DISCRETE COMPONENTS OF SOME COMPLEMENTARY SERIES

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Abstract. We show that Complementary Series representations of $SO(n,1)$ which are sufficiently close to a cohomological representation contain **discretely**, Complementary Series of $SO(m,1)$ sufficiently close to cohomological representations, provided that the degree of the cohomological representation does not exceed $m/2$.

We prove, as a consequence, that the cohomological representation of degree i of the group $SO(n,1)$ contains discretely, the cohomological representation of degree i of the subgroup $SO(m,1)$ if $i \leq m/2$.

As a global application, we show that if $G/\mathbb{Q}$ is a semisimple algebraic group such that $G(\mathbb{R}) = SO(n,1)$ up to compact factors, and if we assume that for all n, the tempered cohomological representations are not limits of complementary series *in the automorphic dual of* $SO(n,1)$, then for all n, non-tempered cohomological representations are isolated in the automorphic dual of G. This reduces conjectures of Bergeron to the case of **tempered** cohomological representations.

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1. Introduction

A famous Theorem of Selberg [Sel] says that the non-zero eigenvalues of the Laplacian acting on functions on quoteints of the upper half plane by *congruence subgroups* of the integral modular group, are bounded away from zero, as the congruence subgroup varies. In fact, Selberg proved that every non-zero eigenvalue λ of the Laplacian on functions on $\Gamma\backslash\mathfrak{h}$, $\Gamma \subset SL_2(\mathbb{Z})$, satisfies the inequality

$$\lambda \geq \frac{3}{16}.$$

A Theorem of Clozel [Clo] generalises this result to any congreunce quotient of any symmetric space of non-compact type: if G is a linear semi-simple group defined over $\mathbb{Q}$, $\Gamma \subset G(\mathbb{Z})$ a congruence subgroup and $X = G(\mathbb{R})/K$ the symmetric space of G, then nonzero eigenvalues λ of the Laplacian *acting on the space of functions on* $\Gamma\backslash X$ satisfy:

$$\lambda \geq \epsilon,$$

where $\epsilon > 0$ is a number **independent** of the congruence subgroup Γ.

Analogous questions on Laplacians acting on differential forms of higher degree (functions may be thought of as differential forms of degree zero) have geometric implications on the cohomology of the locally symmetric space. Concerning the eigenvalues of the Laplacian, Bergeron ([Ber]) has made the following conjecture:

Conjecture 1. (Bergeron) Let X be the real hyperbolic n-space and $\Gamma \subset SO(n,1)$ a congruence arithmetic subgroup. Then non-zero eigenvalues λ of the Laplacian acting on the space $\Omega^i(\Gamma\backslash X)$ of differential forms of degree i satisfy:

$$\lambda \geq \epsilon,$$

for some $\epsilon > 0$ **independent** of the congruence subgroup Γ, provided i is strictly less than the "middle dimension" (i.e. $i < [n/2]$).

If n is even, the above conclusion holds even if $i = [n/2] = n/2$. (When n is odd, there is a slightly more technical statement which we omit).

In this paper, we show, for example, that if the above conjecture holds true in the *middle degree* for all even integers n, then the conjecture holds for arbitrary degrees of the diferential forms (See Theorem 3). For odd n, there is, again, a more technical statement (see Theorem 3, part (2)).

The main Theorem of the present paper is Theorem 1 on the occurrence of some discrete components of certain Complementary Series Representations associated to $SO(n,1)$. The statement on Laplacians may be deduced from the main theorem, using the Burger-Sarnak ([Bu-Sa]) method.

We now describe the main theorem more precisely. Let $i \leq [n/2]-1$ and $G = SO(n,1)$. Let $P = MAN$ be the Langlands decomposition of G, $K \subset G$ a maximal compact subgroup of G containing M. Let $\mathfrak{p}_M$ be the standard representation of M and $\wedge^i$ be its i-th exterior power representation. Denote by ρ_P^2 the character of P acting on the top exterior power of the Lie algebra of N. Consider the representation

$$\widehat{\pi_u(i)} = Ind_P^G(\wedge^i \otimes \rho_P(a)^u)$$

for $0 < u < 1 - \frac{2i}{n-1}$. The representation $\widehat{\pi_u(i)}$ denotes the **completion** of the space π_u of K-finite vectors with respect to the G-invariant metric on $\pi_u(i)$, and is called the **complementary series representation** corresponding to the representation $\wedge^i$ of M and the parameter u.

Let $H = SO(n-1,1)$ be embedded in G such that $P \cap H$ is a maximal parabolic subgroup of H, $A \subset H$, and $K \cap H$ a maximal compact subgroup of H. We now assume that $\frac{1}{n-1} < u < 1 - \frac{2i}{n-1}$. Let $u' = \frac{(n-1)u-1}{n-2}$; then $0 < u' < 1 - \frac{2i}{n-2}$. Denote by $\wedge_H^i$ the i-th exterior power of the standard representation of $M \cap H \simeq O(n-2)$.

We obtain analogously the complementary series representation

$$\widehat{\sigma_{u'} i)} = Ind_{P\cap H}^H(\wedge_H^i \otimes \rho_{P\cap H}(a)^{u'}),$$

of H. The main theorem of the paper is the following.

Theorem 1. *If*

$$\frac{1}{n-1} < u < 1 - \frac{2i}{n-1},$$

then the complementary series representation $\widehat{\sigma_{u'}(i)}$ occurs discretely in the restriction of the complementary series representation $\widehat{\pi_u(i)}$ of $G = SO(n,1)$, to the subgroup $H = SO(n-1,1)$:

$$\widehat{\sigma_{u'}(i)} \subset \widehat{\pi_u(i)}_{|SO(n-1,1)}.$$

Remark 1. The corresponding statement is false for the space of K-finite vectors in both the spaces; this inclusion holds only at the level of completions of the representations involved).

As u tends to the limit $1-\frac{2i}{n-1}$, the representations $\pi_u(i)$ tend (in the Fell topology on the Unitary dual $\widehat{G}$) both to the representation $A_i = A_i(n)$ and to $A_{i+1} = A_{i+1}(n)$; we denote by $A_j(n)$ the unique cohomological representation of G which has cohomology (with trivial coefficients) in degree j. Using this, and the proof of Theorem 1, we obtain

Theorem 2. *The restriction of the cohomological representation $A_i(n)$ of $SO(n,1)$ to the subgroup $H = SO(n-1,1)$ contains discretely, the cohomological representation $A_i(n-1)$ of $SO(n-1,1)$:*

$$A_i(n-1) \subset A_i(n)_{|SO(n-1,1)}.$$

Supose now that G is a semi-simple linear algebraic group defined over $\mathbb{Q}$ and $\mathbb{Q}$-simple, such that

$$G(\mathbb{R}) \simeq SO(n,1)$$

up to compact factors with $n \geq 4$ (if $n = 7$, we assume in addition that G is not the inner form of some trialitarian D_4). Then there exists a $\mathbb{Q}$-simple $\mathbb{Q}$-subgroup $H_1 \subset G$ such that

$$H_1(\mathbb{R}) \simeq SO(n-2,1),$$

up to compact factors.

Denote by $\widehat{G}_{Aut}$ the "automorphic dual" of $SO(n,1)$, in the sense of Burger-Sarnak ([Bu-Sa]). Suppose $A_i = A_i(n)$ is a limit of representations ρ_m in $\widehat{G}_{Aut}$. The structure of the unitary dual of $SO(n,1)$ shows that this means that $\rho_m = \widehat{\pi_{u_m}(i)}$ for some sequence u_m which tends from the left, to $1-\frac{2i}{n-1}$ (or to $1-\frac{2i+2}{n-1}$; we will concentrate only on the first case, for ease of exposition).

Since $\rho_m = \widehat{\pi}_{u_m}(i) \in \widehat{G}_{Aut}$, a result of Burger-Sarnak ([Bu-Sa]), implies that

$$res(\rho_m) \subset \widehat{H}_{Aut}.$$

Applying Theorem 1 twice, we get

$$\sigma_m = \widehat{\sigma}_{u''_m}(i) \in res(\rho_m) \subset \widehat{H}_{Aut}$$

Taking limits as m tends to infinity, we get $A_i(n-2)$ as a limit of representations σ_m in $\widehat{H}_{Aut}$. Therefore, the isolation of $A_i(n)$ in $\widehat{G}_{Aut}$ is reduced to that for $SO(n-2,1)$, and so on. We can finally assume that $A_i(m)$ is a tempered representation of $SO(m,1)$ where $m = 2i$ or $2i+1$.

This proves the following Theorem.

Theorem 3. *If for all m, the* **tempered** *cohomological representation $A_i(m)$ (i.e. when $i = [m/2]$) is not a limit of complementary series in the automorphic dual of $SO(m,1)$, then for all integers n, and for $i < [n/2]$, the cohomological representation $A_i(n)$ is isolated in the automorphic dual of $SO(n,1)$.*

The major part of the paper is devoted to proving the main theorem (Theorem 1). The proof of Theorem 1 is somewhat roundabout and proceeds as follows.

(1) We first prove Theorem 1 when $i = 0$; that is $\widehat{\pi}_u = \widehat{\pi}_u(i)$ is an unramified representation and $\frac{1}{n-1} < u < 1$. In this case, we get a model of the representation $\widehat{\pi}_u$ by restricting the functions (sections of a line bundle) on G/P to the big Bruhat cell $Nw \simeq \mathbb{R}^{n-1}$ and taking their Fourier transforms. The G-invariant metric is particualrly easy to work with on the Fourier transforms (see Theorem 10); it is then easy to see that there is an isometric embedding $J : \widehat{\sigma}_{u'}$ in $\widehat{\pi}_u$ (Theorem 15).

(2) When this is interpreted in the space π_u of K-finite vectors, we have the isomorphism of the intertwining map $I_G(u)$:

$$I_G(u) : \pi_u \simeq \pi_{-u} = Ind_P^G(\rho_P^{-u}).$$

The restriction of the sections on G/P of the line bundle corresponding to π_{-u} to the subspace $H/H \cap P$ is exactly the representation $\sigma_{-u'}$, which is isomorphic, via the inverse of the intertwining map, namely $I_H(u')^{-1}$, to $\sigma_{u'}$. The embedding $J : \widehat{\sigma}_{u'} \subset \widehat{\pi}_u$ is such that its adjoint J^* is precisely the restriction res (Proposition 14). This implies that the restriction map of sections

$$res : \pi_{-u} \to \sigma_{-u'},$$

is continuous for the metric on $\pi_{-u} \simeq \pi_u$ and on $\sigma_{-u'} \simeq \sigma_{u'}$.

(3) The K-irreducible representations occurring in π_u are parametrised by non-negative integers m, each ireducible V_m occurring with multiplicity one (V_m is isomorphic to the space of homogeneous harmonic polynomials of degree m on the sphere $G/P \simeq K/M$): $\pi_u = \oplus_{m \geq 0} V_m$.

Similarly, we write $\sigma_{u'}$ as a direct sum of irreducibles W_l of $K \cap H$ (indexed again by non-negative integers): $\sigma_{u'} = \oplus_{l \geq 0} W_l$. Denote by $V_{m,l}$ the isotypic of W_l in the restrictio of V_m to $K \cap H$ (V_m restricted to $K \cap H$ is in fact multiplicity free). We have the restriction map

$\widehat{r}:\pi=Ind_M^K(triv)\to\sigma=Ind_{M\cap H}^{K\cap H}(Triv)$. This maps $V_{m,l}$ into W_l. Set

$$C(m,l,0)=\frac{||\,\widehat{r}(f)\,||^2_{K\cap H}}{||\,f\,||^2_K},$$

We show in Theorem 29 that the continuity of the map $res:\pi_{-u}\to\sigma_{-u'}$ implies the statement that the series

$$\sum_{m\geq l}C(m,l,0)\frac{l^{(n-2)u'}}{m^{(n-1)u}}<A, \tag{1}$$

where A is a constant independent of the integer l.

The equivalence is proved by calculating the value of the intertwining operator on each K-type V_m (the operator acts by a scalar $\lambda_m(u)$) and obtaining asymptotics as m tends to infinity by using Stirling's Formula.

Since, by Theorem 15 the map is indeed continuous, the estimate of equation (22) is indeed true.

(3) We now describe the proof of Theorem 1 in the ramified case. We have analogously the restriction maps

$$r_u(i):\pi_{-u}(i)\to\sigma_{-u'}(i),$$

where again $u'=\frac{(n-1)u-1}{n-2}$. By the multiplicity one statement for irreducible representations of $SO(n-1,1)$ which are quotients of the representation $\pi_{-u}(i)$, it follows that Theorem 1 is equivalent to the continuity of the restriction map $r_u(i)$. As a K-module, $\pi_u(i)$ is a direct sum of two kinds of irreducible representations with the irreducibles parametrised again by non-negative integers. We can define the numbers $C(m,l,i)$ analogous to the definition of $C(m,l,0)$ above. Theorem 29 says again that the continuity of the restriction $r_u(i)$ is implied by the estimate

$$\sum_{m\geq l}C(m,l,i)\frac{l^{(n-2)u'}}{m^{(n-1)u}}<A, \tag{2}$$

where A is a constant independent of the $K\cap H$ "type" l. The proof of this implication uses some precise estimates of the intertwining operators on the analogues of the spaces V_m in $\pi(i)\simeq Ind_M^K(\wedge^i)$ (and on the analogues of W_l in $\sigma(i)$ which is defined similarly to $\pi(i)$, but for the group H.

(4) We prove that the estimate in equation (26) is indeed true, by analysing the K-types occurring in $\pi(i)$, and showing as a consequence

that

$$C(m, l, i) \leq C\ C(m, l, 0),$$

for some constant C independent of m, l. Since equation (22) is proved to be true, it folows that equation (26) also is true. This proves Theorem 1 for arbitrary i.

2. Preliminary Results and Notation

2.1. **The Group $G = O(n,1)$ and the representations $\pi_u(i)$.** The group $G = O(n,1)$ is the subgroup of $GL_{n+1}(\mathbb{R})$ preserving the quadratic form

$$x_0x_n + x_1^2 + x_2^2 + \cdots + x_{n-1}^2,$$

on $\mathbb{R}^{n+1}$. Denote by $e_0, e_1, \cdots, e_{n-1}, e_n$ the standard basis of $\mathbb{R}^{n+1}$. Denote by w the permutaions matrix in $GL_{n+1}(\mathbb{R})$, which switches the basis elements e_0 and e_n and takes the other basis elements e_i into e_i for $1 \leq i \leq n-1$. Then w has the matrix form

$$\begin{pmatrix} 0 & 0_{n-1} & 1 \\ {}^t0 & 1_{n-1} & 0 \\ 1 & 0_{n-1} & 0 \end{pmatrix}$$

where 0_{n-1} is the zero vector in $\mathbb{R}^{n-1}$ viewed as a row vector of size $n-1$ and ${}^t0$ refers to its transpose (a column vector of size $n-1$ all of whose entries are zero) and 1_{n-1} refers to the identity matrix of size $n-1$. The group G is the matrix group

$$\{g \in GL_{n+1}(\mathbb{R}) :^t wg = w\}.$$

Denote by P the subgroup of G which takes the basis vector e_0 into a multiple of itself. Let A be the group of diagonal matrices in G, M the maximal compact subgroup of the centraliser of A in G. Let N be the unipotent radical of P. Then,

$$A = \{d(a) = \begin{pmatrix} a & 0_{n-1} & 0 \\ {}^t0 & 1_{n-1} & {}^t0 \\ 0 & 0_{n-1} & a^{-1} \end{pmatrix} : a \in \mathbb{R}^*\},$$

Similarly M consists of matrices of the form

$$M = \{\begin{pmatrix} 1 & 0_{n-1} & 0 \\ {}^t0 & m & {}^t0 \\ 0 & 0_{n-1} & 1 \end{pmatrix} : m \in SO(n-1)\}.$$

The group N is

$$N = \{u(x) = \begin{pmatrix} 1 & x & \frac{|x|^2}{2} \\ {}^t0 & 1_{n-1} & {}^tx \\ 0 & 0_{n-1} & 1 \end{pmatrix} : x \in \mathbb{R}^{n-1}\},$$

where $x \in \mathbb{R}^{n-1}$ is thought of as a row vector. Then $P = MAN$ is the Langlands decomposition of P. The map $x \mapsto u(x)$ from $\mathbb{R}^{n-1}$ into N is an isomorphism of algebraic groups.

Let $K = G \cap O(n+1)$. Then K is a maximal compact subgroup of G and is isomorphic to $O(n) \times O(1)$. Let $\rho_P(a)$ be the character whose square is the determinant of $Ad(d(a))$ acting on the complex Lie algebra $\mathfrak{n} = Lie(N) \otimes \mathbb{C}$. Then $\rho_P(a) = a^{\frac{n-1}{2}}$.

Write $\mathfrak{g} = Lie(G) \otimes \mathbb{C}$, $\mathfrak{k} = Lie(K) \otimes \mathbb{C}$ and set $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan Decomposition which is complexified. As a representation of $K = O(n)$, the space $\mathfrak{p} = \mathbb{C}^n$ is the standard representation. Denote by $\mathfrak{p}_M = \mathbb{C}^{n-1}$ the standard representation of $M = SO(n-1)$. Let i be an integer with $0 \leq i \leq [n/2] - 1$ where [] denotes the integral part. Denote by $\wedge^i \mathfrak{p}_M$ the representation of M on the i-th exterior power of $\mathfrak{p}_M$.

If $u \in \mathbb{C}$ and $p = man \in MAN = P$, the map

$$p \mapsto \wedge^i \mathfrak{p}_M(m) \otimes \rho_P(a)^u \otimes triv_N(n) = \wedge^i(m)\rho_P(a)^u$$

is a representation of P. Define

$$\pi_u(i) = Ind_P^G(\wedge^i \mathfrak{p}_M \otimes \rho_P(a)^u \otimes 1_N)$$

to be the space of functions on G with values in the vector space $\wedge^i \mathfrak{p}_M$ which satisfy

$$f(gman) = \wedge^i(m)(f(g))\rho_P(a)^{-(1+u)},$$

for all $(g, m, a, n) \in G \times M \times A \times \times N$. In addition, the function f is assumed to be K-finite. The space $\pi_u(i)$ is a $(\mathfrak{g}, K)$-module. The representation $\pi_u(i)$ is called the complementary series of $SO(n, 1)$ with parameter u and $\wedge^i$.

When $i = 0$, the representation $\pi_u(0)$ has a K-fixed vector, namely the function which satisfies $f(kan) = \rho_P(a)^{-(1+u)}$ for all $kan \in KAN = G$. We denote $\pi_u(0) = \pi_u$; we refer to this an the unramified complementary series of $SO(n, 1)$ with parameter u.

2.2. **Bruhat Decomposition of $O(n, 1)$.** We have the **Bruhat Decomposition** of $O(n, 1)$:

$$G = NwMAN \cup MAN. \tag{3}$$

That is, every matrix $g \in G = O(n, 1)$ which does not lie in the parabolic subgroup P, may be written in the form $g = u(x)wmd(a')u(y)$ for some $x, y \in \mathbb{R}^{n-1}$, $a' \in \mathbb{R}^*$ and $m \in M$ or in the form $g = md(a)u(y)$. In these coordinates, the set $NwP = NwMAN$ is an open subset of G of full Haar measure. The map $x \mapsto u(x)$ is an isomorphism of groups from $\mathbb{R}^{n-1}$ onto N. Let dn denote the image of the Lebesgue measure on $\mathbb{R}^{n-1}$ under this map. Denote by d^*a the measure $\frac{da'}{|a'|}$ for $a' \in \mathbb{R}^* \simeq A$ via the map $a' \mapsto d(a')$. Under the Bruhat decomposition (equation (3), there exists a choice of Haar measure on M such that

$$dg = dx\ d^*a'\ dm\ dy\ \rho_P(a')^2. \tag{4}$$

Given $x = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$, denote by $\mid x \mid^2$ the sum

$$\mid x \mid^2 = \sum_{j=1}^{k} x_j^2.$$

Lemma 4. *If $x \in \mathbb{R}^{n-1}$, then the Bruhat decomposition of the element $g = wu(x)w$ is given by*

$$g = u(z)wm(x)d(a')u(y)$$

with

$$z = \frac{2x}{\mid x \mid^2},\ m = 1_{n-1} - 2\frac{{}^t xx}{\mid x \mid^2},\ a' = \frac{\mid x \mid^2}{2} \text{ and } y = x$$

(note that ${}^t xx$ is a square matrix of size $n-1$ and that m is the reflection on the orthogonal complement of x).

The proof is by multiplying out the right hand side and explicitly computing the matrix product.

We will find it convenient to compute the Jacobian of the transformation $x \mapsto \frac{2x}{|x|^2}$ for $x \in \mathbb{R}^{n-1} \setminus \{0\}$.

Lemma 5. *If $f \in \mathcal{C}_c(\mathbb{R}^{n-1} \setminus \{0\})$ is a continuous function with compact support, then we have the formula*

$$\int_{\mathbb{R}^{n-1}} dx f(\frac{x}{\mid x \mid^2}) = \int_{\mathbb{R}^{n-1}} \frac{dx}{(\frac{|x|^2}{2})^{n-1}} f(x).$$

Proof. In polar coordinates, $x = rp$ with $r > 0$ and p in the unit sphere S^{n-2} of $\mathbb{R}^{n-1}$. Then the Lebesgue measure dx has the form $dx = r^{n-1}\frac{dr}{r}d\mu(p)$ where μ is a rotation invariant measure on S^{n-2}. In polar coordinates, the transformation $x \mapsto \frac{2x}{|x|^2}$ takes the form $(r, p) \mapsto (\frac{2}{r}, p)$. Therefore $d(\frac{x}{|x|^2}) = \frac{2^{n-1}}{r^{n-1}}\frac{dr}{r}d\mu(p) = \frac{dx}{r^{2n-2}/2^{n-1}}$. $\square$

2.3. **Iwasawa Decomposition of** $O(n,1)$**.** We have the **Iwasawa decomposition** of $O(n,1) = G$:

$$G = KAN. \tag{5}$$

That is, every matrix g in $SO(n,1)$ may be written as a product $g = kd(a)u(y)$ with $k \in K$, $a \in \mathbb{R}^*$ and $y \in \mathbb{R}^{n-1}$.

Lemma 6. *The Iwasawa decomposition of the matrix $u(x)w$ is of the form*

$$u(x)w = k(x)d(a)u(y),$$

with $a = 1 + \frac{|x|^2}{2}$ *and* $y = \frac{x}{a} = \frac{x}{1+\frac{|x|^2}{2}}$. *The matrix $k(x)$ is given by the formula*

$$k(x) = u(x)wu(-y)d(a^{-1}).$$

Proof. Put $g = u(x)w = kd(a)u(y)$. Compute the product

$$w^t u(x)u(x)w =^t gg =^t u(y)d(a)d(a)u(y).$$

By comparing the matrix entries on both sides of this equation, we get the Lemma. □

We normalise the Haar measures dg, dk d^*a and dy on G, K, A, N respectively so that

$$dg = \rho_P(a)^2 \; dk \; d^*a \; dy. \tag{6}$$

(dy is the Lebesgue measure on $N \simeq \mathbb{R}^{n-1}$, on $A \simeq \mathbb{R}^*$ the measure is $d^*a = \frac{d_1(a)}{|a|}$ where $d_1 a$ is the Lebesgue measure).

2.4. **The space $\mathcal{F}_1$ and a linear form on $\mathcal{F}_1$.** Define $\mathcal{F}_1$ to be the space of complex valued continuous functions f on G such for all $(g, m, a, n) \in G \times M \times A \times N$ we have

$$f(gman) = \rho_P(a)^{-2} f(g).$$

The group G acts on $\mathcal{F}_1$ by left translations. Denote by $\mathcal{C}_c(G)$ the space of complex valued continuous functions with compact support. The following result is well known.

Lemma 7. *The map*

$$f \mapsto \overline{f}(g) \overset{def}{=} \int_{MAN} f(gman)\rho_P(a)^2 \; dm \; dn \; d^*a,$$

is a surjection from $\mathcal{C}_c(G)$ onto the space $\mathcal{F}_1$.

It follows from this and the Iwasawa decomposition (see equation (6)), that the map

$$L(\overline{f}) = \int_K f(k)dk \tag{7}$$

is a G-invariant linear form on the space $\mathcal{F}_1$.

Lemma 8. *For every function $\phi \in \mathcal{F}_1$ we have the formula*

$$\int_{\mathbb{R}^{n-1}} dx\phi(u(x)w) = \int_K dk\phi(k).$$

Proof. Given $f \in \mathcal{C}_c(G)$, we have from equation (4) that

$$\int_{\mathbb{R}^{n-1}} dx \overline{f}(u(x)w) = \int_G f(g)dg = L(\overline{f}).$$

On the other hand, by equation (6), we have

$$\int_K dk \overline{f}(k) = \int_G f(g)dg = L(\overline{f}).$$

These two equations show that the lemma holds if $\phi = \overline{f}$.

It follows from Lemma 7 that every function $\phi \in \mathcal{F}_1$ is of the form $\overline{f}$ for some f. This proves the Lemma.

□

3. Unramified Complementary Series for $SO(n,1)$

3.1. **A Model for Unramified Complementary Series.** Let $0 < u < 1$, and $W = \mathbb{R}^{n-1}$. Denote by $\mathcal{S}(W)$ the Schwartz Space of W.

Let $K = O(n)$ and $M = O(n-1)$. Then K/M is the unit sphere in W. Denote by w the non-trivial Weyl group element of $SO(n,1)$. The space G/P is homogeneous for the action of K and the isotropy at P is M. Therefore, $G/P = K/M$. The Bruhat decomposition implies that we have an embedding $N \subset K/M$ via the map $n \mapsto nwP \subset G/P = K/M$. Denote by $\mathcal{F}_u$ the space of **continuous** functions on G which transform according to the formula

$$f(gman) = f(g)\rho_P(a)^{-(1+u)},$$

for all $g \in G, m \in M, a \in A$ and $n \in N$. An element of the Schwartz space $\mathcal{S}(W)$ on $W \simeq N$ may be extended to a continuous function on G, which lies in the space $\mathcal{F}_u$.

We will view $\mathcal{S}(W)$ as a subapace of $\mathcal{F}_u$ via this identification.

Lemma 9. *If $0 < u < 1$ and $\phi \in \mathcal{S}(W) \subset \mathcal{F}_u$, the integral*

$$I_u(\phi)(g) = \int_W dx\phi(gu(x)w)$$

converges.

The integral $I(u)(\phi)$ lies in the space $\mathcal{F}_{-u}$.

Proof. The Iwasawa decomposition of $u(x)w$ (Lemma 6) implies that $u(x)w = kman$ with

$$\rho_P(a)^{1+u} = \frac{1}{(1+\frac{|x|^2}{2})^{(n-1)(1+u)/2}}$$

where $\mid x \mid^2$ is the standard inner product of $n = u(x) \in N \simeq W$ in the space W and $k \in K$, $m \in M$ and $n \in N$. Therefore, for large $x \in W$, $\rho_P(a)^{1+u} \simeq \mid x \mid^{(n-1)(1+u)}$, which proves the convergence of the integral. The last statement is routine to check. □

There is a G-invariant pairing between $\mathcal{F}_u$ and $\mathcal{F}_{-u}$ defined by

$$< \phi, \psi >= \int_K dk\overline{\phi(k)}\psi(k),$$

for all $\psi \in \mathcal{F}_u$ and $\psi \in \mathcal{F}_{-u}$. Here $\overline{\phi(k)}$ denotes the complex conjugate of $\phi(k)$.

Observe that the product $\overline{\phi(g)}\psi(g)$ of the elements $\phi \in \mathcal{F}_u$ and $\psi \in \mathcal{F}_{-u}$ lies in the space $\mathcal{F}_1$. Therefore, $< \phi, \psi >= L(\phi\psi)$ where L is the invariant linear form on $\mathcal{F}_1$ as in equation (7).

From Lemma 8, it follows, for $\phi \in \mathcal{F}_u$ and $\psi \in \mathcal{F}_{-u}$, that

$$< \phi, \psi >= \int_{\mathbb{R}^{n-1}} dx\overline{\psi(u(x)w)}\psi(u(x)w).$$

Given $\phi \in \mathcal{S}(W)$,denote by $\widehat{\phi}$ its Fourier transform.

Theorem 10. *For $0 < u < 1$ and $\phi \in \mathcal{S}(W) \subset \mathcal{F}_u$, we have the formula*

$$< \phi, I(u)(\phi) >_u = c \int_W \mid \widehat{\phi}(x) \mid^2 \frac{1}{\mid x \mid^{(n-1)u}} dx,$$

relating the pairing between $\mathcal{F}_u$ and $\mathcal{F}_{-u}$, and a norm on the Schwartz space. (here c is a constant dependeing only on the constant u).

In particular, the pairing $< \phi, I(u)(\phi) >$ is positive definite on the Schwartz space $\mathcal{S}(W)$.

Lemma 11. *If $\phi^* \in \mathcal{S}(W) \subset \mathcal{F}_u$, then we have the formula*

$$\int_N \phi^*(n'wnw)dn = \int_W \phi^*(y+x)\frac{1}{\mid x \mid^{(n-1)(1-u)}} dx,$$

where $n, n' \in N$ correspond respectively to $y, x \in W$.

Proof. We view $\mathcal{S}(W)$ as a subspace of $\mathcal{F}_u$. Thus, the function $\phi^*(x) \in \mathcal{S}(W)$ is identified with the section $x \mapsto \phi^*(u(x)w)$ with $\phi^* \in \mathcal{F}_u$. The integral

$$\int_N \phi^*(nwn'w)dn' = \int_W \phi^*(u(y)wu(x)w)dx.$$

By Lemma 4,

$$wu(x)w = u(\frac{2x}{\mid x \mid^2})m(x)d(\frac{\mid x \mid^2}{2})u(x).$$

Using the fact that ϕ^* lies in $\mathcal{F}_u$, we see that

$$\phi^*(u(y)wu(x)w) = \phi^*(u(y + \frac{2x}{\mid x \mid^2}))(\frac{\mid x \mid^2}{2})^{-(n-1)(1+u)/2}.$$

Therefore, the integral of the lemma is

$$\int_W dx\phi^*(y + \frac{2x}{\mid x \mid^2})(\frac{\mid x \mid^2}{2})^{-(n-1)(1+u)/2)}.$$

We now use the computation of the Jacobian of the map $x \mapsto \frac{2x}{|x|^2}$ (see Lemma 5) to get

$$\int_N \phi^*(nwn'w)dn' = \int_W \frac{dx}{(\mid x \mid^2 /2)^{n-1}}\phi^*(y+x)(\frac{\mid x \mid^2}{2})^{n-1)(1+u)/2},$$

and this is easily seen to be the right hand side of the equation of the Lemma. □

Recall that

$$I(u)(\phi^*)(nw) = \int_N dn(\phi^*(nwn'w).$$

From Lemma 8 we have

$$< \phi^*, I(u)(\phi^*) >= \int_N dn(\phi^*(nw)I(u)(\phi^*)(nw).$$

Therefore, it follows from the preceding Lemma that the G-invariant pairing $< \phi^*, I_u(\phi^*) >$ is the integral

$$\int_W dy \int_W dx \frac{1}{\mid x \mid^{(n-1)u}} \overline{\phi(y)}\phi(y+x).$$

The Fubini Theorem implies that this integral is

$$\int_X dx \frac{1}{\mid x \mid^{(n-1)u}} \int_W dy \overline{\phi(y)}\phi(y+x).$$

The inner integral is the L^2 inner product between ϕ and its translate by x. Since the Fourier transform preserves the inner product and converts translation by x into multiplication by the character $y \mapsto e^{-2ix.y}$, it folows that the integral becomes

$$\int_W dx \frac{1}{\mid x \mid^{(n-1)(1-u)}} \int_W dy \mid \widehat{\phi}(y) \mid^2 e^{-2ix.y}.$$

The integral over y is simply the Fourier transform. Therefore, we have

$$< \phi, I_u(\phi) >= \int_W dx \frac{1}{\mid x \mid^{(n-1)(1-u)}} \mid \widehat{\widehat{\phi}(y) \mid^2}(x). \tag{8}$$

Lemma 12. *Let $f \in \mathcal{S}(W)$, and s a complex number with real part positive and less than $(n-1)/2$. Then we have the functional equation*

$$\Gamma(s) \int_W dx \frac{1}{\mid x \mid^{2s}} \widehat{f}(x) = \Gamma(\frac{n-1}{2} - s) \int_W dx \mid x \mid^{n-1-2s} f(x),$$

where Γ is the classical Gamma function.

Proof. If s is a complex number with positive real part, then $\Gamma(s)$ is defined by the integral

$$\int_0^{+\infty} \frac{dt}{t} t^s e^{-t}.$$

Denote by $L(s, \widehat{f})$ the integral

$$\int_W dx \frac{1}{\mid x \mid^s} \widehat{f}(x).$$

Multiplying $L(2s, \widehat{f})$ by $\Gamma(s)$, and using Fubini, we obtain

$$\Gamma(s)L(2s, \widehat{f}) = \int_W \frac{dx}{\mid x \mid^{2s}} \widehat{f}(x) \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

Chaging the variable t to $\mid x \mid^2 t$ and using Fubini again, we get

$$\Gamma(s)L(2s, \widehat{f}) = \int_0^\infty \frac{dt}{t} t^s \int_W e^{-t|x|^2} \widehat{f}(x) dx.$$

The inner integral is the inner product in $L^2(W)$ of the functions $e^{-t|x|^2}$ and $\widehat{f}$. The Fourier transform preserves this inner product. Therefore, the inner product is the inner product of their Fourier transforms.

The Fourier transform of $e^{-t|x|^2}$ is $\frac{1}{t^{(n-1)/2}} e^{-|x|^2/t}$. The Fourier transform of $\widehat{f}$ is $f(-x)$. Therefore, we have the equality

$$\Gamma(s)L(2s, \widehat{f}) = \int_0^\infty \frac{dt}{t} t^{s-(n-1)/2} \int_W e^{-|x|^2/t} f(-x) dx.$$

In this double integral, change t to $\mid x \mid^2 /t$. We then get

$$\Gamma(s)L(2s, \widehat{f}) = \int_0^\infty \frac{dt}{t} t^{(n-1)/2-s} e^{-t} \int_W \mid x \mid^{2s-(n-1)} f(x) dx.$$

This proves the functional equation.

$\square$

Theorem 10 is now an immediate consequence of equation (8) and of the functional equation in Lemma 12 (applied to the value $s = \frac{(n-1)(1-u)}{2}$).

Notation 1. We denote by $\widehat{\pi_u}$ the **completion** of the space $\mathcal{S}(W)$ under the metric defined by Theorem 10. We then get a metric on $\widehat{\pi_u}$. The G-invariance of the pairing between $\mathcal{F}_u$ and $\mathcal{F}_{-u}$ and the definition of the inner product in Theorem 10 implies that the metric on $\widehat{\pi_u}$ is G-invariant. Note that elements of the completion $\widehat{\pi_u}$ may not be measurable functions on G, but only "generalised functions" or distributions.

3.2. **Embedding of the unramified Complementary series.**

Notation 2. Consider $H = O(n-1,1) \subset O(n,1) = G$ embedded by fixing the $n-1$-th basis vector e_{n-1}. Denote by $\sigma_{u'}$ the representation $\pi_{u'}$ constructed in the last section, not for the group G, but, for the group H. We set $u' = \frac{(n-1)u-1}{n-2}$. We assume that $0 < u' < 1$ which means that $\frac{1}{n-1} < u < 1$.

Denote by W the space $\mathbb{R}^{n-1}$. Let $W' \subset W$ be the subspace $\mathbb{R}^{n-2}$ whose last co-ordinate in $\mathbb{R}^{n-1}$ is zero.

Let $\widehat{W}_u$ be the completion of the Schwarz space of W under the metric

$$\mid \phi \mid_u^2 = \int_W \frac{dx}{\mid x \mid^{(n-1)u}} \mid \phi(x) \mid^2 .$$

Define the space $\widehat{W'}_{u'}$ correspondingly for W' as the completion of the Schwartz space of W' under the metric

$$\mid \psi \mid_{u'}^2 = \int_{W'} \frac{dy}{\mid y \mid^{(n-2)u'}} \mid \psi(y) \mid^2 .$$

Now, $\widehat{W}_u$ consists of all measurable functions ϕ on $W = \mathbb{R}^{n-1}$ which the u-norm, that is, integral

$$\mid \phi \mid_u^2 = \int_W \frac{dx}{(\mid x \mid^2 /2)^{(n-1)u}} \mid \widehat{\phi}(x) \mid^2,$$

is finite. One may define similarly, the u'-norm of a function on W'. Given ϕ in the space $\mathcal{S}(W)$, we define the function $J(\phi)$ on W by setting. for all $(y,t) \in W' \times \mathbb{R} = \mathbb{R}^{n-2} \times \mathbb{R} \simeq \mathbb{R}^{n-1} = W$,

$$J(\phi(y,t) = \phi(y).$$

Then $J(\phi)$ is a bounded measurable function on $\mathbb{R}^{n-1}$.

Proposition 13. *The map J defined on the Schwartz space $\mathcal{S}(W')$ has its image in the Hilbert space $\widehat{W}_u$ and extends to an isometry from $\widehat{W'}_{u'} \to \widehat{W}_u$.*

Proof. We compute tne u-norm of the bounded function $j(\phi)$:

$$\mid J(\phi) \mid_u^2 = \int_{W' \times \mathbb{R}} \frac{dy\ dt}{(\mid y \mid^2 + t^2)^{\frac{n-1)u}{2}}} \mid \phi(y) \mid^2 .$$

By changing the variable t to $\mid y \mid t$ and using te definition of the map J, we see that this u-norm is equal to the integral

$$\int \frac{dt}{(1+t^2)^{\frac{n-1)u}{2}}} \int_{W'} \frac{dy}{\mid y \mid^{(n-1)u-1}} \mid \phi(y) \mid^2 .$$

The integral over t converges when the exponent of the denominator term $\frac{(n-1)u}{2}$ is strictly larger than 1; that is, $u > \frac{1}{n-1}$. The integral over y is simply the u' norm of ϕ since $(n-1)u-1 = (n-2)u'$. Therefore, the u-norm of $J(\phi)$ is the u'-norm of ϕ up to a constant (the integral over the variable t of the function $\frac{1}{(1+t^2)^{(n-1)u}}$). We have therefore proved the proposition. $\square$

The completion π_u (and similarly $\sigma_{u'}$) was defined with respect to the metric $< \phi, I(u)(\phi) >$ (we use Theorem 10 to say that this is indeed a metric). It is clear from this definition that for $\phi \in \mathcal{S}(W) \subset \mathcal{F}_u$, the map $\phi \mapsto \widehat{\phi}$ (the roof over ϕ refers to the Fourier transform) gives an isometry from $\widehat{\pi}_u$ onto $\widehat{W}_u$. Similarly $\sigma_{u'}$ is isometric to $\widehat{W'}_{u'}$. Therefore, Proposition 13 says that the Hilbert space $\widehat{\sigma}_{u'}$ is isometrically embedded in the Hilbert space $\widehat{\pi}_u$. We now show that the map J is equivariant with respect to the action of the subgroup H on both sides.

Notation 3. Let $\phi^* \in \mathcal{S}(W) \subset \mathcal{F}_u$ be a function whose Fourier transform is a smooth compactly supported function $\widehat{\phi^*}$ on $\mathbb{R}^{n-1} \setminus \{0\}$. Denote by $\phi \in \mathcal{F}_{-u}$ the image of ϕ^* under the map $I(u)$. By the definition of $I(u)$, we have the formula (see Lemma 11),

$$\phi(u(y)w) = \int_{\mathbb{R}^{n-1}} \phi^*(u(y)wu(x)w)dx.$$

Using Lemma 12, we now get

$$\phi(y) = \int_{\mathbb{R}^{n-1}} \frac{dx}{\mid x \mid^{(n-1)u}} \widehat{\phi^*}(x) e^{-2ix.y}.$$

Denote by A_G the map $\widehat{\phi^*} \mapsto \phi$ from the space $\mathcal{C}_c^\infty(\mathbb{R}^{n-1} \setminus \{0\})$ into the space $\mathcal{F}_{-u}$, by the preceding formula. By our identifications, this extends to an isometry A_G from $\widehat{W}_u$ onto the Hilbert space $\widehat{\pi}_{-u} \simeq \widehat{\pi}_u$. This map exists for $0 < u < 1$. The G action on the image $\widehat{\pi}_{-u}$ gives, via this isomorphism A_G, the G-action on the Hilbert space $\widehat{W}_u$.

We similarly have a map A_H from the space $\mathcal{C}^\infty(\mathbb{R}^{n-2} \setminus \{0\})$ into $\mathcal{F}^H_{-u'}$ (of corresponding functions on the group H) which extends to an isometry from the space $\widehat{W'}_{-u'}$ onto $\widehat{\sigma}_{-u'} \simeq \widehat{\sigma}_{u'}$. The H action on $\widehat{\sigma}_{-u'}$

gives an action of H on the space $\widehat{W'}_{-u'}$.

Let $J^* : \widehat{W}_u \to \widehat{W}_{u'}$ be the adjoint of the map J.

Proposition 14. *We have the formula for the adjoint J^* of J:*

$$J^* = A_H^{-1} \circ res \circ A_G.$$

(Here $res : \mathcal{F}_{-u} \to \mathcal{F}^H_{-u'}$ is simply restricting the functions on G to the subgroup H (since $(n-1)u - 1 = (n-2)u'$, it follows that the restriction, maps the functions in $\mathcal{F}_{-u}$ into the functions in $\mathcal{F}^H_{-u'}$)).

Proof. The adjoint is defined by the formula

$$< g, J^* f >_{W_{u'}} = < Jg, f >_{w_u},$$

for all functions $f \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1} \setminus \{0\})$ and $g \in \mathcal{C}_c^\infty(\mathbb{R}^{n-2} \setminus \{0\})$. We compute the right hand side of this formula. By definition, the inner product of Jg and f is the integral

$$\int_{\mathbb{R}^{n-1}} \frac{dx}{\mid x \mid^{(n-1)u}} Jg(x) f(x).$$

Write $x \in \mathbb{R}^{n-1} = \mathbb{R}^{n-2} \times \mathbb{R}$ as $x = (y, s)$. Then, by the definition of J, $Jg(x) = Jg(y, s) = g(y)$ and the preceding integral is the integral

$$\int_{\mathbb{R}^{n-2}} dy g(y) \int ds f(y, s) \frac{1}{(\mid y \mid^2 + s^2)^{(n-1)u}}.$$

By changing s to $\mid y \mid s$, the integral becomes

$$\int_{\mathbb{R}^{n-2}} \frac{dy}{\mid y \mid^{(n-1)u-1}} g(y) \int_{\mathbb{R}} \frac{ds}{(1+s^2)^{(n-1)u/2}} f(y, \mid y \mid s).$$

By the definition of the $\widehat{W'}_{u'}$ inner product, this is $< g, h >_{W'_{u'}}$ where h is the function in y which is the integral over the variable s in the preceding equality. We have proved that $< Jg, f > = < g, h >$.

By the definition of the adjoint, this means that $h = J^* f$. We have therefore

$$J^*(f) = \int_{\mathbb{R}} \frac{ds}{(1+s^2)^{n-1)u/2}} f(y, \mid y \mid s).$$

We now compute $A_H J^*(f)(x)$ at a point $x \in \mathbb{R}^{n-2}$. By the definition of A_H, this is the integral

$$\int_{\mathbb{R}^{n-2}} \frac{dy}{\mid y \mid^{(n-2)u'}} J^*(f)(y) e^{-2ix.y}$$

The above integral formula for J^* shows that $A_H J^*(f)(x)$ is the integral

$$A_H J^* f(x) = \int_{\mathbb{R}^{n-2}\times\mathbb{R}} \frac{dy\ ds}{(\mid y \mid)^{(n-2)u'}(1+s^2)^{(n-1)u/2}} f(y, \mid y \mid s) e^{-2ix.y}.$$

Changing the variable s back to $\frac{s}{|y|}$ and substituting in this integral (and using the fact that $(n-2)u' = (n-1)u - 1$), we get

$$A_H J^* f(x) = \int_{\mathbb{R}^{n-2}\times\mathbb{R}} \frac{dy\ ds}{(\mid y \mid^2 + s^2)^{(n-1)u)/2}} f(y, s) e^{-2i(x,0).(y,s)}.$$

The latter is simply $A_G(f)(x)$ for $x \in \mathbb{R}^{n-2}$. By taking f to be $A_G^{-1}\phi$, we get, for all $x \in \mathbb{R}^{n-2}$, and all ϕ which are A_G - images of compactly suported smooth function on $\mathbb{R}^{n-1} \setminus \{0\}$, that

$$A_H J^* A_G^{-1}(\phi)(x) = \phi(x) = res(\phi)(x).$$

This proves the Proposition. □

Corollary 1. *The isometric map $J : \widehat{W'}_{u'} \rightarrow \widehat{W}_u$ is equivariant for the action of H on both sides.*

Proof. It is enough to prove that the adjoint J^* of J is equivariant for H action. It follows from the Proposition, that $J^* = A_H^{-1} \circ res \circ A_G$. All the maps A_H, A_G and res are H-equivariant. The Corollary follows. □

Corollary 2. *The map $\pi_{-u} \rightarrow \sigma_{-u'}$ from the module of K-finite vectors in $\widehat{\pi}_{-u}$ onto the corresponding vectors in $\widehat{\sigma}_{-u'}$ is continuous.*

Proof. The map res is, by the Proposition, essentially the adjoint J^* of the map J. But J is an isometry which means that J^* is a continuous projection. Therefore, res is continuous. □

Theorem 15. *Let $\frac{1}{n-1} < u < 1$ and $u' = \frac{(n-1)u-1}{n-2}$. The complementary series representation $\widehat{\sigma_{u'}}$ of $SO(n-1,1)$ occurs discretely in the restriction of of the complementary series representation $\widehat{\pi_u}$ of $SO(n,1)$.*

Proof. We have the H-equivariant isometry from $\widehat{W}'_u$ into $\widehat{W}_u$. Th former space is by construction, isometric to $\sigma_{u'}$ and the latter, to $\widehat{\pi}_u$.

Therefore, the Theorem follows. □

4. The Unramified Complementary series as a K-module

4.1. **The Harmonic Polynomials.** Let $Q_0 = x_1^2+x_2^2+\cdots+x_n^2$ be the standard quadratic form in n-variables and $K = SO(n)$ the orthogonal group of Q_0. Then K acts on the space $sym(\mathbb{C}^n)^*$ of polynomilas on $\mathbb{C}^n$ and commutes with homothesies. Therefore, the decomposition

$$sym(\mathbb{C}^n)^* = \bigoplus_{m\geq 0} sym^m(\mathbb{C}^n)^*,$$

is K-equivariant. The ideal $Q_0 sym(\mathbb{C}^n)^*$ is clearly K-stable and hence for the quotient ring the decomposition

$$\frac{sym(\mathbb{C}^n)^*}{Q_0 sym(\mathbb{C}^n)^*} = \bigoplus_{m\geq 0} \frac{sym^m(\mathbb{C}^n)^*}{Q_0 sym^{m-2}(\mathbb{C}^n)^*}$$

is a decomposition of K-modules.

It can be shown easily that the quotient $H_m = \frac{sym^m(\mathbb{C}^n)^*}{Q_0 sym^{m-2}(\mathbb{C}^n)^*}$ is an irreducible representation of $SO(n)$; it occurs exactly once in $sym^m(\mathbb{C}^n)^*$; the unique copy of H_m which occurs in $sym^m(\mathbb{C}^n)^*$ is the space of **harmonic** homogeneous polynomials of degree m. Moreover, we have the decomposition

$$sym^m(\mathbb{C}^n)^* = \bigoplus_{l=0}^{[m/2]} Q_0^l H_{m-2l}. \tag{9}$$

of $sym^m(\mathbb{C}^n)^*$ into irreducible representations of K.
The subgroup M of K is isomorphic to $SO(n-1)$ and the quotient K/M is the $n-1$ dimensional unit sphere in $\mathbb{R}^n$. it is well known that the space $Rep(K/M) = Rep(S^{n-1})$ of representation functions on K which are right M invariant decomposes as a K-module into

$$\pi_0 \overset{defn}{=} Rep(K/M) = \bigoplus_{m\geq 0} H_m.$$

We denote H_m sometimes by $H_m(\mathbb{C}^n)$.

We now exhibit some elements of H_m: let $\epsilon_1, \cdots, \epsilon_n$ be the standard basis of $\mathbb{R}^n$. we will view elements of $K = SO(n) \subset GL_n$ as $n\times n$ matrices $k = (k_{ij})$. Fix a Borel subgroup $B(\mathbb{C})$ of the complexification $K(\mathbb{C})$, which contains a Borel subgroup of $M(\mathbb{C})$ and which contains the standard maximal torus in K, namely the torus which leaves the two dimensional subspaces $\mathbb{R}\epsilon_j \oplus \mathbb{R}\epsilon_{n-j}$ for each $j \leq n-1$.

Notation 4. Given $k \in K$, consider the function $z = k_{11} + \sqrt{-1}k_{n1}$ on $k \in K$. We will view $M \subset K$ as the subgroup which fixes the first basis vector ϵ_1 of $\mathbb{R}^n$. Then, it is clear that the function z is right invariant under M. Therefore, $z \in Rep(K/M)$. Moreover, for every integer $m \geq 0$, the function $z^m \in H_m$: the function $z^m \in Rep(K/M) = Rep(S^{n-1})$ extends to the function $z = x_1 + \sqrt{-1}x_n$. Hence z^m is holomorphic on $\mathbb{C} = \mathbb{R}x_1 \oplus \mathbb{R}x_n = \mathbb{R}^2$ and is hence harmonic on $\mathbb{R}^2$; therefore, it is harmonic on $\mathbb{R}^n$. In fact, z^m is a highest weight vector for the Borel subgroup $B(\mathbb{C}) \subset K(\mathbb{C})$.

We then have:

$$z^m = (k_{11} + \sqrt{-1}k_{n1})^m \in H_m \subset Rep(S^{n-1}). \tag{10}$$

4.2. **The Restriction of unramified complementary series to K.** The Iwasawa decomposition shows that if $f \in \pi_u = Ind_P^G(\rho_P(a)^u)$ (the latter is the space of K-finite vectors in the unramified complementary series), then $f(kman) = f(k)\rho_P(a)^{(1+u)}$ is determined completely by its restriction to K; as a function on K, f is right M-invariant. Moreover, given a function $f \in Rep(K/M)$, f may be extended uniquely to an element of π_u because of the formula for f above. Hence the restriction of π_u to K is isomorphic to $\pi_0 = Rep(K/M)$.

Lemma 16. *The restriction of the unramified complementary series $\pi_u = Ind_P^G(\rho_P(a)^u)$ to K is a direct sum of irreducibles H_m each occurring exactly once:*

$$\pi_u|K \simeq \bigoplus_{m\geq 0} H_m.$$

Proof. The Lemma is an immediate consequence of the foregoing decomposiition of $Rep(K/M)$ as a direct sum of the space H_m of homogeneous harmonic polynomials of degree m. $\square$

The group $K = SO(n+1) \cap SO^0(n,1)$ is isomorphic to $SO(n)$ ($SO^0(n,1)$ denotes the connected component of identity). To set up this isomorphism, we conjugate $K \subset SO(n,1)$ into $SO(n)$ where the latter is viewed as a subgroup of $GL_{n+1}(\mathbb{R})$, as the set of elements which fix the last basis vector and preserve the standard quadratic form on $\mathbb{R}^{n+1}$. Recall that $SO(n,1)$ preserves the quadratic form

$$Q = 2x_0x_n + x_1^2 + \cdots + x_{n-1}^2,$$

defined for the basis $e_0, e_1, \cdots, e_{n-1}, e_n$ of $\mathbb{R}^{n+1}$. The vector fixed by K in $\mathbb{R}^{n+1} = \bigoplus_{j=0}^n \mathbb{R}e_j$ is easily seen to be $\epsilon_{n+1} = (e_0 - e_n)/\sqrt{2}$ (up to scalar multiples). Define

$$\epsilon_1 = (e_0 + e_n)/\sqrt{2}, \epsilon_2 = e_1, \cdots \epsilon_n = e_{n-1}.$$

Then the vectors ϵ_j $(1 \leq j \leq n+1)$ form a basis for $\mathbb{R}^{n+1}$.

In Lemma 6, we considered the Iwasawa decomposition of the element $u(x)w = k(x)u(x/a)d(a)$ and hence $k(x) = u(x)wd(a^{-1}u(-x/a)$. Thus, $k(x) \in K$, expressed as a matrix in GL_{n+1} with respect to the basis $e_0, \cdots, e_n$. We compute the matrix $k(x)$ with repsect to the basis $\epsilon_1, \cdots, \epsilon_n$. An explicit computation shows the following Lemma.

Lemma 17. *With respect to the basis $\epsilon_1, \cdots, \epsilon_n$, the matrix of $k(x)$ is*

$$\begin{pmatrix} 1 - \frac{|x|^2}{2a} & -\sqrt{2}x/a & 0 \\ -\sqrt{2}\ {}^t x/a & 1 -^t xx/a & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In the Lemma, $x \in \mathbb{R}^{n-1}$ is viewed as an $1 \times (n-1)$ row vector, and ${}^t x$ is a column vector of size $(n-1) \times 1$. The entry in the middle: $1 -^t xx/a$ is the identity square matrix 1 of size $n-1$ and ${}^t xx$ is a square matrix of size $n-1$.

The function $z = k_{11} + \sqrt{-1}k_{n1}$ of $SO(n)$ (see equation (10)) is then given, thanks to Lemma 17, by

$$z = \frac{1- \mid x \mid^2 /2 + \sqrt{-1}\sqrt{2}x_{n-1}}{1 + x^2/2}. \tag{11}$$

Proposition 18. *Consider the function $f_m : \mathbb{R}^{n-1} \simeq N \to \mathbb{C}$ given by*

$$f_m(x) = (1+ \mid x \mid^2 /2)^{(n-1)(1+u)/2} z^m,$$

with z as in equation (11). Then, f_m extends to a continuous function $f_m \in \pi_u$ which lies in $H_m \subset \pi_u$.

Proof. Consider the Iwasawa decomposition of the element $u(x)w \in G$. By Lemma 6, this is of the form

$$u(x)w = k(x)d(a)u(y,$$

with $a = 1+ \mid x \mid^2 /2$ and $k(x)$ as in Lemma 17. If $\phi \in H_m\pi_u$, then

$$\phi(u(x)w) = a^{(n-1)(1+u)/2}\phi(k(x)).$$

Conversely, any ϕ which is a function on $\mathbb{R}^{n-1}$ which satisfies the above equation extends to a function in π_u, provided the restriction of ϕ to the matrices $k(x)$ extends to a K-finite function on K. We choose ϕ to be the function z^m with z as in equation 11). Clearly, z^m lies in H_m. Then the proposition follows. $\square$

4.3. **The Restriction of the Representation H_m to $SO(n-1) = K_H$.** Let $K_H = K \cap H = SO(n-1)$. The standard representation $\mathbb{C}^n$ decomposes into $\mathbb{C}^{n-1} \oplus \mathbb{C}$ for the action of K_H. If $x_1, x_2, \cdots, x_{n-1}$ and x_n are the coordinate functions on $\mathbb{C}^n$ then K_H fixes the function x_n. Consequently, we have the decomposition of $sym^m(\mathbb{C}^n)^*$ as a K-module:

$$sym^m(\mathbb{C}^n)^* \simeq \bigoplus_{l=0}^{m} sym^l(\mathbb{C}^{n-1})^* \otimes sym^{k-l}(\mathbb{C})^* \simeq \bigoplus_{l=0}^{m} sym^l(\mathbb{C}^{n-1})^* x_n^{m-l}.$$

In the quotient ring $\frac{sym(\mathbb{C}^n)^*}{Q_0 sym(\mathbb{C}^n)^*}$, we have the equation

$$x_n^2 = -x_1^2 - x_2^2 - \cdots - x_{n-1}^2.$$

Consequently, we have the isomorphism

$$H_m = \frac{sym^m(\mathbb{C}^n)^*}{Q_0 sym^{m-2}(\mathbb{C}^n)^*} \simeq sym^m(\mathbb{C}^n)^* \oplus sym^{m-1}(\mathbb{C}^n)^* x_n,$$

as modules for K_H. By equation (9) (applied to K_H in place of K) we see that

$$H_m|_{K\cap H} \simeq \bigoplus_{l\leq l\leq m/2} H_{m-2l} Q_0'^l \bigoplus_{l\leq l\leq (m-1)/2} H_{m-2l-1} Q_0'^l,$$

where Q_0' is the quadratic form in $n-1$ variables preserved by $K \cap H$. This proves the following lemma.

Lemma 19. *The representation H_m restricted to $K_H = SO(n-1)$ decomposes into a direct sum of the irreducible representations $H_l(\mathbb{C}^{n-1})$ for $0 \leq l \leq m/2$ each occurring exactly once. Here, $H_l(\mathbb{C}^{n-1}) = W_l$ is the space of homogeneous harmonic polynomials of degree l in $n-1$ variables.*

We write $H_m = \bigoplus_{l=0}^m H_{m,l}$ where $H_{m,l}$ is the subspace which is the unique copy of the irreducible representation W_l occurring in $H_m(\mathbb{C}^n)$ restricted to $SO(n-1)$.

4.4. **The Geometric Restriction from S^{n-1} to S^{n-2}.** The map $K_H = SO(n-1) \subset K = SO(n)$ induced by the inclusion $\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ takes $M_H = M \cap H$ into M and hence gives the standard inclusion of the spheres $S^{n-2} = K_H/M_H \subset K/M = S^{n-1}$. Therefore, we get a map $\widehat{r} : Rep(K/M) \to Rep(K_H/M_H)$ of restriction of *functions*. Clearly, the map $\widehat{r}$ is K_H-equivariant.
Consider the pre-image of a fixed subspace $W_l \subset Rep(K_H/M_H)$. Then Lemma 19 says that

$$\widehat{r}^{-1}(W_l) = \bigoplus_{m\geq l} (\widehat{r}^{-1}(W_l) \cap H_m),$$

and that the intersection $\widehat{r}^{-1}(W_l)$ is either zero or is the unique copy of the representation W_l in H_m restricted to K_H, namely the space $H_{m,l}$ defined above. (Note that although H_m contains the representation W_l as a subspace, under the *geometric* restriction map $\widehat{r}$, the subspace may map to zero).

If E is an irreducible representation of a compact group L, then there exists a unique (up to positive scalar multiples) G-invariant inner product on E. Therefore, the restriction of the $L^2(K)$ inner product on the space $H_{m,l} \simeq W_l$ must be a multiple of the $L^2(K_H)$-inner product on $W_l \subset Rep(K_H/M_H)$. Hence the number

$$C_{m,l}(0) \stackrel{defn}{=} \frac{||\ \widehat{r}(f)\ ||^2_{L^2(K)}}{||\ f\ ||^2_{L^2(K_H)}} \tag{12}$$

is independent of the function f in $H_{m,l}$ provided $f \neq 0$. From now on, we replace the subscript $L^2(K)$ by K and similarly for K_H.

Remark 2. We can explicitly compute the number $C_{m,l}(0)$ when $G = SO(3,1)$ and $H = SO(2,1)$:

$$C_{m,l}(0) = 0 \text{ if m} \neq \text{l (mod2)}$$

and otherwise,

$$C_{m,l}(0) = \frac{4^m\Gamma(\frac{m-l+1}{2})^2\Gamma(\frac{m+l+1}{2})^2}{\Gamma(m-l+1)\Gamma(m+l+1)}.$$

Using Stirling's approximation for the Gamma function, one can show that as m tends to infinity,

$$C_{m,l}(0) \simeq \frac{Constant}{\sqrt{m-l+1}\sqrt{m+l+1}}.$$

4.5. **The Intertwining map in terms of K finite functions.** Recall that $\widehat{\pi}_u$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{n-1})$ with respect to the metirc

$$< \phi, I(u)(\phi) >= \int_K \phi(k)(I(u)(\phi))(k)\ dk.$$

Now, $I(u) : \mathcal{F}_u \to \mathcal{F}_{-u}$ is G-equivariant and hence preserves the K-types. The space of K-finite vectorsin $\widehat{\pi}_u$ was denoted $\pi_u = Ind_P^G(\rho_P(a)^u$. This is the space of K-finite sections of the homogeneous line bundle

on G/P associated to the character ρ_P^{1+u}.
By Lemma 16, we have

$$\pi_u|_K \simeq \bigoplus_{m\geq 0} H_m \simeq \pi_{-u}|_K.$$

Therefore, $I(u)$ takes the (unique) copy of $H_m \subset \pi_u$ into the copy of $H_m \subset \pi_{-u}$; identifying each of these copies with H_m, the K-intertwining map $I(u)$ operates by a scalar on H_m which we denote by $\lambda_m(u)$. We denote $I(u)$ by $I_G(u)$ to keep track of its dependence on G.

Theorem 20. *With the normalisation of $I_G(u)$ such that $\lambda_0(u) = \frac{\Gamma((n-1)(1-u)/2)}{\Gamma(n-1)(1+u)/2)}$, we have*

$$\lambda_m(u) = \frac{\Gamma(m+(n-1)(1-u)/2)}{\Gamma(m+(n-1)(1+u)/2)}.$$

By using Stirling's approximation that $\Gamma(m+z+1) \simeq Const.\frac{m^{m+z-1/2}}{e^m}$, we get the asymptotic, as m tends to infinity:

$$\lambda_m(u) \simeq \frac{Const.}{m^{(n-1)u}}. \tag{13}$$

We now prove Theorem 20.

Proof. We compute the action of $I(u)$ on a specific element of $H_m \subset \pi_u$. Consider the element $f_m \in H_m \subset \pi_u$ as in Proposition 18. Its restriction to K is the function z^m which has value 1 at the identity. On the other hand, by definition,

$$I(u)f_m = \lambda_m(u)f_m.$$

Therefore,

$$\lambda_m(u) = (I(u)f_m)(1) = \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+\mid x\mid^2/2)^{(n-1)(1-u)/2}} z^m(k(x)),$$

where z is given by

$$z = \frac{1-\mid x\mid^2/2+\sqrt{-1}\sqrt{2}x_{n-1}}{1+\mid x\mid^2/2}. \tag{14}$$

By changing the variable from x to $\sqrt{2}x \in \mathbb{R}^{n-1}$, we see that

$$\lambda_m(u) = \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+\mid x\mid^2)^{m+(n-1)(1+u)/2}}(1-\mid x\mid^2+2\sqrt{-1}x_{n-1})^m.$$

The function $1-\mid x\mid^2+2\sqrt{-1}x_{n-1}$ is the function

$$-x_1^2-\cdots x_{n-2}^2-(x_{n-1}+\sqrt{-1})^2.$$

The Theorem follows form the proposition 21 below on integrals in $\mathbb{R}^{n-1}$. $\square$

we compute some integrals on $\mathbb{R}^{n-1}$. These are closely related to the intertwining operator $I(u)$ acting on some specific elements of $H_m \subset \pi_u$. Assume that $u > 0$ is a real number.

Define $c_m(u)$ by the integral

$$c_m(u) = \int_{\mathbb{R}^{n-1}} \frac{dx\ dy}{(1+\mid y\mid^2 + \mid x\mid^2)^{m+(n-1)(1-u)/2}} (\mid y\mid^2 + (x-\sqrt{-1})^2)^m.$$

In this integral, an element of $\mathbb{R}^{n-1}$ is written in the form $(y,x) \in \mathbb{R}^{n-2} \times \mathbb{R}$. The Lebesgue measure on $\mathbb{R}^{n-1}$ is $dy\ dx$ where dy and dx are the Lebesgue measures on $\mathbb{R}^{n-2}$ and $\mathbb{R}$ respectively. The following proposition shows that the integral $c_m(u)$ is a product of ratios of Gamma functions.

Proposition 21. *Up to a constant $C(u)$ depending only on u, we have for all integers $m \geq 0$, the formula,*

$$c_m(u) = \frac{\Gamma(m+(n-1)(1-u))/2)}{\Gamma(m+(n-1)(1+u)/2)} C(u).$$

Proof. Set $k = n-1$, $s = (n-1)(1+u)/2$, and $s' = (n-1)(1-u)/2$. We will view an element of $\mathbb{R}^k$ as a pair (y,x) with $y \in \mathbb{R}^{k-1}$ and $x \in \mathbb{R}$. Then the integral

$$c_m(u) = \int_{\mathbb{R}^k} \frac{dy\ dx}{(1+\mid y\mid^2 + \mid x\mid^2)^{m+s}} (\mid y\mid^2 + (x-\sqrt{-1})^2)^m.$$

Multiply $c_m(u)$ by $\Gamma(m+s) = \int_0^\infty \frac{dt}{t} t^{m+s} e^{-t}$. Then, change the variable t to $t \times (1+\mid y\mid^2 + \mid x\mid^2)^{m+s}$. Then, $c_m(u)\Gamma(m+s)$ is the integral

$$\int_0^\infty \frac{dt}{t} t^{m+s} e^{-t} \int_{\mathbb{R}^k} dy\ dx\ e^{-t(|y|^2+|x|^2)} (\mid y\mid^2 + (x-\sqrt{-1})^2)^m.$$

Using the Cauchy integral formula (the integral over x is an integral over the real line and may be replaced by the integral over the line $\mathbb{R}+\sqrt{-1}$; it can be easily checked that the integral over vertical lines joining $\pm R$ and $\pm R+\sqrt{-1}$ tend to zero as R tends to infinity), the integral over $\mathbb{R}^k$ is easily seen to be the integral

$$\int_{\mathbb{R}^k} dy\ dx\ e^{-t(|y|^2+(x+\sqrt{-1}))^2} (\mid y\mid^2 + x^2)^m.$$

We now change the variable (y,x) to $(y/\sqrt{t}), x/\sqrt{t})$. Then, the integral over $\mathbb{R}^k$ becomes

$$A(t) = e^t \int_{\mathbb{R}^k} \frac{dy\ dx}{t^{(n-1)/2}}\ e^{-(|y|^2+x^2-\sqrt{t}\sqrt{-1}x)} (\mid y\mid^2 + x^2)^m \frac{1}{t^m}.$$

Now, $A(t) = e^t \frac{1}{t^{m+(n-1)/2}} \widehat{\phi}(\sqrt{t})$, where $\widehat{\phi}$ is the Fourier transform in the variable x of the function

$$\phi(x) = \int_{\mathbb{R}^{k-1}} dy\ e^{-(|y|^2+x^2)}(\mid y \mid^2 + x^2)^m.$$

Hence, some of the powers of t in the integral for $c_m(u)\Gamma(m_s)$ cancel, and

$$c_m(u)\Gamma(m+s) = \int_0^\infty \frac{dt}{t} t^{(n-1)u/2}\ \widehat{\phi}(\sqrt{t}).$$

Changing t to t^2, we have

$$c_m(u)\Gamma(m+s) = \int_0^\infty \frac{dt}{t} t^{(n-1)u}\ \widehat{\phi}(t) = \int_{\mathbb{R}} \frac{dt}{\mid t \mid^{1-(n-1)u}} \widehat{\phi}(t).$$

We now use the functional equation (Lemma 12) to conclude that

$$c_m(u)\Gamma(m+s) = R(u) \int_{\mathbb{R}} \frac{dt}{\mid t \mid^{(n-1)u}} \phi(t) =$$

$$= R(u) \int_{\mathbb{R}} dt \frac{1}{\mid t \mid^{(n-1)u}} \int_{\mathbb{R}^{k-1}} dy\ e^{-(|y|^2+t^2)}(\mid y \mid^2 + t^2)^m,$$

where $R(u)$ is the ratio $\frac{\Gamma((n-1)u/2)}{\Gamma((1-(n-1)u)/2}$.

Change the variable y to ty. Then the integral is

$$R(u) \int_{\mathbb{R}} \frac{dt}{\mid t \mid^{(n-1)u}} \int_{\mathbb{R}^{k-1}} \mid t \mid^{k-1} dy\ e^{-t^2(1+|y|^2)} t^{2m}(1 + \mid y \mid^2)^m =$$

$$= R(u) 2 \int_0^\infty \frac{dt}{t} t^{1-(n-1)u} t^{k-1+m} \int_{\mathbb{R}^{k-1}} dy\ e^{-t^2(1+|y|^2)}(1 + \mid y \mid^2)^m.$$

In the last integral, change t to $\frac{\sqrt{t}}{\sqrt{1+|y|^2}}$. Note that $k = n-1$. We then get

$$c_m(u)\Gamma(m+s) = R(u) \int \frac{dt}{t} t^{m+(n-1)(1-u)/2} e^{-t} \int_{\mathbb{R}^{k-1}} \frac{dy}{(1 + \mid y \mid^2)^{(n-1)(1-u)/2}}.$$

The integral over y is independent of m and depends only on u; the integral over t is $\Gamma(m+s')$. We have therefore proved that, for some function $C(u)$ depending only on u, we have

$$c_m(u) = \frac{\Gamma(m+s')}{\Gamma(m+s)} C(u),$$

with $s = (n-1)(1-u)/2$ and $s' = (n-1)(1-u)/2$. $\square$

Analogous to the numbers $\lambda_m(u)$ for the intertwining operator $I(u)$ for the group $G = SO(n,1)$, we may define for the intertwining operator $I_H(u') : \sigma_{u'} \to \sigma_{-u'}$ a similar sequence of numbers $\delta_l(u')$: the representation $\sigma_{u'}$ restricted to $K_H = SO(n-1)$ is a direct sum of irreducibles $W_l \simeq H_l(\mathbb{C}^{n-1})$ each occurring exactly once. The same holds for $\sigma_{-u'}$ and the operator $I_H(u')$ acts by a scalar $\delta_l(u')$ on the unique copy of W_l in $\pi_{u'}$. Theorem 20 applied to H shows that, up to a constant depending only on u',

$$\delta_l(u') = \frac{\Gamma(l + \frac{(n-2)(1-u')}{2})}{\Gamma(l + \frac{(n-2)(1+u')}{2})}$$

We also have the asymptotic as l tends to infinity:

$$\delta_l(u') \simeq \frac{Constant}{l^{(n-2)u'}} \tag{15}$$

Notation 5. Fix an integer $l \geq 0$ and let $W_l = H_l(\mathbb{C}^{n-1})$ be, as before, the space of homogeneous harmonic polynomials of degree l in $n-1$ variables.

Now,

$$Rep(K/M) = Rep(S^{n-1}) \simeq \bigoplus_{m=0}^{\infty} H_m$$

is a direct sum of irreducible representations H_m. Each H_m restricted to $SO(n-1) = K_H$ contains a unique copy of W_l which we denoted by $H_{m,l}$, provided $m \geq l$. We have therefore, a section $s_m : W_l \to H_{m,l}$ of the restriction map $\widehat{r} : H_{m,l} \to W_l$, *provided the map is non-zero.* Let X be the subset of the set m of nonnegative integers such that the restriction from $H_{m,l}$ into W_l is non-zero (W_l being irreducible, then the map is an isomorphism). Therefore, any K_H -equivariant map $s : W_l \to Rep(K/M)$ is of the form

$$s(w) = \sum_{m \in X} a_m s_m(w),$$

for all $w \in W_l$; the a_m are constants only finitely many of which are non-zero.

Fix a K_H- equivariant map s, an element w of W_l and set $v = s(w) = \sum a_m s_m(w)$. Since the s_m's are sections, $\widehat{r}(v) = (\sum a_m)w$. Since $s_m(W_l) = H_{m,l} \subset H_m \subset \pi_{-u}$ they are all orthogonal to each other in the K-invariant inner product on π_{-u}. Consider the ratio

$$\frac{|| \widehat{r}(v) ||^2_{\sigma_{-u'}}}{|| v ||^2_{\pi_{-u}}} = \frac{| \sum a_m |^2 || w ||^2_{\sigma_{-u'}}}{\sum | a_m |^2 || s_m(w) ||^2_{\pi_{-u}}}.$$

By the Cauchy Schwartz inequality, the latter ratio is bounded above by

$$\sum \frac{1}{|| s_m(w) ||^2_{\pi_{-u}}} || w ||^2_{\pi_{-u}},$$

and equality is attained by choosing the a_m's suitably (note that since $m \in X$, the norms of $s_m(w)$ are non-zero and hence one may divide by them). Now, by the definition of the inner product in π_{-u}, we have

$$|| s_m(w) ||^2_{\pi_{-u}} \frac{1}{\lambda_m(u)} || s_m(w) ||^2_K .$$

Similarly,

$$|| w ||^2_{\sigma_{-u'}} = \frac{1}{\delta_l(u')} || w ||^2_{K_H} .$$

By the definition of the numbers $C_{m,l}(0)$, we have

$$C_{m,l}(0) = \frac{|| res(s_m(w)) ||^2_{K_H}}{|| s_m(w) ||^2_K} = \frac{|| w ||^2_{K_H}}{|| s_m(w) ||^2_K}.$$

The last four equalities and the Cauchy-Schwartz inequality imply that

$$\frac{|| res(v) ||^2_{\sigma_{-u'}}}{|| v ||^2_{\pi_{-u}}} \leq \big(\sum_{m \in X} \lambda_m(u) C_{m,l}(0) \big) \frac{1}{\delta_l(u')}.$$

By choosing v (that is, the numbers a_m) suitably, we may ensure that the equality in the Cauchy- Schwartz inequality holds. Hence we have

$$sup_{v \in \pi_{-u'}} \frac{|| res(v) ||^2_{\sigma_{-u'}}}{|| v ||^2_{\pi_{-u}}} = \big(\sum \lambda_m(u) C_{m,l}(0) \big) \frac{1}{\delta_l(u')}. \tag{16}$$

Theorem 22. *There is a constant A depending only on u so that for all $l \geq 0$, we have the estimate*

$$\sum_{m \geq l} \frac{C_{m,l}(0)}{m^{(n-1)u}} l^{(n-2)u'} < A.$$

Proof. By Corollary 2, the restriction of sections $res : \pi_{-u} \to \sigma_{-u'}$ is continuous. Therefore, there exists a number $B > 0$ such that

$$B = sup \frac{|| res(v) ||^2_{\sigma_{-u'}}}{|| v ||^2_{\pi_{-u}}},$$

where the supremum is over all the non-zero vectors $v \in \pi_{-u}$. We then get from equation (16), that

$$B = \big(\sum \lambda_m(u) C_{m,l}(0)\big) \frac{1}{\delta_l(u')}.$$

We now use the asymptotic (13) for $\lambda_m(u)$ and the asymptotic (15) for $\delta_l(u')$ to conclude the Theorem: A may be taken to be a multiple of B by the constants in the cited asymptotics. $\square$

5. RAMIFIED COMPLEMENTARY SERIES AS A K MODULE

5.1. **The K module $Ind_M^K(\wedge^i)$.** Fix an integer $i \leq n-1$; denote by $\wedge^i$ the representation of M on the ith exterior power of the standard representation of $M = SO(n-1)$. Denote by $\pi(i) = Ind_M^K(\wedge^i)$ the space of K-finite sections of the homogeneous vector bundle on K/M induced by the representation $\wedge^i$ of M. Fix an M invariant inner product $<,>_{\wedge^i}$ on $\wedge^i$. Given two sections $\phi, \psi \in \pi(i)$, define as usual, their (K-invariant) inner product

$$< \phi, \psi >_{\pi(i)} = \int_{K/M} dk \int_M dm < \psi(km), \psi(km) >_{\wedge^i} .$$

Denote by $|| \phi ||_K^2 = < \phi, \phi_{\pi(i)}$.

By the Frobenius reciprocity theorem, an irreducible representation ρ of K occurs in $\pi(i)$ if and only if the restriction of ρ to M contains the dual (in the case of $M = SO(n-1)$, all representations are self dual) of the representation $\wedge^i$. Clearly the restriction to M of the two representations $\wedge^i\mathfrak{p}$ and $\wedge^{i+1}\mathfrak{p}$ both contain $\wedge^i$ ($\mathfrak{p}$ is the standard representation of K on $\mathbb{C}^n$). Therefore, both the K-representations $\wedge^i\mathfrak{p}$ and $\wedge^{i+1}\mathfrak{p}$ occur in $\pi(i)$.

We will view $\wedge^i\mathfrak{p}$ as sections in $\pi(i)$. Let $\omega_1, \omega_2, \cdots, \omega_k$ be a basis of $\wedge^i\mathfrak{p}$. Further, again by Frobenius reciprocity, any irreducible representation ρ occurring in $\pi(i)$ contains $\wedge^i$ as an M submodule hence $\rho \otimes \wedge^i$ contains a vector which is M invariant. But the only irreducible representations of K which have an M invariant vector are the representations H_m of homogeneous harmonic polynomials of degree m ion $\mathbb{C}^n = \mathfrak{p}$. Therefore, we have

$$\rho \subset H_m \otimes \wedge^i\mathfrak{p},$$

as K modules.

Fix a Borel subgroup $B(\mathbb{C})$ of the complexification $K(\mathbb{C}) = SO(n, \mathbb{C})$ of $K = SO(n)$, as in Notation 4. Let $v_m \in H_m$ and $w_i \in \wedge^i\mathfrak{p}$ be highest weight vectors for $B(\mathbb{C})$. Viewed as a function on K, the vector v_m is the function z^m as in equation (10). The vector w_i is simply an $i \times i$ minor of the matrix $k \in K \subset GL_n(\mathbb{R})$. We may take the minor w_i corresponding to the subset $\{2, 3, \cdots, i, i+1\}$; this vector (function on K) has the property that its right translations by elements of M, span a representation isomorphic to $\wedge^i\mathfrak{p}_M$, and upon left translation under K, it generates a representation isomorphic to $\wedge^i\mathfrak{p}$. Then, $v_m \otimes w_i$ is a highest weight vector in $H_m \otimes \wedge^i \mathfrak{p}$ and therefore its K translates span an irreducible representation V_m of K with $V_m \subset H_m \otimes \wedge^i \mathfrak{p}$. We may similarly construct an irreducible representation $V'_m \subset H_{m-1} \otimes \wedge^{i+1} \mathfrak{p}$.

We now describe the decomposition of the representation of $\pi(i)$ into K-irreducibles. The following lemma holds.

Lemma 23. *Each V_m (and each V'_m) is an irreducible representation of K. (The representations V_m and $V'_{m'}$ are all inequivalent). We have the orthogonal decomposition*

$$\pi(i) = Ind_M^K(\wedge^i) = V \oplus V',$$

where $V = \bigoplus V_m$ and $V' = \bigoplus V'_m$.

5.2. **the restriction of the K module to the K_H module.** In the previous subsection, we defined the sections $\pi(i)$ of the homogeneous vector bundle $E_K(i))$ on K/M associated to the representation $\wedge^i(\mathbb{C}^{n-1}$ of M, and analysed the representation $\pi(i)$ as a K module. In this section, we consider the vector bundle $E(i)_{K_H}$ on K_H/M_H, induced by the M_H-representation $\wedge^i \mathbb{C}^{n-2}$. Denote the resulting map of sections by $\sigma(i)$. There is, analogously, a K_H-invariant inner product on $\sigma(i)$.

We construct sub-representations $W_l \subset H_l(\mathbb{C}^{n-1}) \otimes \wedge^i \mathbb{C}^{n-1}$ and $W'_l \subset H_l(\mathbb{C}^{n-1}) \otimes \wedge^{i+1} \mathbb{C}^{n-1}$, similarly to the representations V_m and V'_m in the preceding section, but now for the group $K_H = SO(n-1)$. Here, $H_l(\mathbb{C}^{n-1})$ is the space of harmonic polynomials in $n-1$-variables. Then

Lemma 24. *W_l and W'_l are irreducible representations of $K_H = SO(n-1)$. Moreover,*

$$\sigma(i) = W \oplus W'$$

with $W = \bigoplus W_l$ and $W'_l = \bigoplus W'_l$.

The inclusion $V_m \subset H_m \otimes \wedge^i \mathfrak{p}$, implies that $H_m \subset \wedge^i \mathfrak{p}$. Let $dim(\wedge^i \mathfrak{p}) = k$. We have similarly, the inclusion $W_l \subset H_l(\mathbb{C}^{n-1}) \otimes \wedge^i \mathfrak{p}_M$ and $H_l(\mathbb{C}^{n-1} \subset W_l \otimes \wedge^i \mathfrak{p}_M$. These inclusions give us the following inequalities:

we have for all l and all m,

$$1/k \leq \frac{dim\ V_m}{dim\ H_m} \leq k, \tag{17}$$

and similarly,

$$1/k \leq \frac{dim\ W_l}{dim\ H_l(\mathbb{C}^{n-1})} \leq k, \tag{18}$$

The restriction followed by the projection from $\mathbb{C}^{n-1}$ onto $\mathbb{C}^{n-2}$ gives a K_H -equivariant map res from $\pi(i)$ into $\sigma(i)$. If $f \in \pi(0)$ and $\phi \in \pi(i)$, then clearly,

$$res(f\phi) = res(f)res(\phi).$$

Recall that each the restriction of H_m to $K_H = SO(n-1)$ contains a unique copy of the irreducible representation $H_l(\mathbb{C}^{n-1})$, provided $m \geq l$. From Lemma 23 and Lemma 24, it follows that the restriction of V_m to K_H contains a unique copy of the ireducible representation W_l. We denote this copy by $V_{m,l}$.

Up to positive scalar multiples, there is a unique K_H-invariant inner product on any irreducible of K_H. In particular, there exists a scalar $C_{m,l}(i)$ such that for all non-zero vectors of the form $\phi \in V_{m,l}$ we have:

$$C_{m,l}(i) = \frac{|| \ res(\phi) \ ||^2_{K_H}}{|| \ \phi \ ||^2_K},$$

with $\phi \in V_{m,l}$. We will compute the ratio

$$R = \frac{N}{D},$$

for a $\phi \in V_{m,l}$.

Now, $V_{m,l} \subset V_m \subset H_m \otimes \wedge^i \mathfrak{p}$. Under this embedding, and by the definition of $V_{m,l}$, we have that $V_{m,l} \subset H_{m,l} \otimes \wedge^i \mathfrak{p}$. Hence every vector $\phi \in V_{m,l}$ may be written as a linear combination

$$\phi = f_1 \otimes e_1 + f_2 \otimes e_2 + \cdots f_k \otimes e_k$$

, with e_j an orthonormal basis of $\wedge^i \mathfrak{p}$, a part of which is an orthonormal basis of the subspace $\wedge^i \mathfrak{p}_H$, and with $f_j \in H_{m,l}$. We will assume that ϕ is of norm one in the K-invariant metric on the tensor product. Then we have

$$1 = <\phi, \phi>_{V_m} = \sum_j <f_j, f_j>_{H_m} <e_j, e_j> = \sum_j <f_j, f_j> .$$

This shows in particular, that $<f_j, f_j> \leq 1$.
Consider the matrix coefficient $<\phi, g(\phi)>_{V_m}$ of the representation V_m. Its $L^2(K)$ norm is the reciprocal of the dimension of V_m:

$$D = ||<\phi, g(\phi)>||^2_K = \frac{1}{dimV_m}.$$

Consider the restriction of ϕ to K_H: this is the vector

$$res(\phi) = \sum res(f_j) \otimes e_j,$$

where the sum runs over only those e_j which lie in the subspace $\wedge^i \mathfrak{p}_H$. Consequently, we get the following expression for the umerator:

$$N = || \, res < \phi, h(\phi) > ||^2 = \sum_j || \, (res(f_j, h(f_j)) \, ||^2_{K_H} \, .$$

By the definition of the constant $C_{m,l}(0)$ (this is the unramified case), we have that the latter sum is

$$C_{m,l}(0) \sum || \, (f_j, g(f_j)) \, ||^2_K = \leq k C_{m,l}(0) \frac{1}{dim H_m}.$$

By comparing the dimensions (see equation (17)), we see that the square of the $L^2(K_H)$ norm of the restriction of the matrix coefficient $(\phi, g(\phi))_{V_m}$ is bounded by $k^2 \frac{1}{dim V_m}$ which is k^2- times the $L^2(K)$-norm of the matrix coefficient $(\phi, g(\phi))_{V_m}$.

We have thus proved the following Theorem.

Theorem 25. *There exists a constant γ independent of m, l such that for all integers $m, l \geq 0$, the estimate*

$$C_{m,l}(i) \leq \gamma C_{m,l}(0)$$

holds.

Theorem 26. *Let $\frac{1}{n-1} < u < 1 - \frac{2i}{n-1}$. Set $u' = \frac{(n-1)u-1}{n-2}$. Then, for each i, the series*

$$\sum_{m=l}^{\infty} \frac{C_{m,l}(i)}{m^{(n-1)u}} l^{(n-2)u'},$$

converges and is bounded by a constant A independent of the integer l.

Proof. This is an immediate corollary of Theorem 25 and Theorem 22. ∎

6. Ramified Complementary Series

6.1. **ramified complementary series as a K-module.** .
Assume $n \geq 4$. Let i be an integer with $1 \leq i \leq [n/2]-1$. Let $0 < u < 1 - \frac{2i}{n-1}$. Set

$$\pi_u(i) = Ind_P^G(\wedge^i \mathfrak{p}_M \otimes \rho_P(a)^u).$$

Here, the induction is unitary induction and $\pi_u(i)$ refers to the space of K**-finite functions** (as opposed to the notation of the previous section, where $\widehat{\pi_u}$ refereed to the **completion** of the Schwartz space on $N \simeq Nw \subset G/P$ with respect to a suitable metric. The space $\widehat{\pi_u(i)}$ is a G-module. The K-finite vectors in $\widehat{\pi_u(i)}$ are precisely the elements of $\pi_u(i)$. This is only a $(\mathfrak{g}, K)$-module.

We may define $\pi_{-u}(i)$ similarly, replacing u by $-u$ (minus u). Let $\mathfrak{g}_0$ denote the real Lie algebra of G. We have a $\mathfrak{g}_0$-invariant pairing (by an abuse of notation, we will refer to this pairing as a G-invariant pairing) between $\pi_u(i)$ and $\pi_{-u}(i)$ (since the representations $\wedge^i \mathfrak{p}_M$ of M is self dual).

Lemma 27. *If $u > 0$ and $\phi \in \pi_u(i)$, then the integral*

$$I(u)(\phi)(g) = \int_N \phi(gnw)dn$$

converges and the function $g \mapsto I(u)(\phi)(g)$ lies in $\pi_{-u}(i)$. Moreover, the map $I(u)$ intertwines the $(\mathfrak{g}, K)$- action as a map from $\pi_u(i)$ into $\pi_{-u}(i)$.

Consider the complementary series $\pi_u(i)$. This is a K module. Its elements are K-finite sections of a homogeneous vector bundle on G/P and, by the Iwasawa decomposition, are completely determined by their restriction to K. But restricted to K, these are just elements of π_i. We have thus proved that $\pi_u(i)$, viewed as a K-module is isomorphic to $\pi(i)$.

By Lemma 23, the representation π_i is multiplicity free. Each representation V_m and V'_m occurs in π_i exactly once and no other irreducble representation of K occurs in π_i. By the preceding paragraph, the same holds for $\pi_u(i)$ and $\pi_{-u}(i)$.

By Lemma 27, the map $I(u)$ intertwines the $(\mathfrak{g}, K)$-action between $\pi_u(i)$ and $\pi_{-u}(i)$. As we have seen in the last paragraph, both these are isomorphic as K-modules to π_i. The latter is multiplicity free as a K-module. Hence $I(u)$ maps the copy of V_m inside $\pi_u(i)$ into the copy of V_m inside $\pi_{-u}(i)$. Since V_m is K-irreducible, the intertwining map $I(u)$ acts by a scalar on V_m. Denote this scalar by $\lambda_m(u)$. Define similarly the scalar $\lambda'_m(u)$ for the module W_m.

Theorem 28. *After a suitable normalisation of the intertwining operator, we have the following formula expressing the scalar $\lambda_m(u)$ in terms of Γ functions:*

$$\lambda_m(u,i) = \frac{\Gamma(m + \frac{(n-1)(1-u)}{2})}{\Gamma(m + \frac{(n-1)(1+u)}{2})}(A - B_i \frac{m + (n-1)(1-u)/2}{m + (n-1)(1+u)/2}),$$

with A and B_i constants depending only on u (B_i depends on the integer i) and independent of m.

The asymptotics of the Γ function now imply that as m tends to infinity,

$$\lambda_m(u,i) \simeq \frac{Constant}{m^{(n-1)u}}. \tag{19}$$

Replacing G by $H = SO(n-1,1)$ we obtain, as l tends to infinity, the asymptotic $\delta_l(u',i) \simeq \frac{constant}{l^{(n-2)u'}}$, where $\delta_l(u',i)$ is the scalar by which an analogous intertwining operator acts on the K_H-type W_l in $\sigma_{-u'}$.

Notation 6. Given an integer $l \geq 0$, we have the irreducible representation of $SO(n-1,1) = K_H$ occurring in $W \subset \sigma(i)$. If $m \geq l$, then there is a unique copy $V_{m,l}$ of W_l in the restriction of $\pi(i)$ to K_H. Denote by $s_m : W_l \to V_{m,l}$ a section of the restriction map $res : V_{m,l} \to W_l$ provided the restriction map is non-zero (this construction is entirely analogous to the unramified case). Then any K_H equivariant map s from W_l into $\pi(i)$ is of the form

$$s(w) = \sum a_m s_m(w),$$

for all $w \in W$.

Compute the ratio

$$\frac{\| \, res(s(w)) \, \|^2_{\sigma_{-u'}}}{\| \, s(w) \, \|^2_{\pi_{-u}}}.$$

Using arguments similar to those in the subsection 5, we see that

$$\frac{\| \, res(s(w)) \, \|^2_{\sigma_{-u'}}}{\| \, s(w) \, \|^2_{\pi_{-u}}} \leq \sum_{m \geq l} C_{m,l}(i) \lambda_m(u,i) \frac{1}{\delta_l(u',i)}, \tag{20}$$

for every integer l.

Theorem 29. *The representation on the* **Hilbert space** $\widehat{\sigma_{u'}(i)}$ *occurs discretely in the* **Hilbert space** $\widehat{\pi_u(i)}$ *if* $\frac{1}{n-1} < u < 1 - \frac{2i}{n-1}$.

Proof. We prove in fact that the map

$$res : \pi_{-u} \to \sigma_{-u'}$$

is continuous for the norms on the representations involved. Since the map res is equivariant for K_H and distinct irreducible K_H-isotypicals on both sides are mutually orthogonal, it is sufficient to check that the map is bounded on each K_H-type W_l occurring in π_{-u} and $\sigma_{-u'}$, the bound being independent of the integer l. That is, For each non-zero $v \in \pi_{-u}(i)$, we must prove that the ratio

$$R_v = \frac{|| \, res(v) \, ||^2_{\sigma_{-u'}}}{|| \, v \, ||^2_{\pi_{-u}}}$$

is bounded by a constant independent of l.

We see from equation (25) that the ratio R_v is bounded from above by the sum

$$C_{m,l}(i)\lambda_m(u,i)\frac{1}{\delta_l(u',i)}.$$

The asymptotics for $\delta_l(u,i)$ are the smae as those for $\delta_l(u') \simeq \frac{Const}{l^{(n-2)u'}}$; similarly $\lambda_m(u,i)$ has the same asymptotic as $\lambda_m(u) \simeq \frac{Const}{m^{(n-1)u}}$. Therefore, the ratio R_v is bounded above by

$$\sum C_{m,l}(i)\frac{1}{m^{(n-1)u}} l^{(n-2)u'}.$$

Now, Theorem 26 shows that R_v is bounded by a constant independent of l and v. This proves the continuity of the restriction map.

$\square$

7. The Main Theorem and Corollaries

Theorem 30. *Let $n \geq 3$ and $G = SO(n,1)$. Let i be an integer with $0 \leq i \leq [n/2]-1$ where $[x]$ denotes the integral part of x. Let $\frac{1}{n-1} < u < 1 - \frac{2i}{n-1}$. Then the complementary series representation*

$$\pi_u(i) = Ind_P^G(\wedge^i \mathfrak{p}_M \otimes \rho_P(a)^u),$$

contains discretely, the complementary series representation

$$\sigma_{u'}(i) = Ind_{P\cap H}^H(\wedge^i \mathfrak{p}_{M\cap H} \otimes \rho_{P\cap H}^{u'}),$$

where $u' = \frac{(n-1)u-1}{n-2}$. Further, as u tends to $1-\frac{2i}{n-1}$, u' tends to $1-\frac{2i}{n-2}$.

Proof. The Theorem follows if part (2) of Theorem 29 holds, because of the equivalence in Theorem 29. Now, part (2) folows if the corresponding statement of part (2)is true when i is replaced by $i=0$. This follows from Theorem 25.

But Statement (2) of Theorem 29 follows from part (1); however, by Theorem 15, statement (1) is indeed true. Therefore statement (2) of the Theorem 29 is also true for $i=0$. This implies statement (2) for arbitrary i and hence statement (1) is true for arbitrary i. This proves the main theorem.

□

Corollary 3. *Let $i \leq [n/2]-1$. Then the cohomological representation $A_i(n)$ of $SO(n,1)$ of degree i restricted to $SO(n-1,1)$ contains discretely, the cohomological representation $A_i(n-1)$.*

The discrete series representation $A_i(2i)$ of $SO(2i,1)$ is contained discretely in the representation A_i.

Proof. To prove the first part, we use Theorem 30 and let u tend to the limit $1-\frac{2i}{n-1}$. This gives: $A_i(n)$ restricted to $SO(n-1,1)$, contains $A_i(n-1)$ discretely. The rest of the corollary follows by induction. □

Corollary 4. *Suppose that for all n, the* **tempered** *cohomological representations $A_i(n)$ (i.e. $i=[n/2]$) are not limits of complementary series* **in the automorphic dual** *of $SO(n,1)$. Then, for any n, all nontempered cohomological representations of $SO(n,1)$ are isolated in the automorphic dual.*

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