0(u', h)$ into $\chi_0(-u', h)$. This normalised intertwining operator will, by an abuse of notation, stil be denoted $I_h(u')$. One computes that after this normalisation, we have, for all integers $l \neq 0$ and all $k \in K_H$,

$$I_H(u')(\chi_l(u', k)) = d_l(u')\chi_l(-u', k),$$

where

$$d_l(u') = \frac{(1-u')(3-u')\cdots(2\mid l\mid -1-u')}{(1+u')(3+u')\cdots(2\mid l\mid -1+u')} \tag{1}$$

and $d_0(u') = 1$. We note that for $\chi_l(u', h)$ we have

$$\begin{aligned} \|\chi_l(u', .)\|^2_{\sigma_{u'}} &= < \chi_l(u', .), I_H(u')(\chi_l(u', .)) > \\ &= d_l(u') \| \chi_l \|^2_{L^2(K_H)} . \end{aligned} \tag{2}$$

Therefore, the norm on $\sigma_{-u'}$ is given by

$$|| \chi_l(-u',.) ||^2_{\sigma_{-u'}} = \frac{1}{d_l(u')} || \chi_l ||^2_{L^2(K_H)} . \tag{3}$$

Note that the space $Rep(\{\pm 1)\backslash K_H)$ is a direct sum (over integers $l \geq 1$ of the spaces $\sigma_l = \mathbb{C}\chi_l \oplus \mathbb{C}\chi_{-l}$ and of $\mathbb{C}\chi_0$. The elements of σ_l may be thought of as the space of **Harmonic Polynomials** in the circle of degree $2l$.

It is clear that $d_l(u') < 1$. It can be shown, as $\mid l \mid$ tends to infinity, that $d_l(u')$ satisfies the following asymptotic: there exists a constant C such that $d_l(u') \simeq C\frac{1}{|l|^{u'}}$.

Notice that if $u' = 2u - 1$ and $l \geq 1$ then

$$d_l(u') = \frac{(1-u)(2-u)\cdots(l-u)}{1+u)(2+u)\cdots(l+u)}\frac{(l+u)}{u}.$$

Define $\lambda_l(u)$ by the formula

$$d_l(u') = \lambda_l(u)\frac{l+u}{u}.$$

We have already noted that if $0 < u' < 1$, the pairing $<,>$ on $\sigma_{u'}$ is positive definite. This easily follows from the formula (1) for $d_l(u')$, which shows that $d_l(u') > 0$, and the equation (2).

2.2. **Complementary Series of** $SL(2,\mathbb{C})$**.** Let $G = SL(2,\mathbb{C})$ and denote by $B(\mathbb{C})$ the (Borel-)subgroup of upper triangular matrices in G, by $N(\mathbb{C})$ the subgroup of unipotent upper triangular matrices in G. Given an element $b = \begin{pmatrix} a & n \\ 0 & a^{-1} \end{pmatrix}$ of $B(\mathbb{C})$, denote by $\rho(b) =\mid a \mid^2$. The group $K = SU(2)$ is a maximal compact subgroup of G. Given a complex number u, denote by π_u the space of functions on G which satisfy, for all $b \in B(\mathbb{C})$ and all $g \in G(\mathbb{C})$ the formula

$$f(bg) = \rho(b)^{1+u}f(g)$$

and in addition are K-finite under the action of K by right translations. Given $f \in \pi_u$ and $g \in \pi_{-u}$, define the pairing $(f,g) = \int_K f(k)g(k)dk$. It can be shown that this is a (right) G-invariant pairing.

If $Re(u) > 0$ define the map $I_G(u) : \pi_u \to \pi_{-u}$ by the formula (for $x \in G$),

$$(I_G(u)(f))(x) = \int_{N(\mathbb{C})} dnf(w_0nx).$$

The integral converges (for $Re(u) > 0$).

For any $u \in \mathbb{C}$, the restriction of the representation π_u to the maximal compact subgroup K is isomorphic to $Rep(T\backslash K)$ where T is the group of diagonal matrices in K, and $Rep(T\backslash K)$ denotes the space of representation functions on $T\backslash K$ on which K acts by right translations. It is known that as a representation of K, we have

$$Rep(T\backslash K) = \oplus_{m\geq 0}\rho_m,$$

where $\rho_m = Sym^{2m}(\mathbb{C}^2)$ is the $2m$-th symmetric power of $\mathbb{C}^2$, the standard two dimensional representation of K; ρ_m is irreducible and occurs exactly once in $Rep(T\backslash K)$. The same decomposition holds if π_u is replaced by π_{-u}. The operator $I_G(u)$ may be normalised so that under the identification of the K-representations

$$\pi_u \simeq R(T\backslash K) \simeq \pi_{-u},$$

it acts on each ρ_m by the scalar

$$\lambda_m(u) = \frac{(1-u)(2-u)\cdots(m-u)}{(1+u)(2+u)\cdots(m+u)}.$$

The formula for $\lambda_m(u)$ shows that if u is real and $0 < u < 1$, then the pairing

$$< f, f >= (\overline{f}, I_G(u)f)$$

defines a positive definite G-**invariant** inner product on K-finite functions in π_u as can be easily checked using the formula for $I_G(u) = \lambda_m(u)$ on ρ_m. If $0 < u < 1$, then the operator $I_G(u)\pi_u \to \pi_{-u}$ is an isomorphism, and hence the inner product on π_u defines, via the isomorphism $I_G(u)$, a G-invariant inner product on π_{-u}. Denote the completions of π_u and π_{-u} with respect to these inner products, by $\widehat{\pi}_u$ and $\widehat{\pi}_{-u}$.

Lemma 2.1. *Let $\rho_m = Sym^{2m}(\mathbb{C}^2)$ be the $2m$-th symmetric power of the standard representation of $K = SU(2)$.Let $(,)$ be a K-invariant inner product on ρ_m and v, w vectors in ρ_m of norm one with respect to $(,)$ such that v is invariant under the diagonals T on K and the group $K_H = SO(2)$ acts by the character χ_l on the vector w. Then, the formula*

$$\mid (v, w) \mid = \frac{2^m\Gamma(\frac{m-l+1}{2})\Gamma(\frac{m+l+1}{2})}{\sqrt{(m-l)!(m+l)!}}$$

holds.

Proof. The formula clearly does not depend on the K-invariant metric chosen, since any two invariant inner products are scalar multiples of each other. We will view elements of ρ_m as homogeneous polynomials of degree $2m$ with complex coefficients in two variables X and Y such that if $k = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$, then k acts on $X$and Y by $k(X) = \alpha X - \overline{\beta} Y$ and $k(Y) = \beta X + \overline{\alpha} Y$. The vector $v' = X^m Y^m \in \rho_m$ is invariant under the diagonal subgroup T of $SU(2)$.

The subgroup $K_H = SO(2)$ is conjugate to T by the element $k_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. That is, $SO(2) = k_0 T k_0^{-1}$. If $-m \leq l \leq m$ then the element $w'' = X^{m+l} Y^{m-l}$ is an eigenvector for T with eigencharacter $\chi_l : \theta \mapsto e^{4\pi i l}$. Consequently, the vector $w' = k_0(w'')$ is an eigenvector of $SO(2)$ with eigencharacter χ_l.

If

$$f = \sum_{\mu=-m}^{m} a_\mu X^{m+\mu} Y^{m-\mu}, \; g = \sum_{\mu=-m}^{m} b_\mu X^{m+\mu} Y^{m-\mu} \in \rho_m,$$

then the inner product

$$(f, g) = \sum_{\mu=-m}^{m} a_\mu \overline{b_\mu} (m+\mu)!(m-\mu)!$$

is easily shown to be K-invariant (see p. 44 of [6]). Therefore, the vectors

$$w = \frac{X^m Y^m}{m!}, \; v = k_0\left(\frac{X^{m+l} Y^{m-l}}{\sqrt{(m+l)!(m-l)!}}\right) \tag{4}$$

satisfy the conditions of Lemma 2.1. We compute

$$k_0(X^{m+l} Y^{m-l}) = \left(\frac{X + iY}{\sqrt{2}}\right)^{m+l} \left(\frac{iX + Y}{\sqrt{2}}\right)^{m-l}$$

$$= \left(\sum_{a=0}^{m+l} \binom{m+l}{a} X^a (iY)^{m+l-a}\right)\left(\sum_{b=0}^{m-l} \binom{m-l}{b} (iX)^b Y^{m-l-b}\right).$$

Using the fact that the vectors $X^{m+l} Y^{m-l}$ are orthogonal for varying l, we find that the inner product of $X^m Y^m$ with $k_0(X^{m+l} Y^{m-l})$ is the sum (over $a \leq m+l$ and $b \leq m-l$)

$$\sum_{a+b=m} \frac{(m!)^2}{2^m} i^{m+l-a} i^b \binom{m+l}{a} \binom{m-l}{b}.$$

This sum is [because of Lemma 2.2 below], equal (in absolute value) to

$$\frac{1}{\pi}\frac{m!}{2^m}4^m\Gamma(\frac{m+l+1}{2})\Gamma(\frac{m-l+1}{2}). \tag{5}$$

if $m+l$ is even (and 0 if $m+l$ is odd).

The Lemma follows from equations (4) and (5). □

Lemma 2.2. *The equality*

$$\frac{m!}{2^m}\sum_{a+b=m}\binom{m+l}{a}\binom{m-l}{b}(-1)^b = \frac{1}{\pi}2^m\Gamma(\frac{m+l+1}{2})\Gamma(\frac{m-l+1}{2})$$

holds if $m+l$ is even; the sum on the left hand side is 0 *if $m+l$ is odd.*

Proof. If $f(z) = \sum a_k z^k$ is a polynomial with complex coefficients, then the coefficient a_m is given by the formula

$$a_m = \frac{1}{2\pi}\int_0^{2\pi} d\theta f(e^{i\theta})e^{-im\theta}.$$

The sum Σ on the left hand side of the statement of the Lemma is clearly ($\frac{m!}{2^m}$ times) the m^{th}-coefficient of the polynomial

$$f(z) = (1+z)^{m+l}(1-z)^{m-l}.$$

We use the foregoing formula for the m^{th} coefficient to deduce that

$$\Sigma = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-im\theta}(1+e^{i\theta})^{m+l}(1-e^{i\theta})^{m-l}.$$

After a few elementary manipulations, the integral becomes

$$\frac{i^{m-l}4^m}{\pi}\int_0^2 \pi d\theta(cos(\frac{\theta}{2})sin(\frac{\theta}{2}))^m(\frac{sin(\frac{\theta}{2})}{cos(\frac{\theta}{2})})^l.$$

Substituting $t = tan(\theta/2)$ the integral becomes

$$\frac{2i^{m-l}4^m}{\pi}\int_0^\infty dt\frac{t^{m-l}}{(1+t^2)^{m+1}}$$

and the latter, when multiplied by $\frac{m!}{2^m} = \frac{\Gamma(m+1)}{2^m}$, is the right side of the Lemma 2.2. □

We now collect some estimates for the Gamma function which will be needed later.

Lemma 2.3. *If $Re(z) > 0$, then we have, as m tends to infinity, the asymptotic relation*

$$\Gamma(m+z) \simeq Constant\ m^{m+z-\frac{1}{2}}.\frac{1}{e^m}.$$

In particular, as m tends to infinity through integers,

$$m! = \Gamma(m+1) \simeq Constant\ m^{m+\frac{1}{2}} \frac{1}{e^m}.$$

The formula for the inner product in Lemma 2.1 is unchanged if we replace l by $-l$. We may therefore assume that $l \geq 0$. Let $m \geq 0$ and $0 \leq l \leq m$. Put $m = k + l$. From Lemmas 2.3 and 2.1 we obtain (notation as in Lemma 2.1), as m tends to infinity and l is arbitrary, the asymptotic

$$| (v, w) | \simeq \frac{Constant\ 2^{k+l}}{(k+2l+1)^{\frac{k+2l+(1/2)}{2}}(k+1)^{\frac{k+(1/2)}{2}}} \left(\frac{k+2l+1}{2}\right)^{\frac{k+2l+1}{2}} \left(\frac{k+1}{2}\right)^{\frac{k}{2}}$$

$$\simeq \frac{Constant}{(k+2l+1)^{1/4}(k+1)^{1/4}}.$$

Moreover, the constant is independent of l.

This proves:

Lemma 2.4. *Let $m \geq 0$ be an integer and $(,)$ a $SU(2)$-invariant inner product on the representation $\rho_m = Sym^{2m}(\mathbb{C}^2)$. Let $0 \leq l \leq m$ and put $m = k + l$. Let v_m a vector of norm 1 in ρ_m invariant under the diagonals T in $SU(2)$ and $w_{m,l} \in \rho_m$ a vector of norm 1 on which $SO(2)$ acts by the character χ_l. We have the following asymptotic as $m = k + l$ tends to infinity:*

$$| (v_m, w_{m,l}) | \simeq \frac{Constant}{(k+2l+1)^{1/4}(k+1)^{1/4}}.$$

Notation 1. Given $m \geq 0$ and $-m \leq l \leq m$, define the function for $k \in K = SU(2)$ by the formula

$$\psi_{m,l}(k) = (v_m, \rho_m(k) w_{m,l}).$$

The functions $\psi_{m,l}$ form a complete orthogonal set for $Rep(T\backslash K)$. The norm of $\psi_{m,l}$ with respect to the L^2 norm on functions K, is, by the Orthogonality Relations for matrix coefficients of ρ_m, equal to $\sqrt{2m+1}$.

If ψ is a function on K in $Rep(T\backslash K)$, denote by $|| \psi ||_K^2$ the integral (dk is the Haar measure on K)

$$\int_K | \psi(k) |^2 \, dk.$$

Define similarly the number $|| \phi ||_{K_H}^2$ for $\phi \in Rep(K_H)$, where $K_H = SO(2)$.

The restriction of the function $\psi_{m,l}$ to $K_H = SO(2)$ is, by the choice of the vector $w_{m,l}$, a multiple of the character χ_l: for $k_H \in K_H$, we have $\psi_{m,l}(k_H) = \psi_{m,l}(1)\chi_l(k_H)$. By Lemma 2.4, we have, for $k+l = m$ tending to infinity, the asymptotic

$$|\ \psi_{m,l}(1)\ |^2 \simeq \frac{Constant}{\sqrt{(k+2l+1)(k+1)}} \tag{6}$$

Notation 2. Let $\frac{1}{2} < u < 1$ and π_{-u} the complementary series representation of $G = SL(2,\mathbb{C})$ as before. Set $u' = 2u-1$. Then $0 < u' < 1$. If $\sigma_{-u'}$ is the complementary series representation of $SL(2,\mathbb{R})$ as before, the restriction of the functions (sections) in π_{-u} on $G/B(\mathbb{C})$ to the subspace $H/B(\mathbb{R})$ lies in $\sigma_{-u'}$, as is easily seen. Denote by $res : \pi_{-u} \to \sigma_{-u'}$ this restriction of sections.

Note that if $\psi \in \rho_m \subset Rep(T\backslash K) \simeq \pi_{-u}$ (the latter isomorphism is of K modules), then

$$||\ \psi\ ||^2_{\pi_{-u}} = \frac{1}{\lambda_l(u)}\ ||\ \psi\ ||^2_K\ .$$

Similarly, if $\phi \in \mathbb{C}\chi_l \subset Rep(\{\pm 1\}\backslash K_H) \simeq \sigma_{-u'}$ (the last isomporphism is of K_H-modules), then

$$||\ \phi\ ||^2_{\sigma_{-u'}} = \frac{1}{d_l(u')}\ ||\ \phi\ ||^2_{K_H}\ .$$

Moreover, we have the asymptotics

$$\lambda_m(u) \simeq \frac{Constant}{m^{2u}},\ d_l(u') \simeq \frac{Constant}{|\ l\ |^{2u-1}}, \tag{7}$$

as m and $|\ l\ |$ tends to infinity.

Theorem 2.5. *Let $\frac{1}{2} < u < 1$. The map $res : \pi_{-u} \to \sigma_{-(2u-1)}$ of the complementary series for $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$, is continuous with respect to the invariant metrics on the complementary series.*

Proof. We must prove the existence of a constant C such that for all $\psi \in \pi_{-u}$, the estimate

$$||\ \psi\ ||^2_{\pi_{-u}} \leq C\ ||\ res(\psi)\ ||^2_{\sigma_{-(2u-1)}}\ .$$

The map res is equivariant for the action of H and in particular, for the action of K_H. Therefore we need only prove this estimate when ψ is an eigenvector for the acton of K_H; however, the constant C must be proved to be independent of the eigencharacter.

Assume then that ψ is an eigenvector for K_H with eigencharacter χ_l. The function ψ is a linear combination of the functions $\psi_{m,l}$ ($m \geq |\ l\ |$).

Write

$$\psi = \sum_{m \geq |l|} x_m \psi_{m,l},$$

where the sum is over a finite set of the m's; the finite set could be arbitrarily large.

The orthogonality of $\psi_{m,l}$ and the equalities in Notation 2 imply

$$\| \psi \|^2_{\pi_{-u}} = \sum_{m \geq |l|} | x_m |^2 \| \psi_{m,l} \|^2_{\pi_{-u}} = \sum | x_m |^2 \frac{1}{\lambda_m(u)} \| \psi_{m,l} \|^2_K .$$

We therefore get, for $\psi \in \pi_{-u}$,

$$\| \psi \|^2_{\pi_{-u}} = \sum_{m \geq |l|} | x_m |^2 \frac{1}{(2m+1)\lambda_m(u)}. \tag{8}$$

We now compute $res(\psi)$. Since ψ is an eigenvector for K_H with eigencharacter χ_l, we have

$$res(\psi) = \psi(1)\chi_l = (\sum_{m \geq |l|} x_m \psi_{m,l}(1))\chi_l.$$

Therefore,

$$\| res(\psi) \|^2_{\sigma_{-(2u-1)}} = | (\sum x_m \psi_{m,l}(1)) |^2 \frac{1}{d_l(2u-1)}.$$

The Cauchy -Schwartz inequality implies

$$\| res(\psi) \|^2_{\sigma_{-(2u-1)}} \leq$$

$$(\sum | x_m |^2 \frac{1}{\lambda_m(u)(2m+1)})(\sum (2m+1)\lambda_m(u) | \psi_{m,l}(1) |^2)\frac{1}{d_l(u')}.$$

Assume for convenience that $l \geq 0$. Put $k = m + l$. Then $k \geq 0$. The estimate (6) and the equality (8) imply that (write σ for $\sigma_{-(2u-1)}$ and π for π_{-u}),

$$\| res(\psi) \|^2_\sigma \leq \| \psi \|^2_\pi (\sum_{k \geq 0} \frac{2k+2l+1}{\sqrt{(k+2l+1)(k+1)}} \lambda_{k+l}(u) \frac{1}{d_l(u')}).$$

Let Σ denote the sum in brackets in the above equation. To prove Theorem 2.5, we must show that Σ is bounded above by a constant independent of l. We now use the asymptotics 7 to get a constant C such that

$$\Sigma \leq C \sum_{k \geq 0} \frac{2k+2l+1}{\sqrt{(k+2l+1)(k+1)}} \frac{l^{2u-1}}{(k+l)^{2u}}.$$

This is a *decreasing* series in k and therefore bounded above by the sum of the $k=0$ term and the integral

$$\int_0^\infty dk \frac{2k+2l+1}{\sqrt{(k+2l+1)(k+1)}} \frac{l^{2u-1}}{(k+l)^{2u}}.$$

We first compute the $k=0$ term: this is

$$\frac{2l+1}{\sqrt{(2l+1)}} \frac{l^{2u-1}}{l^{2u}} \leq \frac{2}{\sqrt{2l+1}}$$

which therefore tends to 0 for large l and is bounded for all l.

To estimate the integral, we first change the variable from k to kl. The integral becomes

$$\int_0^\infty l dk \frac{2kl+2l+1}{\sqrt{(kl+2l+1)(kl+1)}} \frac{l^{2u-1}}{(kl+l)^{2u}}$$

$$\leq \int_0^\infty dk \frac{2k+3}{\sqrt{(k+2)(k)}} \frac{1}{(k+1)^{2u}},$$

and since $2u>1$, the latter integral is finite (and is independent of l).

We have therefore checked that both the $k=0$ term and the integral are bounded by constants independent of l and this proves Theorem 2.5.

$\square$

Theorem 2.6. *Let $\frac{1}{2} < u < 1$ and $\widehat{\pi}_u$ denote the completion of the complementary series representation of $SL(2,\mathbb{C})$. Define similarly, the completion $\widehat{\sigma}_{u'}$ of the complementary series $\sigma_{u'}$for $SL(2,\mathbb{R})$. If $u' = 2u-1$, then $\widehat{\sigma}_{u'}$ is a direct summand of $\widehat{\pi}_u$ restricted to $SL(2,\mathbb{R})$.*

Proof. We may replace π_u and $\sigma_{u'}$ by the isomorphic (and isometric) representations π_{-u} and $\sigma_{-u'}$. By Theorem 2.5, the restriction map $\pi_{-u} \to \sigma_{-u'}$ is continuous. Therefore this map extends to the completions. Hence $\widehat{\pi}_{-u}$ is, as a representation of $SL(2,\mathbb{R})$, the direct sum of the kernel of this restriction map and of $\widehat{\sigma}_{-u'}$. This completes the proof. $\square$

Remark 1. Theorem 2.6 is proved in [5]; the point of the proof in the present paper is that the "abstract" projection map is realised as a simple geometric map of sections of a line bundle on the flag varieties of $G = SL(2,\mathbb{C})$ and $H = SL(2,\mathbb{R})$.

3. **Branching laws for the Steinberg representation**

Let $G = SL_2(\mathbb{C})$ and $H = SL_2(\mathbb{R})$. Let π be the Steinberg Representation of G.

3.1. **The Representation π_0 and a G-invariant linear form.**

Consider the representation $\pi_0 = ind_B^G(\rho^2)$. In this equality, ind refers to **non-unitary** induction and π_o is the space of all continuous complex valued functions on G such that for all $g \in G$ and $man \in MAN = B$, we have

$$(\phi(mang) = \rho^2(a)\phi(g).$$

Here, ρ^2 is the product of all the positive roots of the split torus A occurring in the Lie algebra of the unipotent radical N of B and M is a maximal compact subgroup of the centraliser of A in G.

Now, π_0 a non-unitary representation, but has a G-invariant linear form L defined on it as follows. The map $\mathcal{C}_c(G) \to \pi_0$ given by integration with respect to a **left** invariant Haar measure on B is surjective. Given an element $\phi \in \pi_0$ select any function $\phi^* \in \mathcal{C}_c(G)$ in the preimage of ϕ and define $L(\phi)$ as the integral of ϕ^* with respect to the Haar measure on G. This is well defined (i.e. independent of the function ϕ^* chosen) and yields a linear form L. Moreover, if a function $\phi \in \pi_0$ is a positive function on G, then $L(\phi)$ is positive.

Under the action of the subgroup H on the G-space G/B, the space G/B has three disjoint orbits: the upper half plane, the lower half plane and the space $H/B \cap H$. The upper and lower half planes form open orbits. Given a function $\phi \in \mathcal{C}_c(\mathfrak{h})$ we may view it as a function in π_0 as follows. The restriction of the character ρ^2 to the maximal compact subgroup of H is trivial, therefore, the restriction of any element of π_0 to H yields a function on $\mathfrak{h}$ (also on $\mathfrak{h}^-$). Conversely, given $\phi \in \mathcal{C}_c(\mathfrak{h})$, extend ϕ by zero outside $\mathfrak{h}$; we get a function (we will again denote it ϕ) on all of π_0. The linear form L applied to $\mathcal{C}_c(\mathfrak{h})$ yields a positive linear functional which is H-invariant. Hence the positive linear functional L is a Haar measure on $\mathfrak{h}$.

3.2. **The metric on the Steinberg Representation of G.**

Consider the Steinberg representation $\pi = Ind_B^G(\chi)$. Here, Ind refers to **unitary** induction from a unitary character χ of B. Given two functions $f, f' \in \pi$, the product $\phi = f\overline{f'}$ ($\overline{f'}$ is the complex conjugate of f') lies in π_0. The linear form L applied to ϕ gives a pairing

$$< f, f' >= L(f\overline{f'})$$

on π which is clearly G-invariant. This is the G-invariant inner product on π.

Given a compactly supported function f on H which, under the left action of $K \cap H$ acts via the restriction of character χ to $K \cap H$, we can extend it by zero to an element of π. Then, the inner product $< f, f >$ is, by the conclusion of the last paragraph in (2.1), just the Haar integral on H applied to the function $\mid f \mid^2 \in \mathcal{C}_c(\mathfrak{h})$. Consequently, the metric on π restricted to $\mathcal{C}_c(H) \cap \pi$ is just the restriction of the L^2-metric on $\mathcal{C}_c(H)$.

Remark 2. We know that the Steinberg representation of G is tempered (and is induced by a unitary character from the Borel subgroup of upper triangular matrices). The tempered dual of G does not contain isolated points (G does not have discrete series representations). Moreover, the entire tempered dual is automorphic (Burger-Sarnak). Consequently, the Steinberg representation (which is cohomological) is not isolated in the automorphic dual of G.

3.3. **Decomposition of the Steinberg Representation π.**

Proposition 3.1. *The restriction to H of π contains the Steinberg Representation of H. More precisely, the restriction is a sum of the Steinberg representation σ of H, its complex conjugate $\overline{\sigma}$, and a sum of two copies of $L^2(H/K \cap H)$ where $K \cap H$ is a maximal compact subgroup of H.*

Proof. The Steinberg Representation π is (unitarily) induced from a **unitary** character χ of the Borel Subgroup $B = B(\mathbb{C})$ of upper triangular matrices in $G = SL_2(\mathbb{C})$. Now, the space G/B is the Riemann sphere $\mathbf{P}^1(\mathbb{C})$. The group H has three orbits, the upper half plane, the lower half plane and the projective line $\mathbf{P}^1(\mathbb{R})$ over $\mathbb{R}$. The first two are open orbits and $\mathbf{P}^1(\mathbb{R})$ has zero measure in G/B. From this, it is clear from section (2.2), that π is the direct sum of $L^2(\mathfrak{h}, \chi_{K\cap H})$ and $L^2(\mathfrak{h}^-, \chi^*_{K\cap H})$, where the subscript denotes the restriction of the character χ to the subgroup $K \cap H$ and χ^* denotes the complex conjugate of χ.

The representation χ^* is such that its restriction to $K \cap H$ is the minimal K-type of the Steinberg of $H = SL(2, \mathbb{R})$. The space $L^2(\mathfrak{h}, \chi)$ is therefore a direct sum of the Steinberg representation π and the full unramified tempered spectrum (any unramified representation contains χ as a $K \cap H$-type).

The Proposition now follows immediately. □

Remark 3. The Steinberg representation π is unitarily induced from the unitary character χ. Thus, it is (non-unitarily) induced from the character $\delta_{\mathbb{C}}\chi$ whose restriction to $B(\mathbb{R})$ is $\delta_{\mathbb{R}}^2$ ($\delta_{\mathbb{R}}^2$ is the character by which the split torus $S(\mathbb{R})$ acts on the Lie albegra of the unipotent radical of $B(\mathbb{R})$). Similarly $\delta_{\mathbb{C}}^2$ is the **square** of the character by which the split real torus in $S(\mathbb{C})$ acts on the complex Lie algebra of the unipotent radical of $B(\mathbb{C})$).

The Proposition was proved by restricting π to the open orbit; we may instead restrict π to the **closed** orbit $G(\mathbb{R})/B(\mathbb{R})$. We thus get a surjection of π onto the space of $K \cap H$- finite sections of the line bundle on $G(\mathbb{R})/B(\mathbb{R})$ which is induced from the character $\delta_{\mathbb{R}}^2$ on $B(\mathbb{R})$.

The latter representation contains the trivial representation as a quotient. We have therefore obtained that the trivial representation is a quotient of the restriction of π to the subgroup $SL_2(\mathbb{R})$. This shows that there is a mapping of the $(\mathfrak{h}, K \cap H)$- modules from π restricted to H, onto the trivial module of H; however, this cannot give a map of Hilbert spaces (their completions) since the Howe-Moore Theorem implies that the matrix coefficients of π restricted to the non-compact subgroup H must tend to zero at infinity.

Suppose that $G = SO(2m+1, 1)$ and $H = SO(2m, 1)$. Let $\pi_m = A_{\mathfrak{q}}$ be the cohomological representation of G which has non-zero cohomology in degree m, and vanishing cohomology in lower degrees. Then $A_{\mathfrak{q}}$ is a tempered representation. Define the representation σ of H which is cohomological and the lowest degree in which σ has cohomology is m. Then m is a discrete series representation. Following the proof of Proposition 3.1, we obtain the following proposition.

Proposition 3.2. *The representation σ is a direct summand of the restriction to H of the G- representation $A_{\mathfrak{q}}$.*

Remark 4. Here again, if $G = SO(2m+1, 1)$, then G hs no compact Cartan Subgroup, and hence $L^2(G)$ does not have discrete spectrum. Let Γ be an arithmetic (congruence) subgroup of G. The notion of "automorphic spectrum" of G with respect to the $\mathbb{Q}$-structure defined by —Γ was defined by Burger and Sarnak. Since all the tempered dual of G is automorphic by Burger-Sarnak, it follows that the representation

$\pi_m = A_{\mathfrak{q}}$ is not isolated in the automorphic spectrum of G. Thus, representations with cohomology need not be isolated in the automorphic dual.

3.4. **Explicit Space of Functions in $\sigma \subset \pi$.** Denote by $St_{\mathbb{R}}$ the space of $K \cap H$-finite functions in the Steinberg representation σ of $SL_2(\mathbb{R})$. By Proposition 3.1, this space of functions restricts trivially to the lower half plane. Moreover, in the space of L^2-functions on the upper half plane, the representation σ occurs with multiplicity one. In this subsection, we describe explicitly, elements in $St_{\mathbb{R}}$ viewed as functions on the upper half plane.

We will now replace $H = SL_2(\mathbb{R})$ with the subgroup $SU(1,1)$ of $G = SL_2(\mathbb{C})$. Since $SU(1,1)$ is conjugate to H, this does not affect the statement and proof of Proposition 3.1. The upper and lower half planes are then replaced respectively, by the open unit ball in $\mathbb{C}$ and the complement of the closed unit ball in $\mathbb{P}^1(\mathbb{C})$. With this notation, elements of $St_{\mathbb{R}}$ are now thought of as functions on $SU(1,1)$ with the equivariance property

$$f(ht) = \chi(t)f(h)\forall t \in K \cap H, \forall h \in SU(1,1).$$

The elements of $St_{\mathbb{R}}$ are explicitly described in [4] (Chapter IX, section 2, Theorem 1 in p. 181 of Lang). The eigenvectors of $K \cap H$ in $St_{\mathbb{R}}$ are

$$\phi_{2+2r} = \alpha^{-2}(\frac{\beta}{\alpha})^r,$$

with $r = 0, 1, 2, \cdots$. In this formula, an element of $SU(1,1)$ is of the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{C}$ such that

$$\mid \alpha \mid^2 - \mid \beta \mid^2 = 1.$$

Note that the integer m in Lang's book is 2 for the representation $St_{\mathbb{R}}$.

Further, the function ϕ_2 vanishes on the complement of the closed disc. That is, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\mid \frac{c}{d} \mid > 1$, then $\phi_2(g) = 0$.

It follows from the last two paragraphs that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, then one of the following two conditions hold:

Proposition 3.3. *If* $\mid \frac{d}{c} \mid < 1$*, then for any matrix* $h = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1,1)$ *with* $(\infty)h = (\infty)g$ *(the inequality satisfied by* g *ensures that there exists an* h *with this property), we have,*

$$\phi_2(g) = \alpha^{-2} \frac{1}{\mid d \mid^2 - \mid c \mid^2}.$$

If $\mid \frac{d}{c} \mid > 1$ *then* $\phi_2(g) = 0$.

Proof. The points on the open unit disc are obtained as translates of the point at infinity by an element of $SU(1,1)$. Therefore, if $\frac{d}{c}$ has modulus less than one, there exists an element $h \in SU(1,1)$ such that $(\infty)g = \frac{d}{c} = \infty(h)$. This means that

$$g = \begin{pmatrix} u & n \\ 0 & u^{-1} \end{pmatrix} h$$

for some element $b = \begin{pmatrix} u & n \\ 0 & u^{-1} \end{pmatrix} \imath SL_2(\mathbb{C})$ (elements of type b form the isotropy of G at infinity).

The intersection of the isotropy at infinity with $SU(1,1)$ is the space of diagonal matrices whose entries have absolute value one. Therefore, we may assume that the entry u above of the matrix b is real and positive. Then it follows that

$$\chi\delta(b) = u^2 = \frac{1}{\mid d \mid^2 - \mid c \mid^2},$$

and this proves the first part of the proposition.

The second part was already proved, as we noted that the restriction of $St_{\mathbb{R}}$ to the complement of the closed unit disc vanishes. □

Consider the decomposition

$$\pi = \sigma \oplus \overline{\sigma} \oplus L^2(K \cap H \backslash H) \oplus L^2(K \cap H \backslash H),$$

of π as a representation of the group H. It can be proved that the space π^∞ of smooth vectors for the action of $G = SL_2(\mathbb{C})$ is simply the space of smooth functions on G which lie in π, by proving the corresponding statement for the maximal comact subgroup $K = SU(2)$ of G. A natural question that arises is whether the subresentation σ for the action of H, contains any smooth vectors in π. we answer this n the negative.

Proposition 3.4. *The intersection*

$$\sigma \cap \pi^{\infty} = 0.$$

That is σ does not contain any nonzero smooth vectors in π.

Proof. The intersection of the proposition is stable under H and hence under the maximal compact subgroup $K \cap H$. If the intersection is non-zero, then it contains nonzero $K \cap H$-finite vectors. Since the representation σ is irreducible for H, the space $St_{\mathbb{R}}$ of $K \cap H$-finite vectors is irreducible as a $(\mathfrak{h}, K \cap H)$-module. Therefore, the space f smooth vectors in σ contains all of $St_{\mathbb{R}}$ and in particular, contains the function $f = \phi_2$ introduced above. That is, the function ϕ_2 is smooth on G (and hence on K).

We will now view ϕ_2 as a function on the group

$$SO(2) = \{k_\theta = \begin{pmatrix} cos\theta & sin\theta \\ -\sin\theta & cos\theta \end{pmatrix} : 0 \leq \theta \leq 2\pi\}.$$

If $\mid \frac{cos\theta}{-sin\theta} \mid < 1$, then there exists a real number t such that

$$\frac{cos\theta}{-sin\theta} = \frac{cosht}{sinht}.$$

By Proposition 3.3,

$$\phi_2(k_\theta) = \alpha^{-2}u^-2 = cosht^{-2}\frac{1}{cos^2\theta - \sin^2\theta}.$$

Moreover, it follows from the fact that $h = bg$ (in the notation of Proposition 3.3) that $u^-1cosht = cos\theta$ and hence that $cosh^2tu^{-2} = cos\theta^{-2}$. We have then:

$$\phi_2(k_\theta) = \frac{1}{cos^2\theta}$$

if $0 < \theta < \pi/4$ and 0 if $\pi/4 < \theta < \pi/2$. This contradicts the smootheness of ϕ_2 as a function of θ and proves Proposition 3.4.

$\square$

Remark 5. The Proposition says that although the **completion** of the Steinberg module of $SL(2, \mathbb{C})$, contains discretely the completion of the Steinberg module of $SL(2, \mathbb{R})$, this decomposition does not hold at the level of K-finite vectors. In contrast, in the situation of Kobayashi (see [3]), the decomposition is not at the level of K-finite vectors.

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