Noncollision Singularities in the $n$-Body Problem

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The $n$-Body Problem

Introduction  Suppose we have $n$ points in $\mathbb{R}^3$ interacting through the gravitational force. We want to find the position and the velocity of each point over time.

For the $i$th point, let $m_i$ be its mass, $q_i$ its position, and $\dot{q}_i$ its velocity ($m_i \in \mathbb{R}^+, q_i \in \mathbb{R}^3, \dot{q}_i \in \mathbb{R}^3$). Newton’s law tells us that

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$$

(1)

where $U$ is potential energy. Thus we have

$$m\ddot{q} = \frac{\partial U}{\partial q}$$

and we arrive at the following system of $6n$ first-order differential equations:

$$\dot{q} = v, \quad \dot{v} = \frac{\partial U}{\partial q} m^{-1}$$

(2)

This is the setup for the $n$-body problem given in [1]. It is a dynamical system composed of a phase space $\mathbb{R}^{6n}$ together with the solution $q$ (which is a function of time $t \in \mathbb{R}$), so it is called a flow.
History  The problem of predicting motions of celestial bodies dates back to the ancient Greeks, although the $n$-body problem as we know it comes from Newton’s work on forces and gravitation in the 17th century. In 1710, the 2-body problem was completely solved by Johann Bernoulli. As we have seen, for $n \geq 3$ matters become much more complicated. Except for special cases, the $n$-body problem remains to this day unsolved.

Singularities  The standard existence and uniqueness theorems of ordinary differential equations guarantees us a solution to the system (2) that is analytic in a certain neighborhood. We say that the solution experiences a singularity at time $\sigma < \infty$ if the solution cannot be extended analytically beyond $\sigma$.

Formally, we use the definition from [2] to classify singularities:

Definition. Let $\Delta = \bigcup_{i<j}^{n} \{ q = (q_1, q_2, \ldots, q_n) \mid q_i = q_j \}$, a subset of the phase space that contains all collisions. Suppose $q(t)$ has a singularity at time $\sigma$. Then if there is a $q \in \Delta$ such that $q(t) \to q$ as $t \to \sigma$, it is called a collision singularity. Otherwise, it is called a noncollision singularity.

To see this definition in action, consider $U$ given in (1). It is real analytic everywhere except when $q_i = q_j$ for some $j$ and $i$ (since $r_{ij} = 0$), whereupon it experiences a collision singularity.

Paul Painlevé proved that for $n = 3$, all singularities are collisions. This is a fundamental to Sundman’s theorem and the power series solution for the 3-body problem [1]. In 1895, Painlevé gave a conjecture during a lecture in Stockholm:

Proposition (Painlevé). For $n \geq 4$, there are solutions of (2) with noncollision singularities.

He did not actively explore the question after 1917, as he was the Prime Minister of France.

In 1908, little more than a decade after this conjecture, Edvard Hugo von Ziepel showed that a noncollision singularity occurs if the particles go off to infinity in finite time [2]. Recall the moment of inertia $I$ and the Lagrange–Jacobi equation:

$$I = \sum_{i=1}^{n} \frac{1}{2} m_i |q_i|^2 \quad \ddot{I} = U + 2h$$

Theorem (von Zeipel). If a singularity occurs at $\sigma$, then $\lim_{t \to \sigma} I = \infty$. If $\lim I < \infty$, then $\sigma$ is a collision singularity. Otherwise, $\sigma$ is a noncollision singularity and $\lim I = \infty$ as $t \to \sigma$. 

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In this paper, we will explore the answers to Painlevé’s conjecture for $n = 4$ and 5. The strategy in both cases is to construct a system that becomes unbounded in finite time, and then using von Zeipel’s theorem to show the existence of a noncollision singularity.

The $n = 4$ Case

In 2003, Joseph Gerver developed a model for a special case of the 4-body problem, where all points are restricted to the plane in the following configuration [3]:

Let $Q_1$, $Q_2$, $Q_3$, $Q_4$ be the particles labeled above. Let $\mu$ be the gravitational constant, $\mu << 1$. Suppose $m_1 = \mu^{-1} = m_2$ and $m_3 = m_4 = 1$. $Q_3$ starts out in an elliptical orbit around $Q_2$, and suppose that both the energy and the angular momentum of the system are 0.

The main idea is this: $Q_4$ oscillates between $Q_1$ and the $Q_2$-$Q_3$ pair. It moves back and forth faster and faster, eventually going an infinite distance in finite time as $Q_1$ and $Q_2$ move apart (but much more slowly). According to Gerver’s model, we can find a set of initial conditions such that $Q_4$ and $Q_3$ have a near collision every time $Q_4$ “bounces” to that region. It turns out that this near collision can be modeled as an elastic collision, as shown in Figure 2.

After the elastic collision, $Q_4$ travels on a hyperbolic path towards $Q_1$. Then it comes back around to the $Q_2$-$Q_3$ system. Every time $Q_4$ gets close to $Q_2$ and $Q_3$, it alternates between extracting energy or angular momentum.
Fig. 2

**Extracting Angular Momentum**  Every other time $Q_4$ encounters the pair, angular momentum is transferred to $Q_3$.

To see why, suppose that $\epsilon_0$ is the eccentricity of $Q_3$’s orbit ($0 < \epsilon_0 < \frac{1}{2} \sqrt{2}$) going counter-clockwise, and let $\epsilon_1 = \sqrt{1 - \epsilon_0^2}$. This means that $\epsilon_1 > \epsilon_0$, a fact which we will use at the end of this section.

First, we can adjust the initial conditions so that the elastic collision described above can happen anywhere along the ellipse, so let us say it happens at $(X,Y)$, where

$$X = -\epsilon_0 \epsilon_1 \quad Y = \epsilon_0 + \epsilon_1$$

Then the old orbit of $Q_3$ is given by

$$\frac{x^2}{\epsilon_1^2} + (y - \epsilon_0)^2 = 1 \quad (3)$$

and the new orbit is

$$\frac{x^2}{\epsilon_0^2} + (y - \epsilon_1)^2 = 1 \quad (4)$$

This is found by letting $(v_x, v_y)$ be the velocity of $Q_4$ before the collision and $(u_x, u_y)$ the velocity afterwards. Along with the constraints that $X v_x + Y v_y < 0$ and $X u_x + Y u_y > 0$, we can uniquely determine $u_x, u_y, v_x,$ and $v_y$. (See Figure 2).

To check that angular momentum is indeed extracted, observe that (3) and (4) describe $Q_3$ in elliptical orbits around $Q_2$. Recall $h$, the specific
**relative angular momentum** of two masses in an elliptical orbit:

\[ h = b \sqrt{\frac{\mu M}{a}} \approx b \]

where \( M \) is the sum of the masses, \( a \) is the semi-major axis of the elliptic orbit, and \( b \) is the semi-minor axis. The approximation of \( h \) to \( b \) is justified by the fact that in this case \( M = m_3 + m_2 = 1 + \mu^{-1} \approx \mu^{-1} \), and \( a = 1 \).

Since \( \sqrt{\frac{\mu M}{a}} \) remains invariant between (3) and (4), the change in \( h \) depends solely on the change in \( b \). Indeed, \( b \) goes from \( \epsilon_1 \) to \( \epsilon_0 \), so the angular momentum of \( Q_3 \) beforehand is \( \epsilon_1 \), and afterward is \( -\epsilon_0 \) (the opposite sign indicates opposite spin.)

The **specific energy** \( \phi \) of an elliptic orbit is given by

\[-\frac{\mu M}{2a} \approx -\frac{1}{2}\]

which stays invariant during the elastic collision. Since total energy and angular momentum are conserved, we conclude that in this interaction \( Q_4 \) has extracted only angular momentum and no energy.

**Fig. 3**

**Extracting Energy** On turns when \( Q_4 \) is not extracting angular momentum, it is extracting energy. The collision happens at \((\bar{X}, \bar{Y})\), where

\[ \bar{X} = \epsilon_0^2 \quad \bar{Y} = 0 \]
Here, the old orbit of \( Q_3 \) is given by
\[
\frac{x^2}{\epsilon_0^2} + (y - \epsilon_1)^2 = 1 \tag{5}
\]
and the new orbit is
\[
\frac{\epsilon_2^2}{\epsilon_0^2} x^2 + \left(\frac{\epsilon_2^2}{\epsilon_0^2} y + \epsilon_0\right)^2 = 1 \tag{6}
\]

Once again, the above statements combined with the constraints that \( \bar{X}_v x + \bar{Y}_v y < 0 \) and \( \bar{X}_u x + \bar{Y}_u y > 0 \) uniquely determine the velocity of \( Q_3 \) before and after the collision. (See Figure 3).

As before, one can check that \( Q_4 \) extracts energy while angular momentum stays the same.

**Off to Infinity** During every energy extraction, the orbit of \( Q_3 \) shrinks by a factor of \( \frac{\epsilon_2^2}{\epsilon_0^2} \) while the energy of \( Q_4 \) grows by the same factor. Letting \( Q_1 \) and \( Q_2 \) separate very slowly (by a factor of \( 1 + O(\mu) \)), we see \( Q_4 \) bouncing back and forth faster and faster, so that the time between every 2 round trips decreases geometrically. Thus \( Q_4 \) makes infinitely many round trips in a finite time interval, during which the distance between \( Q_1 \) and \( Q_2 \) has gone to infinity. And so we have a noncollision singularity.

**The n = 5 Case**

While Gerver’s model only works for bodies confined to the plane, Zhihong Xia showed that there was a noncollision singularity in the general 5-body problem, using the configuration in Figure 4 [2]. His original proof spans 57 pages and gets quite technical, so here I will just give the main arguments.

The particles \( Q_1, Q_2, Q_3, Q_4, \) and \( Q_5 \) have masses \( m_1 = m_2, m_3, \) and \( m_4 = m_5. \) \( Q_1 \) and \( Q_2 \) are symmetric with respect to the \( z \) axis and are in a highly elliptical orbit going clockwise. \( Q_4 \) and \( Q_5 \) are similarly arranged, except counter-clockwise. \( Q_3 \) is restricted to the \( z \) axis and moves back and forth between the 2 binary systems. Because of the opposite spins, the angular momentum \( L \) of the system is 0 and it has some fixed total energy \( E. \)

There are 12 degrees of freedom in this system, giving us a flow where the phase space is a manifold of dimension 12. Since we know that \( L \) and \( E \) are integrals of motion, however, we can restrict ourselves to a 10-dimensional hypersurface \( \Omega \) inside the phase space.
Now consider a triple collision between $Q_3$ and either of the binary systems. We know that as 3 bodies approach a collision, they form either an equilateral triangle or a straight line. Figure 5 below demonstrates some of the configurations that we are interested in:

Let $\Sigma_1$ be the subset of $\Omega$ consisting of initial conditions so that the corresponding solution results in configuration (a). Likewise define $\Sigma_4$ but with configuration (b). Now we state a few theorems, all due to Xia:

**Theorem (1a).** There exist positive masses $m_1 = m_2, m_3, m_4 = m_5$ so that the following is true. For $x^* \in \Sigma_1$, let $t^*$ be the time of triple collision. Define $q_i(x^*, t)$ to be the solution beginning at $x^*$ evaluated at time $t$. Then exist $x^*$ such that while $q_1(x^*, t^*) = q_2(x^*, t^*) = q_3(x^*, t^*)$, we have $q_4(x^*, t^*) = q_5(x^*, t^*)$ as well.
In other words, there are initial conditions such that when 3 particles collide, the other 2 also do.

**Theorem (1b).** Given a 3-dimensional hypersurface $\Pi$ in $\Omega$ crossing $\Sigma_1$ at $x^*$, there is an uncountable set $\Lambda$ of points on $\Pi$ such that the following is true: for $x \in \Lambda$, there is some time $t_{\infty} (0 < t_{\infty} < \infty)$ so that the solution beginning at $x$ is defined for all $t$, $0 \leq t < t_{\infty} < \infty$. Also,

$$z_1(t) = z_2(t) \to \infty, z_4(t) = z_5(t) \to -\infty$$ as $t \to t_{\infty}$

What this means is that we can find initial conditions which result in $Q_1$ and $Q_2$ going to $\infty$ in the positive $z$ direction ("up" in Figure 4), and $Q_4$ and $Q_5$ similarly going to $-\infty$. Moreover, this happens in finite time.

The argument here is essentially the same as the one used by Gerver. A single particle bounces between 2 binaries, going faster after each bounce. Meanwhile, the binary systems $Q_1$-$Q_2$ and $Q_3$-$Q_4$ are pushed farther and farther apart. Eventually, $Q_3$ has made infinitely many round trips in a finite time interval, and the two binaries are separated by an infinite distance.

The difficulty, as in the last model, is showing that this action is actually possible, and then explaining exactly what goes on when particles approach each other very closely. For example, as the $Q_1$-$Q_2$ system gets more and more elliptic, $Q_1$ and $Q_2$ can get so close that the distance between them is infinitesimal. In that case, it is difficult to tell whether the binaries collide or not.

To prove the theorem, we introduce the idea of a wedge: a region in the phase space enclosed by curves. Xia shows that there is a wedge $W$ with the following property: in any trajectory starting from a point in $W$, we can make $Q_3$ travel from $Q_1$-$Q_2$ to $Q_4$-$Q_5$ as quickly as we like. Also, there are infinitely many initial points in $W$ so that their corresponding trajectories result in a triple collision between $Q_1$, $Q_2$, and $Q_3$ and a binary collision between $Q_4$ and $Q_5$.

**Theorem (2).** There exist $x^* \in \Sigma_1$, $\Pi \subset \Omega$ such that the following is true: there is an uncountable subset $\Lambda_0 \subset \Lambda$ so that for any $x \in \Lambda_0$, $q_1(t) \neq q_2(t)$, $q_4(t) \neq q_5(t)$ for all $t$ where $0 \leq t < t_{\infty} < \infty$.

So $\Lambda_0$ is a set of initial points such that their corresponding trajectories does not result in collisions.

To see why this is true, consider the previous solution where the binaries become unbounded in finite time, and all it $q^*(t)$. Let $w_{12} = |h_{12}c_{12}|c_{12}$, where $h_{12}$ is the energy of the $Q_1$-$Q_2$ system and $c_{12}$ is the angular momentum of
that system. Define $w_{45}$ similarly. Then for $q \in \Lambda$ as defined in Theorem 1, let $t_{\infty}$ be when the trajectory starting at $q$ ends. Then

$$w_{12}(q^*(t)) \to 0 \quad w_{45}(q^*(t)) \to 0 \quad \text{as } t \to t_{\infty}$$

In fact, as $t \to t_{\infty}$, the elliptic axes of the 2 binary systems also has a limit. If, in the limit, the 2 axes are neither parallel nor perpendicular, then for $t$ in a neighborhood of $t_{\infty}$, no binary collision can exist. Finally, to prove the theorem, one need only show that there is some uncountable subset $\Lambda_0 \in \Lambda$ such that for all $x \in \Lambda_0$, those elliptic axes are never parallel or perpendicular as $t \to t_{\infty}$.

**References**


