Cuspidal representations of reductive groups joint work with Dan Barbasch

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G semisimple noncompact Lie group

Γ discrete subgroup with finite covolume

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$L^2_{dis}(\Gamma\backslash G)$ discrete spectrum.

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$L^2_0(\Gamma\backslash G) \subset L^2_{dis}(\Gamma\backslash G)$ cusp forms

An irreducible automorphic representation of G is called cuspidal if it occurs discretely with finite multiplicity in $L^2_{cusp}(\Gamma\backslash G)$. It is called residual if it occurs in in the complement.

General Problem: Determine multiplicities of irreducible representations in the cuspidal spectrum

More precise problem: Are there irreducible cuspidal representations with the following properties: "Favorite list"

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My "Favorite list" :
Integral infinitesimal character and invariant under certain automorphisms.

The setup and Notation

G algebraic reductive group connected defined over $\mathbb{Q}$
Assume that $G(\mathbb{R})$ has no compact factor

$\mathfrak{g}$ Lie algebra of $G(\mathbb{R})$

$K_\infty \subset G(\mathbb{R})$ max compact

$\tau : G \to G$ rational automorphism

F finite dimensional irreducible representation of $G(\mathbb{R}) \ltimes \{1, \tau\}$

Note: $tr(F(\tau)) \neq 0$ implies $F_{|G(\mathbb{R})}$ irreducible. So F has an infinitesimal character.

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We have

$$\mathcal{A}_{cusp}(G(\mathbb{A})/G(\mathbb{Q})) = \oplus \pi_{\mathbb{A}}$$

An irreducible representation is called cuspidal if $\mathrm{Hom}(\pi_{\mathbb{A}}, \mathcal{A}_{cusp}(G(\mathbb{A})/G(\mathbb{Q})) \neq 0$

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Problem: For a given F and τ are there τ-stable irreducible cuspidal representations with the same infinitesimal character as F ?

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A look at the example G=GL(n),

$\pi_{\mathbb{A}}$ cuspidal representation of $G(\mathbb{A})$.

$L(s, \pi_{\mathbb{A}}, S^2\mathbb{C}^n)$ symmetric square L-function of $\pi_{\mathbb{A}}$,

$L(s, \pi_{\mathbb{A}}, \wedge^2 \mathbb{C}^n)$ exterior square L-function of $\pi_{\mathbb{A}}$.

Then $L(s, \pi_{\mathbb{A}}, S^2\mathbb{C}^n)L(s, \pi_{\mathbb{A}}, \wedge^2 \mathbb{C}^n)$ has a pole at s=1 precisely if $\pi_{\mathbb{A}}$ is stable under τ_c, i.e is self dual.

Theorem 1.
Let G be a connected reductive linear algebraic group defined over $\mathbb{Q}$, Assume that $G(\mathbb{R})$ has no compact factors and that the derived group is simple. Let F be a finite dimensional irreducible representation of $G(\mathbb{R}) \ltimes \{1, \tau\}$, and assume that the centralizer of τ in $\mathfrak{g}$ is of equal rank. If tr $F(\tau) \neq 0$, then there exists a cuspidal automorphic representation $\pi_{\mathbb{A}}$ of $G(\mathbf{A})$ stable under τ, with the same infinitesimal character as F.

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In addition

$$H^*(\mathfrak{g}(\mathbb{R}), K_\infty, \pi_{\mathbb{A}} \otimes F) \neq 0.$$

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tr $F(\tau) \neq 0$ for infinitely many representations.

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Previous results:

Clozel,1986, $G(\mathbb{R})$ equal rank, τ trivial

Speh-Rohlfs , 1989, cocompact lattice, τ Cartan like

Borel-Labesse-Schwermer, 1996, S-arithmetic subgroups of reductive groups over number fields, τ Cartan like

and some special cases.

An example:

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Theorem 2.
There exist cuspidal representations $\pi_{\mathbb{A}}$ of GL(n,$\mathbb{A}$) with trivial infinitesimal character invariant under the Cartan involution τ_c.

If n= 2m there also exist cuspidal representations $\pi_{\mathbb{A}}$ of GL(n,$\mathbb{A}$) with trivial infinitesimal character invariant under τ_s.

Application:

K_f small compact subgroup

A_G connected component of maximal split torus of center of $G(\mathbb{R})$.

$S(K_f) := K_\infty \; K_f \backslash G(\mathbb{A})/A_G G(\mathbb{Q})$ locally symmetric space

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Example:
$\mathbb{A}$ adels of $\mathbb{Q}$
G= GL(2)
K_∞ = 0(2)
$K_f = \prod_p GL(2, O_p)$. Then

$$S(K_f) = H/SL(2, \mathbb{Z})$$

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By Franke

$$H^*(S(K_f), F) = H^*(\mathfrak{g}, K_\infty, \mathcal{A}(G(\mathbb{A})/A_G G(\mathbb{Q})) \otimes F)^{K_f}.$$

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By Borel

$$\begin{aligned} &H^*(\mathfrak{g}, K_\infty, \mathcal{A}_{cusp}(G(\mathbb{A})/A_G G(\mathbb{Q})) \otimes F))^{K_f} \\ &\quad \hookrightarrow H^*(\mathfrak{g}, K_\infty, \mathcal{A}(G(\mathbb{A})/A_G G(\mathbb{Q})) \otimes F))^{K_f}. \end{aligned}$$

The image is the cuspidal cohomology $H^*_{cusp}(S(K_f), F)$.

Theorem 3.
Let G be a connected reductive linear algebraic group defined over $\mathbb{Q}$ whose derived group is simple. Suppose that K_f and τ satisfy the assumptions satisfies of the main theorem. Then

$$H^*_{cusp}(S(K_f), \mathbb{C}) \neq 0.$$

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Previous work by Clozel, Rohlfs-Speh, Borel-Labesse-Schwermer and others

The map

$$H^*(\mathfrak{g}, K_\infty, \mathcal{A}_{res}(G(\mathbb{A})/A_G G(\mathbb{Q})) \otimes F))^{K_f}$$
$$\to H^*(\mathfrak{g}, K_\infty, \mathcal{A}(G(\mathbb{Q})A_G \backslash G(\mathbb{A})) \otimes F))^{K_f}.$$

NOT INJECTIVE. It image is the residual cohomology $H^*_{res}(S(K_f), F)$.

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Theorem 4. *(joint with J.Rohlfs) Suppose that $G = Gl(n)$ and that $n = r\,m$. Then for K_f small enough*

$$H^j_{res}(S(K_f), \mathbb{C}) \neq 0$$

if

1. *r and m even and* $j = \frac{r(r+1)m}{4} + \frac{r^2 m(m+2)}{2}$
2. *r even and m odd and* $j = \frac{r(r+1)(m-1)}{4} + \frac{r^2(m-1)(1+m)}{2}$,
3. *m=2 and* $j = r(r+1)/2$.

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G^* generated by G and τ.

$F : G(\mathbb{R}) \to \mathrm{End}(V)$ τ-invariant irreducible, extends to representation of $G^*(\mathbb{R})$.

Results at the real Places

π representation of $G^*(\mathbb{R})$

Definition: $L(\tau,\pi) = \sum(-1)^{-1} tr\tau^i H^i(\mathfrak{g}, K, \pi)$

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A function $f_F \in C_c^\infty(G(\mathbb{R})\tau)$ called a Lefschetz function for F,τ if

a.) $f_F(kgk^{-1}) = f_F(g)$,

b.) tr$\pi(f) = L(\tau,\pi\otimes F)$ for all representations π.

c.) $f_F^P = 0$ for P a real parabolic whose conjugacy class is stable under τ

Theorem 5.

Let f_F be the Lefschetz function for F, τ. Suppose that $\gamma \in G(\mathbb{R})\tau$ is an elliptic element. Then

$$O_\gamma(f_F) := \int_{G(\mathbb{R})/G(\mathbb{R})(\gamma)} f_F(g\gamma g^{-1})\ dg = (-1)^{q(\gamma)} e(\tau) tr\ F^*(\gamma)$$

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Theorem 6.
Let f_F be the Lefschetz function for F, τ. Suppose that $\gamma \in G(\mathbb{R})\tau$ has a nontrivial hyperbolic part. Then

$$|det(I - Ad\gamma)|^{1/2}_{\mathfrak{g}/\mathfrak{g}(\gamma)} O_\gamma(f_F) = 0$$

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Theorem 7. *(Borel Labesse Schwermer Kottwitz) Assume the derived group of G is simple. The only irreducible unitary representations for which tr $\pi(fL) \neq 0$, are one dimensional or the Steinberg representations.*

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Theorem 7. *(Borel Labesse Schwermer Kottwitz) Assume the derived group of G is simple. The only irreducible unitary representations for which tr $\pi(fL) \neq 0$, are one dimensional or the Steinberg representations.*

Theorem 8. *The orbital integrals of f_L are*

$$O_\gamma(f_L) = \begin{cases} 1 & \text{if } \gamma \text{ is elliptic,} \\ 0 & \text{otherwise.} \end{cases}$$

Global Setup:

Define

$$f_{\mathbb{A}} = \prod_{\nu} f_{\nu}$$

so that

- At ∞ place f_ν Lefschetz function f_τ.
- At 2 finite places f_ν Lefschetz function f_L.
- All other places characteristic function of open compact subgroup K_ν.

By inserting $f_{\mathbb{A}}$ into the twisted trace formula we prove the theorem.

At several places the subgroup K_ν has be chosen very carefully and may have to be smaller to ensure that we get a positive contribution on the geometric side of the trace formula.