NOTIONS OF DIMENSION

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A quick overview of some basic notions of dimension for a summer REU program run at UConn in 2010 with a view towards using dimension as a tool in attempting to define fractals.

1. Topological Dimension

An inductive definition given in [3]. This is in the context of a separable metric space X.

Definition 1. *A set is of dimension zero if for any point $p \in X$ if there are arbitrarily small neighborhoods of p whose boundary is empty. A set is of dimension n if there are arbitrarily small neighborhoods of any point p whose boundary is of dimension $\leq n-1$.*

Notice that this definition implicitly defines the dimension of the empty set as zero.

Example 1. *The set of rational numbers in $\mathbb{R}$ is of dimension zero. As are the irrational numbers and in fact any totally disconnected set.*

Under this definition we have that $\mathbb{R}$ is the union of two dimension zero subsets yet it has dimension one itself. This is a source of dissatisfaction with this definition of dimension. This next definition is given in [4] and also depends only on the topological structure of the space X. This notion was originally Lebesgue's "covering dimension."

Definition 2. *A collection $\mathcal{A}$ of subsets of X is of order $m+1$ if some point of X lies in $m+1$ elements of $\mathcal{A}$, and no point of Z lies in more than $m+1$ elements of $\mathcal{A}$.*

Definition 3. *A space X is said to be finite dimensional if there is some integer m such that for every open covering $\mathcal{A}$ of x, there is an open covering $\mathcal{B}$ that refines[1] $\mathcal{A}$ and has order at most $m+1$. The topological dimension is defined as the smallest value of m for which this statement holds. We denote it by $dim_T(X)$.*

Using this definition there is a much more satisfactory relationship between the dimension of a space and the dimension of it's constituent pieces.

Theorem 1. *Let $X = Y \cup Z$, where Y, Z are close subspaces of X having finite topological dimension. Then*

$$dim_T(X) = \max\{dim_T(Y), dim_T(Z)\}. \tag{1.1}$$

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[1]Refinement simply means that any element of $\mathcal{B}$ lies inside of an element of $\mathcal{A}$.

While the “covering dimension” is more satisfactory than the inductive topological dimension it still has a few failings that we encourage us to look for something more sophisticated. First, it does not reflect the amount of fine detail present in X. This is because dim_T takes values in the natural numbers. Second, given some cover $\mathcal{A}$ of X the search through all possible refinements $\mathcal{B}$ of $\mathcal{A}$ is a daunting and infinite task. It will actually turn out that this kind of problem never leaves us, it can be mitigated later on by asking for only upper bounds on dimensions and not exact values.

2. Similarity Dimension

This notion of dimension is given first for sets that are strictly self-similar. Even relaxing the condition that the contractions mappings be affine ruins the proofs and invalidates the general results. It is possible to consider some particularly nice self-affine sets though. Recall that the difference between affine and similar is the uneven scaling in different directions, this is not the same as the different similarities having different scaling ratios. See Section 9.4 of [1]. So we will assume that are considering self similar sets produced from similarities.

Definition 4. *A self-similar set satisfies the open set condition if there exists a non-empty open set V such that*

$$V \supset \bigcup_{i=1}^{N} S_i(V) \tag{2.1}$$

with the union being disjoint.

Definition 5. *Suppose that F is a strictly self-similar set such that $F = \bigcup_{i=1}^{N} S_i(F)$ where the S_i are similarities and satisfies the open set condition and the contraction ratios of the similarities are r_i then the similarity dimension of F is the solution to the equation*

$$\sum_{i=1}^{N} r_i^s = 1. \tag{2.2}$$

In the case where all of the similarities have equal contraction ratios there is a simpler formula:

$$dim_S(F) = -\frac{\log(N)}{\log(r)}. \tag{2.3}$$

Similarity dimension has the peculiar distinction of being easy to calculate. This is also what many, but not all, people will call fractal dimension. Although that is often mistaken for a non-integer value of the Hausdorff dimension. One serious disadvantage though is that the realm of applicability is rather limited. For topological dimension we needed only a topological space, here we need a space with two very special properties. This is often good enough if we have nice enough fractals to work with which is why you’ve probably seen this notion of dimension before.

3. Minkowski Dimension

Also called “box-counting” dimension is based on a method of measuring the size of sets if one already knew the correct exponent (for example 2 in the plane). Assume that F is a non-empty subset of $\mathbb{R}^n$.

Definition 6. *Let $N_\delta(F)$ be the smallest number of diameter precisely δ sets it takes to cover F. The let*

$$dim^+_M(F) = \limsup_{\delta\to 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \tag{3.1}$$

be the upper Minkowski dimension of F. Similarly

$$dim^+_M(F) = \liminf_{\delta\to 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \tag{3.2}$$

be the lower Minkowski dimension of F. If both the upper and lower Minkowski dimensions are equal then their common value is the Minkowski dimension of F.

In fact the use of sets of diameter precisely δ can be altered to be boxes of side length δ, closed sets of diameter exactly δ, largest number of disjoint balls of radius δ with centers inside F. So in practice one gets to choose the precise definition of $N_\delta(F)$ in accordance with the situation.

Minkowski dimension has some noteworthy properties. If $E \subset F$ then the dimension of F is larger than or equal to that of E. The upper Minkowski dimension has the stability property as in Theorem 1 but lower Minkowski dimension doesn't.

Proposition 1. *Both F and it's closure have the same upper and lower Minkowski dimension.*

Example 2. *Let $\mathbb{Q}$ be the rational numbers. Which then have the same Minkowski dimension as it's closure, with is all of $\mathbb{R}$. Recall that the topological dimension of the rational numbers of zero.*

In case you thought things would be simpler with closed sets...

Example 3. *Let $F = \{0, 1, 1/2, 1/3, \ldots\}$. Then $dim_M(F) = \frac{1}{2}$. See [1] page 45. The idea is to take $\delta \in \left[\frac{1}{k(k+1)}, \frac{1}{k(k-1)}\right)$. This set has topological dimension zero as well.*

To get an upper estimate for $dim_M(F)$ let $\delta < \frac{1}{2}$. Then there exists a k such that $\delta \in \left[\frac{1}{k(k+1)}, \frac{1}{k(k-1)}\right)$. Using intervals of length δ it takes $k+1$ of these intervals to the interval $[0.k^{-1}]$. This leaves $k-1$ points all separated by a distance of at least δ so it takes another $k-1$ intervals of length δ to cover F. So $N_\delta(F) \le 2k$. The limit is then

$$dim^+_M(F) = \limsup_{\delta\to 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \le \limsup_{k\to\infty} \frac{\log(2k)}{\log(k(k-1))} = \frac{1}{2}. \tag{3.3}$$

The lower bound on $dim^-_M(F)$is shown along similar lines by the argument that k intervals of length δ will never be sufficient to cover all of F because of the number of widely separated points in the set. Thus $N_\delta(F) > k$ and

$$dim^-_M(F) = \liminf_{\delta\to 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \le \liminf_{k\to\infty} \frac{\log(k)}{\log(k(k+1))} = \frac{1}{2}. \tag{3.4}$$

From these two estimates it is clear that $dim_M(F) = \frac{1}{2}$.

Despite these worries Minkowski dimension is still a useful tool. In many cases (which will be touched on later) it is possible to show from general principles that the Minkowski dimension and the Hausdorff dimension are the same where the Minkowski dimension is much simpler to calculate.

4. Hausdorff Measure and Dimension

The notion of Hausdorff dimension is applicable to any set in any Euclidean space and can even be extended to subsets of any metric space as well. This feature is particularly useful in the study of fractals where the natural metric on a fractal does not interact well with the normal Euclidean metric.

4.1. **Hausdorff Measure.** A measure, μ, is a function from some set of sets, $\mathcal{A}$ to $[0, \infty]$ with the following nice properties:

(1) $\mu(\emptyset) = 0$ and
(2) if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ then $\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j)$.

There is a comprehensive and important theory of measures for which there are many introductory texts such as [2]. What is important to know at present about measures is that they provide a way of giving a "size" to sets in a self-consistent manner but not always for every possible set and not always a non-trivial value.

Let $\delta-$cover of a set F be a collection of sets $\mathcal{U} = \{U_i\}$ each of whose diameter, denoted $|U_i|$, is less than δ. Notice that this is different than the situation for Minkowski dimension where the diameters had to be precisely δ.

Definition 7. *The $s-$dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ is given by*

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) = \lim_{\delta\to 0} \inf_{\mathcal{U}} \left\{ \sum_{i=1}^\infty |U_i|^s : \bigcup U_i \supset F \right\}. \tag{4.1}$$

For any set in $\mathbb{R}^n$ this will have a limit although the limit will generally be either zero or infinity.

Proposition 2. *If F is a subset of $\mathbb{R}^n$ and λF is all of the points of F multiplied by λ then $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$.*

Proof. (Sketch) If you have a $\delta-$cover of F then it gives you a $\lambda\delta-$cover of λF so there is a correspondence between covers from which one can conclude that the limits are the same. □

4.2. **Hausdorff Dimension.** Given a set $F \subset \mathbb{R}^n$ then for most choices of s you have something strange happening. For small s and any set F you get that $\mathcal{H}^s(F) = \infty$ and then for much larger s you get $\mathcal{H}^s(F) = 0$.

Proposition 3. *Let F be a subset of $\mathbb{R}^n$. Then for $t > s$*

$$\mathcal{H}^s(F) \geq \mathcal{H}^t(F). \tag{4.2}$$

Proof. This follows from Proposition 2 by setting up the inequality

$$\sum_{j=1}^\infty |U_i|^t \leq \delta^{t-s} \sum_{i=1}^\infty |U_i|^s. \tag{4.3}$$

□

This tells us that $\mathcal{H}^s(F)$ is a non-increasing function of s so once the switch from a value of infinity to zero occurs the Hausdorff measure stays zero.

Definition 8. *Let*

$$dim_H(F) = \sup_s \{s : \mathcal{H}^s(F) = \infty\} = \inf_s \{s : \mathcal{H}^s(F) = 0\} \tag{4.4}$$

be the Hausdorff dimension of F.

It is possible for the dim_H−dimensional Hausdorff measure of a set F to be zero, positive, or infinite and is typically difficult to calculate. In the cases where F is a n−dimensional manifold the n−dimensional Hausdorff measure is a constant multiple of Lebesgue measure.

Proposition 4. *A set $F \subset \mathbb{R}^n$ with $dim_h F < 1$ is totally disconnected.*

Example 4. *Let F be the middle third Cantor set. If $s = \log(2)/\log(3)$ then $dim_H F = s$ and $\frac{1}{2} \leq \mathcal{H}^s(F) \leq 1$. In fact, $\mathcal{H}^s(F) = 1$ but that calculation is harder.*

Cover the sets E_k by 2^k intervals of length 3^{-k}. This gives $\mathcal{H}^s_{3^{-k}}(F) = 2^k 3^{-sk}$ which is equal to one if $s = \log(2)/\log(3)$. Letting k go to infinity gives an upper bound on $\mathcal{H}^s(F) \leq 1$.

The lower bound is more difficult. It is enough to assume that the U_i that cover F since we only need a bound. By expanding the U_i slightly and using the compactness of F we can even assume that there are only finitely many U_i. For each U_i let k be such that $3^{-(k+1)} \leq |U_i| \leq 3^{-k}$. Each U_i can only intersect one of the intervals in E_k since they are separated by a distance of at least 3^{-k}. If $j > k$ then by construction U_i intersects as most

$$2^{j-k} = 2^j 3^{-sk} \leq 2^j 3^s |U_i|^s \tag{4.5}$$

of the basic intervals of E_j. Take j large enough so that $3^{-j-1} \leq |U_i|$ we get that $2^j \leq \sum 2^j 3^s |U_i|$ since the collection $\{U_i\}$ intersect all 2^j intervals with length 3^{-j} of E_j. But this implies that

$$\frac{1}{2} \leq \sum_{i=1}^{M} |U_i|^s \tag{4.6}$$

thus $\mathcal{H}^s(F) \geq \frac{1}{2}$. Recall that $\mathcal{H}^s(F)$ can take a value in $(0, \infty)$ when $s = dim_H(F)$.

5. Relations Between the Notions

Theorem 2. *For any subset of $\mathbb{R}^n$*

$$dim_H(F) \leq dim^-_M(F) \leq dim^+_M(F). \tag{5.1}$$

Theorem 3. *Let F be as described in Definition 5 then*

$$dim_S(F) = dim_M(F) = dim_H(F). \tag{5.2}$$

Moreover $0 < \mathcal{H}^s < \infty$ for $s = dim_H(F)$.

References

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