1 Introduction

The three-body problem in celestial mechanics consists in determining the positions and trajectories of distinct bodies in past and future times. Beyond studying the displacement of celestial bodies, many researchers on the topic have investigated the behavior of the bodies in the neighborhood of a singularity, i.e. a point in time in which the equations of motion satisfied by the particle coordinates are no longer defined. Among discussed singularities are collisions. Celestial bodies undergo collision when their positions coincide at a given time \( t \). Solutions to the double collision (collision of two bodies) problem have been successfully determined. However, triple collision orbits have long been considered far less tangible. According to Sundmans theorem, triple collision orbits approach a central configuration. A central configuration, by definition, is attained either when the three bodies are disposed in a straight line, or when they form an equilateral triangle. The first case will henceforth be referred to as the collinear case, while the second will be called the equilateral case. This paper will further investigate the behavior of the three particles undergoing triple collision. All researchers on the topic inquire whether the triple collision singularities are regularizable. Physicists such as Easton define regularization as extending the orbits through the singularities to functions that are continuous to other functions within a sufficiently small neighborhood of the singularity. To verify if such regularization is possible, they use topological methods, such as examining the flow on the triple collision manifold (topological space that resembles Euclidean space near every point). In particular, they
determine whether every orbit ending in triple collision can be connected to another beginning in triple collision, in such a way that a flow results. Others, such as Sundman and Siegel, view regularization as extending solutions in the neighborhood of the singularity as analytic functions of time. [2], [3]. Our discussion, based on the work of Siegel, a prime researcher on analytical regularization of triple collision singularities, will explore the latter method, and thus will be purely analytical. Although triple collision generates essential singularities in the coordinates, Siegel nevertheless derives general solutions for their orbits in terms of suitable series expansions. Following Siegel’s approach, we begin by confirming that both the equilateral and collinear cases of a triple collision do occur, and in fact are special solutions for the triple collision orbits. This is accomplished through the construction of a special solution to the planar 3-body problem. Siegel then derives the general triple collision orbits by regularizing the coordinates. This regularization, as we show, takes the form of convergent series expansions. At this point, no explicit form can be derived for the general triple collision orbits, so they are only derived for the collinear and equilateral cases. Afterwards, we reach the established assumption that triple collisions must occur in definite directions. From there, Siegel establishes that the trajectory of the particles in the collinear case is invariant, i.e. the particles move along a fixed line, which does not shift with time.

2 Central configuration

The discussion of triple collision orbits begins with the following theorem:

**Theorem 1:** The central configurations that arise in triple collision for the three-body problem are special solutions to the general triple collision orbits.

**Proof:**

In this demonstration, we will create a mathematical model to represent triple collision at time \( t = 0 \). We will study the behavior of the coordinates in the time interval \([0, \tau], \tau < \infty\). Since the collision occurs at time \( t = 0 \), it is enough to study the problem in finite time. Defining the six coordinates \( q_k, k = 1, \ldots, 6 \), as functions of time:

\[
q_k = \hat{q} \ast w(t), \hat{q} \in \mathbb{R}.
\]
This representation explicitly shows that the particles’ positions vary with time. w(t) is chosen to be strictly positive, to ensure that the particles’ positions grow further from the origin for \( t > 0 \). More specifically, for all \( t > 0 \), the particles move away from the triple collision. Furthermore, w(t) must be twice differentiable, so that we can apply Newton’s Second Equation to the \( q_k \). w(t) approaches 0 as \( t \to 0 \), which represents the particles approaching their center of mass at the collision at \( t = 0 \). The \( \hat{q} \) are chosen such that all coordinates are distinct for \( t > 0 \), i.e. no collision occurs after \( t = 0 \).

Now consider the transformed potential function \( \hat{U} = Uw(t) \), where U is the original potential function. Since \( \hat{U} \) is of degree -1 in \( \hat{q} \), we can apply Euler’s equation:

\[
\sum \hat{q}\hat{U} = -\hat{U}
\]  

(2)

Yet, \( -\hat{U} < 0 \), so there exists a coordinate \( q_k \) such that \( \hat{q}\hat{U}_q < 0 \).

For this coordinate, we apply Newton’s second law, which under the transformation \( q = \hat{q}w \), becomes:

\[
m\hat{q}\ddot{w} = \hat{U}q^2w^{-2}.
\]  

(3)

From \( \hat{q}\hat{U}_q < 0 \), it follows that \( \dot{w}w = C, C \in \mathbb{R}^- \).

To relate this equation to those derived for the collinear and equilateral cases [1], w(t) is normalized by some constant k to obtain \( C = -2/9 \), the constant present in the collinear and equilateral cases.

Upon solving the resulting differential equation and omitting all integration constants, we obtain \( w(t) = t^{2/3} \). This is consistent with Sundman’s conclusion that particles undergoing triple collision have coordinates of the order \( t^{2/3} \) [1]. Furthermore, defining the abcissa axis of the coordinate system in which the particles lie in the direction of \( P_3P_1 \), one obtains the same coordinate values \( \hat{X}_k, \hat{Y}_k \) as those for the collinear and equilateral cases [1].
3 Power series representation of triple collision orbits

According to Siegel, the triple collision orbits can be extended in a sufficiently small neighborhood of a triple collision. Before proving this, Siegel places $P_1$, $P_2$, and $P_3$ in a new Cartesian coordinate system with $P_3$ at the origin, $P_3P_1$ the direction of the abcissa axis, and $p_4$ the angle between the old and new abcissa axes. After performing a series of canonical transformations, he obtains new coordinates $p_k, q_k$ (the $q_k$ are different from those defined in the previous section), $k = 1, ..., 4$. The $p_k$ and $q_k$ satisfy the system of Hamilton differential equations:

\[
\dot{p}_k = (E q_k)_{q_4} = 0, \quad \dot{q}_k = (E p_k)_{q_4} = 0, (k = 1, 2, 3), \quad \dot{p}_4 = (E q_4)_{q_4} = 0
\] (4)

This Hamilton system leads to the determination of the triple collision orbits.

Henceforth, with the use of relative coordinates $-p_k = p_k* t^{-2/3}, -q_k = q_k* t^{1/3}(k = 1, 2, 3), q_4 = -q_4* t^{1/3}, p_4 = -p_4$, Siegel defines $t = e^{-u}$, so that $t \to 0$ as $u$ increases, as the particles approach the triple collision at $t = 0$. Since the transformation implies $E = -E* t^{-2/3}$, Siegel obtains the new system:

\[
d\bar{p}_k/du = 2/3 \bar{p}_k - E \bar{q}_k
\] (5)

\[
d\bar{q}_k/du = -1/3 \bar{q}_k + E \bar{p}_k (k = 1, 2, 3)
\] (6)

\[
d\bar{p}_4/du = -E q_4
\] (7)

\[
d\bar{q}_4/du = 1/3 \bar{q}_4
\] (8)

According to Chapter 12 of Siegel, the $p_k$ and $q_k$ approach limiting values as $t \to 0$. The proof is omitted from this discussion as it has little significance to the topic. This result allows for the definition of the coordinates as functions of variables $\delta_k (k = 1, ..., 8)$, defined in a sufficiently small complex neighborhood of $t = 0$. These will serve for the power series derivation:

\[
\bar{p}_k = \hat{p}_k + \delta_k, \bar{q}_k = \hat{q}_k + \delta_k + 2(k = 1, 2)
\] (9)

\[
\bar{p}_3 = \hat{p}_3 + \delta_5, \bar{q}_3 = \hat{q}_3 + \delta_6
\] (10)
\[ \eta_4 = \delta_7, \phi_4 = \delta_8 \] (11)

We see that the functions of \( \delta_k \) are smooth, so the new differential equation system above can be expressed in terms of \( \delta_k \):

\[ \frac{d\delta_k}{du} = \sum_{l=1}^{8} (l = 1 to 8) a_{kl} + \phi_k, (k = 1, \ldots, 8) \] (12)

\( \phi_k \) is itself a power series beginning in quadratic terms. \( \delta_7 = q_4 t^{-1/3} = 0 \), so this variable will be omitted from the remaining analysis.

Given this system of equations in terms of \( \delta_k \)s, we can prove the following theorem:

**Theorem 2:** The triple collision orbits can be expressed in terms of convergent power series.

**Proof:**
\( \delta_7 = 0 \) and \( \delta_8 = \phi_4 = \phi_4 \) is merely the angle between the old and new coordinate systems. Hence, these 2 variables give no indication on the triple collision orbits, so we focus on \( \delta_1, \ldots, \delta_6 \) for the remainder of the proof. Let \( A \in M_{6,6} \) be the coefficient matrix of the linear part of the system above, for \( k = 1, \ldots, 6 \). Let \( \lambda_k, k = 1, \ldots, 6, \) be the eigenvalues of \( A \), i.e. the roots of the polynomial \( |\lambda I - A| \). Choose \( \lambda_1, \ldots, \lambda_f, f \leq 6 \) to be the eigenvalues which have a negative real part. We assume that the \( \lambda_k \) are simple (have algebraic multiplicity 1), non-zero, and not purely imaginary. This ensures that the coefficients of the differential equation are distinct, and that each \( \delta_k \) term appears in the right-hand side of the differential equations. We impose the additional restriction \( \sum_{l=1}^{f} n_l \lambda_l \neq \lambda_k \). According to a theorem proven by Liapunov, this implies that the solutions of the system are convergent power series: \( \delta_k = \xi_k(v_1, \ldots, v_f), k = 1, \ldots, 6, v_l = c_l e^{\lambda_l u}, l = 1, \ldots, f, c_l < \epsilon \), for some small \( \epsilon > 0 \).
This concludes the proof.

We now shift our attention to \( \delta_8 \) and prove the following result:

**Theorem 3:** Collision of the three particles takes place in definite directions.

**Proof:**
This statement is equivalent to the condition that \( \phi_4 \) has a finite limit \( \hat{\phi}_4 \) as \( t \to 0 \). Therefore, we will prove the latter mathematical statement. Given the solutions \( \delta_1, \ldots, \delta_6 \), we substitute these
into the differential equation for $\delta_8$ and obtain the following:

$$d\vec{p}_4/du = -(1/m_1 + 1/m_3)\vec{q}_0/\vec{p}_1 - 1/m_3\vec{q}_3/\vec{p}_1$$  \hspace{1cm} (13)$$

where $\vec{q}_0 = \vec{q}_2\vec{p}_3 - \vec{p}_2\vec{q}_3/\vec{p}_1$.

$m_1$ and $m_3$ are the respective masses of particles $\vec{p}_1$ and $\vec{p}_3$. As $p_4$ is essentially a function of $\delta_1, ..., \delta_6$, the right-hand side of the differential equation is also a power series in $v_1, ..., v_f$. Furthermore, $\lambda_1, ..., \lambda_f$ have negative real parts, so by integrating the equation we see that the right-hand side converges as $u \to \infty$, and thus $t \to 0$. This implies that $p_4$ has a finite value in the limit $t \to 0$.

4 Additional results in the limit $t \to 0$

The above demonstrated the general results for triple collision orbits, namely the convergent power series solutions. Up to this date, success in deriving specific forms of these solutions has been limited. Easton, in investigating whether triple collision orbits can be extended as functions that are continuous with respect to nearby solutions, concluded that this was impossible, at least for some specific mass values[2]. Many researchers on the topic claim that most triple collision orbits cannot be regularized, i.e. extended through the singularity. However, in the limit $t \to 0$, Siegel is able to derive specific, complete solutions for the triple collision orbits, namely for the collinear and equilateral cases.

4.1 Specific behavior of coordinates in the collinear and equilateral cases

Upon solving for the eigenvalues $\lambda_1, ..., \lambda_6$, we find that in the equilateral case, the 6 eigenvalues are all real, with $\lambda_1, ..., \lambda_3$ negative and $\lambda_4, ..., \lambda_6$ positive. As a result, the triple collision coordinates are power series in the three variables $v_1 = c_1t^{2/3}, v_2 = c_2t^{-\lambda_2}, v_3 = c_3t^{-\lambda_3}, c_1, c_2, c_3 \in \mathbb{R}$.

By determining the angle $p_4$, we obtain another constant $c_4$, which fixes the direction of collision for the 3 particles. To obtain the general solution for the equilateral case, we can subject any arbitrary orientation of the $\mathbb{R}^2$ plane to a translation with constant velocity. This transforma-
tion generates 8 additional constants. In total, the general equilateral case has 12 independent real parameters. In the collinear case, $\lambda_1$ and $\lambda_2$ are real and negative, $\lambda_5$ and $\lambda_6$ are real and positive, and $\lambda_3$ and $\lambda_4$ are either real and positive or complex conjugates with real part 1/6. Therefore, the coordinates are power series in $v_1 = c_1 t^{2/3}, v_2 = c_2 t^{-\lambda_2}, c_1, c_2, \in \mathbb{R}$. To obtain a general solution, it is enough to translate the existing orbital line to any straight line in the original coordinate system moving parallel to itself at constant velocity. This yields 7 new constants, giving us a total of 10 independent real parameters, including the angle $p_4$.

Siegel ends the discussion by showing that in the collinear case, the angle $p_4$ is invariant.

4.2 Restrictions on the collinear case

Siegel proves the following theorem:

**Theorem 4:** In the collinear case of a triple collision, the angle $p_4$ is constant, so that the 3 particles move along a fixed line.

**Proof:**

By studying the differential equations for $\mathbf{p}_3, \mathbf{q}_3$ [see section 2] and $p_4$, we see that the right hand sides of the equations are of the form $\phi \delta_5 + \xi \delta_6$, (since $\delta_7 = 0$, the variable is omitted from the equations). $\phi$ and $\xi$ are convergent power series in $\delta_1, \ldots, \delta_6$. The two eigenvalues, $\lambda_3$ and $\lambda_4$, corresponding to the linear parts right-hand sides of the equations can be calculated by using the previous differential equations (give numbers) involving the $\delta_k$, but we will omit the details here. We find that $\lambda_3$ and $\lambda_4$ both have positive real parts, which implies that $\delta_5$ and $\delta_6$, subject to the condition $\delta_k \to 0$ as $u \to \infty$, must vanish. As a result, $\mathbf{p}_3 = \mathbf{q}_3 = 0$. From equation (31) [see section 2], $\mathbf{q}_0 = 0$ as well, and thus $p_4 = \mathbf{\hat{p}}_4$ is constant.

5 Conclusion

In this discussion, we presented Siegel’s demonstration that shows that the central configurations in the three-body problem derived by Sundman are special solutions for the triple collision orbits. Furthermore, they correspond precisely to solutions in the limit $t \to 0$. Siegel demonstrates that triple collision orbits can be regularized. More specifically, in the neighborhood of triple collisions, the general solutions can be extended as convergent power series. Though little more could be
concluded for the general solutions, Siegel successfully characterizes the special solutions for the collinear and equilateral cases more extensively. The solutions of the equilateral case are power series of terms of the order \( v_1 = c_1t^{2/3}, v_2 = c_2t^{-\lambda_2}, v_3 = c_3t^{-\lambda_3} \), while those of the collinear are of the order \( v_1 = c_1t^{2/3}, v_2 = c_2t^{-\lambda_2} \). Furthermore, Siegel confirms that triple collisions must take place in definite directions. The angle between the new and old coordinate systems attains a limiting value as \( t \to 0 \). For the collinear case in particular, the angle is constant for all times \( t \), thus ensuring that the three particles move along a straight line.

6 References


7 Honor Code

"This paper represents my own work in accordance with University regulations".

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