Research Statement

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September 25, 2012

The classical Laplacian ($\Delta = -\frac{d^2}{dx^2}$ on $\mathbb{R}$) is a very well studied object. It is a natural object in physics coming from the heat and wave equations. In probability theory it arises as the generator of Brownian motion. The fundamental solution of $\Delta u = u$ is the Gaussian density e^{-x^2} that is the ubiquitous bell curve. On nice fractals like the standard Sierpinski gasket which can be defined by graph approximations, like approximating $\mathbb{R}^2$ by $2^{-n}\mathbb{Z}^2$, a Laplacian can be defined by finite difference quotients. Because the Sierpinski gasket does not have neat set of orthogonal directions the first difference quotient is more complicated than the second difference quotient which compares the value of a function at one point to all of its neighbors so there is no need to pick a particular direction.

For many classes of self-similar fractals the existence of a Laplacian, and equivalently a diffusion, are well known. So are bounds on the fundamental solution to the heat equation which are sub-Gaussian, see Eq. **??**. This is an indication that diffusion on these fractals moves slower than diffusion on a manifold. Similarly the eigenvalues of the Laplacian on these fractals grow much faster than on a manifold. Physicists [ADT, EFRS] have taken advantage of these differences in exploring statistical mechanics and quantum mechanics, for example [ADT] explores the physical consequences of the growth of the eigenvalue counting function on the diamond fractal using the heat kernel estimates from [HK]. The classes of fractals for which heat kernel bounds are known do not include all fractals.

One broad class of fractals for which heat kernel bounds are only partially known are hierarchical fractals. These are fractals which have a geometry that can be described in layers and the different layers do not have to be similar to each other. A good example is a non-self-similar Sierpinski carpet where the size of the holes introduced at each step of the construction do not have any necessary relationship while in the self-similar Sierpinski carpet the size of the holes decreases geometrically.

The central theme of my research is to extend the spectral theory on "nice" self-similar fractals to fractals with hierarchical but not self-similar geometries. There are three representative model spaces that I usually work with, Laakso spaces, non-self-similar Sierpinski carpets, and hexacarpets. The construction of these spaces are described below. The initial questions that I am exploring on these spaces are:

1. The existence of Laplacians (equivalently diffusion processes and Dirichlet forms),
2. The properties of harmonic functions, the properties of the spectra of these Laplacians, and
3. The possible estimates for the heat kernels of the Laplacians.

In the case of Laakso spaces the exact spectrum of the Laplacian can be written down, including multiplicities. So I have been able to study an analog to the Casimir effect in a fractal geometry (the existence of a repulsive force between two large, perfectly conducting plates held very close to each

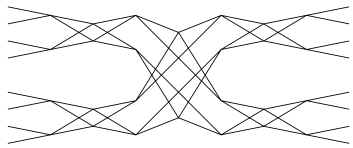

Figure 1: The third approximating graph to a Laakso space with $j_i = 2$.

other). The results showed that the strength of the effect did not depended on the growth rate of the eigenvalue counting function of the Laplacian. In classical other fractal cases, such as the diamond fractal and Sierpinski carpet, it does so this was a genuinely new phenomenon. Both the non-self-similar Sierpinski carpet and the hexacarpet are less well developed. Even the most basic existence question for a Laplacian is unanswered for these fractals. However, there is numerical data that suggests that Laplacian does exist.

For self-similar fractals the existence of a log-periodic oscillatory main term in the eigenvalue counting function's asymptotics has been known for several years, [FS, KL]. The proofs for the existence of the log-periodic function have relied crucially on the self-similarity of the fractal and the Laplacian on it. Is there an analogous result for hierarchical but not self-similar fractals such as Laakso spaces and non-self-similar Sierpinski gaskets? The notion of spectral segmentation was first discussed in [DS] where instead of a log-periodic function there is a segmentation of $[0, \infty)$ where each segment is growing multiplicatively and the $n'th$ segment is representative of the $n'th$ level of the hierarchical geometry. Laakso spaces provide a perfect test case for how spectral segmentation can work since the calculations can be performed exactly, even in the non-self-similar case because of the formulae for the spectrum and multiplicities.

I have begun to think about percolation on infinite fractal graphs in the last year. The model I have been working with is bond percolation where given a graph $G = (V, E)$ with some vertex and edge set, the edges are randomly chosen to be open with probability p and closed with probability $1 - p$. The first question is what is the threshold that p must be above for the random subgraph consisting of open edges and the original vertex set to have a connected component taking up a positive proportion of the graph? The fractal graphs that I have so far considered are a modified Sierpinski gasket where at each vertex point unboundedly many cells meet, the hexacarpet, and the infinitely iterated barycentric subdivision of a triangle.

Model Spaces

Laakso Spaces

Laakso spaces were originally defined as the product of $[0, 1]$ and a Cantor set K, modulo a hierarchical equivalence realtion [L]. The resulting space is a geodesic metric space that is not bi-Lipschitz embeddable into any finite dimensional Euclidean space. Following Barlow and Evans [BE], Laakso spaces can be recast as projective limits of metric graphs, each of which encodes an additional level of the equivalence relation defining the Laakso space. The data that is necessary to construct a Laakso space is only a single sequence of integers drawn from a finite set and a Cantor set. The metric and measure properties of Laakso spaces are extracted from this sequence. For example if the constant sequence $3, 3, \ldots$ is chosen the Hausdorff dimension is $1 + \frac{\log(2)}{\log(3)}$.

The main reason why Laakso spaces serve as useful model spaces for my work is that by the choice of a particular Cantor set and sequence of integers any Hausdorff dimension greater than one can be obtained. A chosen Hausdorff dimension is attained while at the same time the Laplacian remains essentially one-dimensional, that is it acts as a negative second derivative in only one "direction."

Theorem 1. *[RS] For any Laakso space L there exists a Laplacian with a discrete spectrum that is given explicitly by the data used in the construction of the Laakso space.*

It is this explicit description that makes Laasko spaces practical places to work out spectral behavior in a non-self-similar setting. This information is also useful in formulating physically inspired calculations such as an analog to the Casimir effect which is discussed below.

In a recent paper I have also shown that under a local symmetry condition that Brownian motion on Laakso spaces is unique up to constant time scaling [S2]. The local symmetry condition states that when restricted to a small domain that has a symmetry that the laws of the process and the process transformed by the symmetry are equal. This follows the work of Barlow, Bass, Kumagai, and Teplyaev [BBKT] on self-similar Sierpinski carpets and serves to illuminate which pieces of their argument depend only on the hierarchical geometry and not the self-similarity.

Non-self-similar Sierpinski Carpets

The existence of, and heat kernel estimates for Brownian motion on generalized, but still self-similar, Sierpinski carpets is well established [BB]. For some random Sierpinski carpet constructions where at each stage of the construction the next set of contraction maps is chosen from a finite set of possibilities existence and heat kernel estimates are known [HKKZ]. It is also known that Brownian motion which respects the local symmetries of deterministic carpets is unique up to deterministic time change [BBKT]. Mackay, Tyson, and Wildrick [MTW] have demonstrated a construction of Sierpinski-like carpets which produces positive 2-dimensional measure and possess naturally defined Laplacians but which are not self-similar.

In the construction of the standard Sierpinski carpet one takes a square and subdivides it into $9 = 3^2$ sub-squares, removing the central square. This is then repeated in each of the $8 = 3^2 - 1$ remaining sub-squares. Mackay, Tyson, and Wildrick [MTW] propose that instead one subdivides into a_1^2 sub-squares and removes the central one. Then with the $a_1^2 - 1$ remaining squares subdivide each into a_2^2 sub-squares, again removing the central sub-square from each. This produces a fractal with a hierarchical geometry which is only self-similar if the sequence $\{a_i\}$ is periodic. They show that $\{a_i^{-1}\} \in \ell^2$ is equivalent to the existence of a Laplacian with Gaussian heat kernel bounds. The standard carpet is included in this construction by taking $a_i = 3$. Similarly the randomly generated carpets in [HKKZ] have a_i independent and identically distributed as random variables from a finite set. There are many choices for $\{a_i\}$ that are neither chosen from a finite set or ℓ^2 for which even the existence of a Laplacian is not known.

Theorem 2. *(in preparation) There exists a locally symmetric Brownian motion on a non-self-similar Sierpinski carpet if*

$$\lim_{i \to \infty} a_i^{-1} = 0. \tag{1}$$

Most of the proof of existence proof follows the proof for self-similar carpets except for the analysis of crossing time estimates which require a more careful approach. The available analysis of crossing times is not detailed enough to yield any heat kernel estimates for when $\{a_i^{-1}\} \notin \ell^2$.

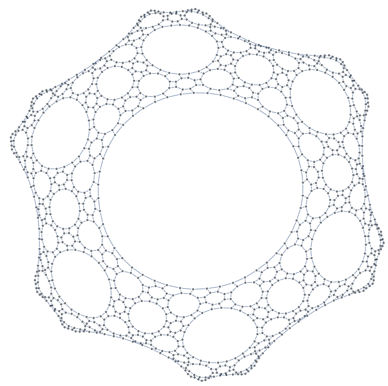

Figure 2: Fourth level approximation to the hexacarpet. Permission kindly granted by the authors of [BKNPPT]

Question 1. *The decay of the sequence a_i^{-1} is primarily used to characterize how much area is left after each step of the construction. If, instead of removing the central square at each step, one removes some pattern of m_i squares, possibly different at each level. Does there exist a Brownian motion? For which patterns of removed squares?*

There are some results in this direction. Let m_i be the number of squares retained at each step of the construction. Then $\{m_i/a_i\} \in \ell^2$ implies the existence of a Laplacian [MTW]. If a_i and the patterns of removed squares are periodic then there exists a Laplacian. I have shown that if the pattern of removed squares is always "near the center" that the usual constructions of Brownian motion on Sierpinski carpets can be extended to this case. This is still a significant assumption on the geometry of the non-self-similar carpet compared to what the general construction can produce.

Hexacarpets

Hexacarpets appeared quite naturally in the percolation model described at the end of this statement. To construct the hexacarpet let T be a triangle and $B(T)$ the barycentric subdivision of T. Then $B(T)$ is a CW-complex consisting of six triangles, each of which can be barycentrically subdivided to produce $B^2(T)$. Iterating we have $B^n(T)$. Define H^n to be the planar dual graph of $B^n(T)$, that is for each triangular face of the graph $B^n(T)$ put a vertex of H^n and two vertices are connected by an edge if the faces they are associated to share a common edge in $B^n(T)$. Barycentric subdivision of triangles is an important idea in many fields, such as in [DM] and references therein.

The graphs H^n have two natural scalings. The outer circumference grows like $n2^n$ while the inner circumference around the barycenter of the original triangle grows like 2^n. When a scaling limit is taken, there is a choice. Choosing to keep the outer circumference bounded forces the inner circumference to shrink to a single point in the limit. On the other hand a scaling limit can also be taken keeping the circumference around the inner barycenter fixed. This results in an unbounded hexacarpet. Both of these limits will be referred to as "hexacarpets." Having two scaling limits raises the question:

Question 2. *In either of the scaling limits does there exist a locally symmetric Brownian motion (Laplacian, Dirichlet form equivalently)? If there is Brownian motion on both limits do they have similar properties such as heat kernel estimates?*

Hexacarpets have recently appeared in the work of [BKNPPT] where basic geometric properties and numerical estimates of the graph Laplacians on H^n were explored. In the case of the compact hexacarpet the obstacles are a dense set of points with infinite negative curvature rather than components of the exterior. The result is that H is topologically homeomorphic to a planar region but is not isometric to a planar region. Thus the obstacles are geometric and not topological. To compensate for this lack of smooth structure self-similarity is used instead to define a metric and a reference measure. Because of these differences proving that there is a locally symmetric Brownian motion on the compact hexacarpet will require different techniques than those used for the Sierpinski carpet.

Spectra and Heat Kernels

Suppose that Δ is a self-adjoint non-negative operator with discrete spectrum, $\sigma(\Delta) = \{\lambda_n\}_{n=0}^{\infty}$ where the eigenvalues are repeated according to their multiplicity. Suppose that Δ has a (heat) kernel $p_\Delta(t, x, y)$. First define two functions directly from the spectrum of Δ :

$$\begin{aligned} \zeta_\Delta(s, \gamma) &= \sum_{n=0}^{\infty} (\lambda_n + \gamma)^{-s/2} \\ N_\Delta(t) &= \#\{\lambda_n \in \sigma(\Delta) |\ \lambda_n \leq t\}. \end{aligned}$$

These are the spectral zeta function and the eigenvalue counting function of Δ respectively. The Mellin transform of the spectral zeta function yields the trace of the heat kernel

$$Z_\Delta(t) = \int_p^\infty p_\Delta(t, x, x)dx = \sum_{n=0}^{\infty} e^{t\lambda_n}.$$

The four functions, the heat kernel, its trace, the eigenvalue counting function, and the spectral zeta function are the basis for the work presented in this section.

For many fractals the existence of the heat kernel as a continuous function is known, such as for the Sierpinski gasket, Sierpinski carpet, the diamond fractal and many others. These fractals generically have Laplacians with heat kernels that are sub-Gaussian. That is

$$p_\Delta(t, x, y) \sim t^{-\alpha/2} e^{\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}}. \tag{2}$$

For Laakso spaces the heat kernel exists and actually has Gaussian heat kernel bounds ($\beta = 2$). However, for non-self-similar Sierpinski carpets and for the hexacarpets what the bounds are is still an open question. The existence proof for a Laplacian on non-self-similar Sierpinski carpets suggests, but does not prove, that the heat kernel bounds lie somewhere between sub-Gaussian and Gaussian. Whether this means different β's for upper and lower estimates or some other correction term is not yet clear.

The connection between the spectral zeta function and its Mellin transform, the trace of the heat kernel, allows asymptotic data about one to be transferred to the other. For example on Laakso spaces, where the spectral zeta function can be computed exactly so can the trace of the heat kernel. Kigami and Lapidus [KL] showed that for a class of fractals that the trace of the heat kernel has a power function asymptotic formula and that in the formula below that G_1 is positive and periodic

$$Z_\Delta(t) \sim c_1 t^{-d_1}(c_2 + c_3 G_1(c_4 \log(t))) + c_4 t^{-d_2}(c_5 + c_6 G_2(c_7 \log(t))) + R(t). \tag{3}$$

In [ST1] A. Teplyaev and I have shown that the periodic functions G_1 and G_2 actually enlarge the domain on which $\zeta_\Delta(s,\gamma)$ converges. If $R(t)$ has exponential decay as $t \to 0$ then $\zeta_\Delta(s,\gamma)$ can be extended meromorphically to the entire complex plane. For Sierprinski carpets, N. Kajino [K] has shown that the trace of the heat kernel has such a power function decomposition and that $R(t)$ does have exponential decay. The state of the art for Sierpinski gaskets is that G_1 is positive, while G_2 is negative. On self-similar Laakso spaces it is in fact possible to calculate exactly the spectral zeta function as a meromorphic function on the complex plane and from this to obtain the spectral asymptotics that show the this kind of oscillatory behavior directly and give concrete formula for G_i.

Question 3. *What is the asymptotic behavior for $Z_\Delta(t)$ when the underlying fractal is not self-similar? Is there are replacement for the log-periodic term that may be decomposable into segments that each represent a particular level in the hierarchical geometry?*

This notion of "spectral segmentation" was introduced by Robert Strichartz in [DS] as an extension of the log-periodicity that had been seen in the numerical data for self-similar fractals produced by his students. In the 2012 REU at Cornell University a student produced a large volume of data that indicates that some variety of spectral segmentation occurs. Laakso spaces, with their hierarchical geometry and explicitly known spectrum, are an ideal model for exploring spectral segmentation. The most likely form for the segmentation on Laakso spaces appears to be the following decomposition.

Theorem 3. *(in preparation) For a Laakso spaces with sequence $\{a_i\}$ let $N_\Delta(t)$ be the eigenvalue counting function for the Laplacian Δ. Then*

$$N_\Delta(t) = \sum_{i=0}^{\infty} C_i N_{a_i}(\pi^{-2} d_i^{-2} t).$$

Where $d_i = \prod_{j=0}^{i} a_j$ and $N_{a_i}(t)$ are copies of the eigenvalue counting function for the standard Laplacian over the unit interval.

In this theorem it is important to note that for a given t the sum is over finitely many terms and each will have a characteristic log-periodic contribution to $N_\Delta(t)$. The long time asymptotics of N_Δ are the same as the short time asymptotics of Z_Δ so having this result implies that both functions have a segmented structure.

Question 4. *For what non-self-similar fractals does a locally symmetric Laplacian have an eigenvalue counting function that can be decomposed into a sum as in the previous theorem?*

On some Laakso spaces the spectral zeta function has a closed form formula, this makes it possible to emulate calculations for physically "inspired" situations. For example, with a group of REU students I was able to calculate the strength of an analog to the Casimir effect. The Casimir effect is the existence of a repulsive force between two large, perfectly conducting plates held very close to each other. The effect has been experimentally verified in the laboratory in $\mathbb{R}^3$. In a Euclidean setting the plates can be thought to be moving through space or as deforming the space around them. In a fractal setting we must pin the plates to a specific location in the geometry and view the fractal space as deforming. J. Chen [C] has recently used the oscillating terms in the asymptotics of the trace of the heat kernel on standard Sierpinski carpets to analyze the Casimir effect. To reproduce these results on other, not necessarily self-similar, fractals it will be necessary to show how the spectrum depends on the deformation induced by the plates. As a model for the physics community Laakso spaces provide another interesting feature, namely that even their finite level approximations exhibit a non-Euclidean behavior i.e. the location of the pole of the spectral zeta function. The exciting consequence of this is that the influence of fractal geometry on physically realistic configurations can be observable with only a finitely many layers of self-similarity.

Theorem 4. *[KS] The spectral zeta function on F_n, a finite level approximation to a self-similar Laakso space, has non-trivial poles off of the real axis which contribute to the strength of the Casimir effect on F_n.*

Percolation

Let $G = (V, E)$ be a graph with countably many edges, choose independently for each edge for it to be open with probability p or closed with probability $1 - p$. Let $G_p = (V, E_p)$. The configuration space,$\Omega = \{0, 1\}^{|E|}$, is a probability space with the product measure denoted $\mathbb{P}_p$. A cluster is a connected collection of open edges. The event of there existing an infinite cluster is a tail event and so has either $\mathbb{P}_p$ probability zero or one depending only on p. Let $G = \mathbb{Z}^d$ and define

$$\theta(p) = \mathbb{E}_p(\text{ there exists a path of open edges from the origin to “infinity” }).$$

Obviously where there is no infinite cluster $\theta(p) = 0$ but when there is an infinite cluster it is possible for it not to contain the origin so $\theta(p)$ may only be positive. As a function $\theta(p)$ is always right continuous and non-decreasing [G]. The critical value p_c is defined as $p_c = \sup\{p : \theta(p) = 0\}$. Studying the properties of G_p for $p < p_c$, $p = p_c$ and $p > p_c$ gives rise to models of random environments.

For graphs coming from approximations to fractals there has been good progress in the case where these graphs are embeddable into $\mathbb{Z}^d$ for some d. These include Sierpinski gaskets (where $p_c = 1$) and Sierpinski carpets (where $0 < p_c < 1$). With a student I have been looking at bond percolation on graph approximations to finitely ramified but non-post critically finite fractals. These are precisely the fractals which can be broken in to disconnected components by removing a finite number of points but for which the graph approximations at those points have unbounded degree. Hambly and Kumagai [HK] began this line of inquiry with the “diamond” fractal which has also been of interest recently in the physics literature [ADT].

I have been considering a non-post critically finite (non-p.c.f.) variation of the Sierpinski gasket [B et al]. Non-post critically finite means that the vertex degrees in the approximating graphs are unbounded. With a student I have shown that $0 < p_c < 0.282$. Interestingly, in the proof it became clear that non-self-similar but still finitely ramified fractals have dual graphs that also appear in natural percolation problems. These carpet-like duals are useful because upper bounds on their p_c give lower bounds on p_c for the original graphs and upper bounds are much easier to prove.

Theorem 5. *(in preparation) For the almost-dual pair of fractals, the non-p.c.f. Sierpinski gasket and the hexacarpet there is a non-trivial phase transition.*

This theorem suggests a line of inquiry trying to extend this result to other fractals.

Question 5. *Is there a class of non-p.c.f. but finitely ramified fractals that can are dual to carpet-like fractals so that the existence of a phase transition in the percolation process can be transferred between the carpet-like fractals and the non-p.c.f. fractals?*

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