VORONOÏ SUMMATION FOR GL(2)

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Abstract. We give a proof of a Voronoï summation formula attached to automorphic forms on GL(2) over number fields. The article can serve as a synthesis and an exposition of some previous works on the subject since it includes a discussion of classical versions of the Voronoï formula and how to establish them in a natural way from considerations in local harmonic analysis and representation theory. This approach also allows for common ramification of the modulus of the additive twist and of the level of the form.

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1. Introduction

In this article we revisit the Voronoï summation formula attached to automorphic forms on GL(2). We shall develop a general framework in which the formula is derived in a natural manner from considerations in harmonic analysis and representation theory. The Voronoï formula for GL(2) is a fundamental tool for the study of analytic properties of automorphic forms.

1.1. Origins. Let $\tau(n) := \sum_{ab=n} 1$ be the divisor function. Dirichlet showed in 1849 that

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x)$$

(1.1)

where $\Delta(x) = O(x^{\frac{1}{2}})$ and $\gamma$ is Euler constant. The sum (1.1) counts the number of integer points $(a, b) \in \mathbb{Z}^2_{\geq 1}$ inside the hyperbola $ab \leq x$. The proof is known as Dirichlet hyperbola method. The Dirichlet divisor problem is the deep conjecture that $\Delta(x) = O_{\epsilon}(x^{\frac{1}{3} + \epsilon})$.

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The first improvement on Dirichlet’s result is due to Voronoï [63] in 1904 who showed
\[ \Delta(x) = O(x^{\frac{1}{2}} \log x), \quad \text{as } x \to \infty. \]
For the proof he established what is now known as the Voronoï formula which is an exact summation formula for \( \Delta(x) \) dual to (1.1).
Hardy and Landau later showed [35], also using the Voronoï formula, that \( \Delta(x) = \Omega(x^{\frac{1}{4}}) \) where the \( \Omega \)-symbol means that
\[ \liminf_{x \to \infty} \Delta(x)x^{-\frac{1}{4}} = -\infty, \quad \limsup_{x \to \infty} \Delta(x)x^{-\frac{1}{4}} = +\infty. \]
Currently the best known bound due to Huxley [37] is \( \Delta(x) = O(\epsilon x^{\frac{131}{416}}) \), obtained by extending a method of Bombieri-Iwaniec to estimate exponent pairs.

The Voronoï formula [63] states that
\[ \Delta(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \tau(n) \left( \frac{x}{n} \right)^{\frac{1}{2}} \left( Y_1(4\pi\sqrt{nx}) + K_1(4\pi\sqrt{nx}) \right) \]
for all \( x \) which is not integer, where \( Y_\nu \) and \( K_\nu \) are Bessel functions. This formula and its generalizations are the subject of the present paper. The formula is the basis of the proof of the upper-bound (1.2) and of the \( \Omega \)-result (1.3). A smoothed version of the Voronoï formula which is equivalent states that
\[ \sum_{n=1}^{\infty} \tau(n)w(n) = \int_0^\infty w(x)(\log x + 2\gamma)dx + \sum_{n=1}^{\infty} \tau(n)\tilde{w}(n) \]
where \( w \) is any smooth function of compact support on \((0, \infty)\) and
\[ \tilde{w}(y) := \int_0^\infty w(x) \left( 4K_0(4\pi\sqrt{xy}) - 2\pi Y_0(4\pi\sqrt{xy}) \right) dy \]
is an Hankel transform of \( w \).
Ramanujan recorded the following formula in his lost notebook. For \( x > 0 \) non-integer and \( 0 < \theta < 1 \),
\[ \sum_{n=1}^{\infty} \left[ \frac{x}{n} \right] \cos(2\pi n\theta) = \frac{1}{4} - x \log(2\sin(\pi\theta)) + \]
\[ + \frac{1}{2} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{I_1 \left( 4\pi\sqrt{m(n+\theta)x} \right)}{\sqrt{m(n+\theta)}} + \frac{I_1 \left( 4\pi\sqrt{m(n+1-\theta)x} \right)}{\sqrt{m(n+1-\theta)}} \right\}. \]
The relation with (1.4) is when \( \theta \to 0 \). The question of convergence of the double series is delicate and studied in detail in a series of articles by Berndt–Kim–Zaharescu [5].
The modern interpretation of the Voronoï formula is that it is related to the functional equation for the Riemann zeta function
\[ \xi(s) := \pi^{-s/2}\Gamma \left( \frac{s}{2} \right) \zeta(s) = \xi(1-s) \]
in view of the identity
\[ \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \zeta(s)^2. \]

In this respect it is natural that there is a Voronoï summation formula, due to Oppenheim, for the generalized divisor function \( \tau_\alpha(n) = \sum_{d|n} d^\alpha \) whose associated Dirichlet series is \( \zeta(s)\zeta(s - \alpha) \).

Furthermore \( \tau_\alpha(n) \) are the Fourier coefficients attached to an Eisenstein series on \( \text{GL}(2) \) and \( \zeta(s)\zeta(s - \alpha) \) is the \( L \)-function attached to this Eisenstein series. More generally the \( L \)-functions \( L(s, f) \) attached to automorphic forms on \( \text{GL}(2) \) have a functional equation and there is a Voronoï summation formula attached to the coefficients \( \lambda_f(n) \) which we are going to explain below. We also mention a different generalization of the Voronoï formula by Beineke-Bump [4] which involves the divisor function of sublattices of \( \mathbb{Z}^n \).

1.2. Georgy Voronoï (1868-1908). This subsection contains brief historical notes taken from the recent book [23]. Voronoï was born in Zhurovka, Poltava province, Ukraine. He attended the University of St Petersburg and notably the lectures by Y.-K. Sokhotsky and A. Markov. He wrote two dissertations and a thesis on Bernoulli numbers, cubic algebraic integers and continued fractions, respectively. Interesting is the following quote from his student diary (1888): “I get up at five in the morning and go in for mathematics. What a marvelous thing it is! Even though it abounds in formulas, all of them are so symmetric that they can be easily memorized”. In this article we will try to highlight the symmetries present in the Voronoï formula. Voronoï was appointed professor at Warsaw University in 1894 and unfortunately died at an early age. His academic environment and family life was marked by difficult circumstances. He had only twelve publications but these are of the highest quality with a considerable impact in many branches of mathematics to this day, such as the Voronoï diagrams and tessellations, Voronoï algorithm to compute units in cubic fields, Voronoï polyhedra.

Voronoï’s research was very much modern. He set to work on great achievements of the past and on making the next big step. This is what happened in the context of the Voronoï algorithm conceived as a generalization of the periodic continued fraction of quadratic units; Markov was so surprised by the result that he made multiple tests to be sure of its correctness before approving it and granting the doctoral thesis diploma. Similarly the upper-bound (1.2) was the first improvement on the Dirichlet hyperbola method in fifty-five years.

We conclude this historical tour with the following formula of Ramanujan mentioned by Hardy in [35]. For \( a, b \in \mathbb{R}_{>0} \),
\[
\sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n + a}} e^{-2\pi \sqrt{(n + a)b}} = \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n + b}} e^{-2\pi \sqrt{(n + b)a}}
\]

This is a variant of the Voronoï summation formula for the number \( r_2(n) \) of representations of \( n \) as a sum of two squares. In this form the formula only involves elementary functions and is particularly symmetric!
1.3. A general formula. In this subsection we state the Voronoï formula in the case
where \( f \) is holomorphic of weight \( k \geq 2 \) and level \( N \geq 1 \) with Nebentypus \( \chi \). Recall
that \( \Gamma_0(N) \) is the subgroup of \( \text{SL}(2, \mathbb{Z}) \) consisting of matrices \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) with \( N|c \) and
that for all \( \gamma \in \Gamma_0(N) \),

\[
\left. f \right|_\gamma(z) := \frac{f(\gamma z)}{j(\gamma, z)^{k}} = \chi(d)f(z),
\]

where \( j(\gamma, z) := cz + d \) and \( \gamma z = \frac{az + b}{cz + d} \). We assume that \( f \) is a primitive newform which implies that \( f \) is an eigenvalue of all the Hecke operators and that

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)
\]

with \( \lambda_f(1) = 1 \). The \( L \)-function

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}
\]

admits an analytic continuation to an entire function, and satisfies the functional
equation

\[
\Lambda(s, f) := (2\pi)^{-s-\frac{k-1}{2}} \Gamma(s + \frac{k-1}{2})L(s, f) = \epsilon(f) N^{\frac{1}{2}-s} \Lambda(1-s, \overline{f})
\]

where \( \epsilon(f) \) is the root number which is a root of unity. Let \( c \geq 1 \), \( a \in \mathbb{Z} \) coprime with \( c \) and \( w \) be a smooth function on \((0, \infty)\) of compact support. The general Voronoï formula involves the sum

\[
(1.8) \quad \sum_{n=1}^{\infty} \lambda_f(n) e \left( \frac{an}{c} \right) w(n).
\]

The difference with (1.5) is that we have replaced \( \tau(n) \) by \( \lambda_f(n) \) and we are now essentially summing over \( n \) in arithmetic progressions modulo \( c \).

Let \( N_1 = (c, N) \) and \( N_2 = N/N_1 \). Assume that \( (N_1, N_2) = 1 \), or equivalently \( (c, N_2) = 1 \) and let \( aN_2 \) be the inverse of \( aN_2 \) modulo \( c \). Under this condition the Theorem A.4 in [48] states that (1.8) is equal to the following dual sum:

\[
(1.9) \quad \frac{2\pi \eta}{c\sqrt{N_2}} \sum_{n=1}^{\infty} \lambda_g(n) e \left( -\frac{an}{c} \frac{\overline{aN_2}}{c} \right) \tilde{w}(n).
\]

Here \( \eta \) is a root of unity and \( g \) is an automorphic newform on \( \text{GL}(2) \) of the same level \( N \) and weight \( k \), and

\[
(1.10) \quad \tilde{w}(y) := \int_{0}^{\infty} \frac{w(t)J_{k-1} \left( \frac{4\pi \sqrt{ty}}{c\sqrt{N_2}} \right)}{c\sqrt{N_2}} dt.
\]

The form \( g \) and the scalar \( \eta \) are derived from \( f \) using Atkin–Lehner operators. It is a theorem of Deligne that \( |\lambda_f(n)|, |\lambda_g(n)| \ll_{\varepsilon} n^{\varepsilon} \) for all \( \varepsilon > 0 \).
1.4. Methodology. We summarize here the different approaches to the Voronoï formula stated in the previous subsection. For the divisor function and related functions, there are several proofs available in the literature, see e.g. [36, 45], [40, §4.5] and the references there.

For automorphic cusp forms of full level, Duke–Iwaniec [18–22] derive the formula from the functional equations of the $L$-functions twisted by a character on $GL(1)$. The formula for trivial central character may be found in Kowalski–Michel–VanderKam [48]. The proof in [48] combines methods of Jutila [45] and Duke–Iwaniec where the analytic continuation and functional equation of the associated Dirichlet series is established first, from a direct consideration of the action of the modular group, and then the summation formula follows from Mellin inversion. The general formula for Maass forms with arbitrary central character is established in Harcos–Michel [34]. We shall recall these results and mention a unified treatment in Section 7.

We summarize recent progress in higher rank. The Voronoï formula for $SL(n,\mathbb{Z})$ is established by Miller–Schmid [53, 54] who develop at the same time the framework of automorphic distributions. The expository articles [51, 52] contain an account of this framework in the $GL(2)$ case. Another proof appears in Goldfeld–Li [30, 31] using the Duke–Iwaniec method. A general version has been given in our previous work with Ichino [38] where we derive the Voronoï formula from the classical theory of integral representation and local/global functional equations of $L$-functions on $GL(n)$, as may be found in the work of Jacquet–Piatetskii-Shapiro and Shalika. The advantage of this method is that the exact relationship with the functional equation may be clearly seen. Hence situations where ramification comes in can be analysed.

The present paper can be seen as a sequel of [38] since we now explore the method in details for $GL(2)$. The theory for $GL(2)$ is somewhat different than for $GL(n)$, $n \geq 3$ due to the presence of additional symmetries. One of the purposes of the present paper is to identify these features which are particular to $GL(2)$, and then explain some classical phenomenon from this broader perspective. Also we wish to mention an unpublished manuscript by Cogdell [10] in which the method using Whittaker expansion is similar to the present paper.

The analogue of (1.8) in the adelic framework is

$$\sum_{\gamma \in \mathcal{F}_\times} \psi(\gamma \zeta) w(\gamma),$$

where $\zeta \in \mathbb{A}_\times$ and $w \in \mathcal{K}(\pi, \psi)$ belongs to the Kirillov model and is a newvector over all nonarchimedean places. The dual sum (1.9) should take the form $\sum_{\gamma \in \mathcal{F}_\times} \psi(\gamma \zeta^{-1}) \tilde{w}(\gamma)$. We shall see in Section 3 that the Fourier–Whittaker expansion of automorphic forms provide a proof of such kind of identities. The question that remains is under which conditions on $\zeta$ and the ramification of $\pi$ are all the terms of the dual sum explicit? Therefore we will want to find a convenient expression for $\tilde{w}$. Ideally $\tilde{w}$ does not depend on $\zeta$. Also we would like that $\tilde{w}$ depend in a direct manner on the representation $\pi$. Over archimedean places we find Bessel transforms such as (1.10). Over non-archimedean places we find a formula for $\tilde{w}_v$ which depends directly on the $L$-factors $L(s, \pi_v)$ and
epsilon factors $\epsilon(s, \pi_v, \psi_v)$. Specifically the $p$-adic Bessel function $j_{\pi_v}$ will be the exact analogue.

Finally we make the observation that the Voronoï summation formula can be related to a Gelfand formation in the sense of [55]:

$$\text{GL}(2)$$

\[
\begin{array}{c}
N^-, \psi \\
\downarrow \downarrow \downarrow \\
\{e\} \\
N, \psi
\end{array}
\]

Here $N = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and $N^- = (\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix})$ is the opposite unipotent and each line represents a Gelfand pair. The Voronoï formula coincides with the identity obtained by expanding the linear form $f \mapsto f(e)$ into Fourier-Whittaker periods for $N$ and for $N^-$. There is a similar diagram for $\text{GL}(n)$.

### 1.5. Applications.

We shall not touch in this article the numerous applications of the Voronoï formula which include subconvexity [34], non-vanishing of $L$-values, estimates for shifted convolution sums; the $\text{GL}(3)$ Voronoï formula also has been used to establish subconvexity [50]. The pattern is that the sums (1.8) occur in many contexts in number theory and a central question is to produce uniform upper-bounds. For that purpose the Voronoï formula is widely used: in favorable situations the length of the dual sum is shorter and hence can be bounded effectively. Typically one is reduced to the asymptotic behavior of the Bessel transform $\tilde{w}$ which is a problem in analysis.

In relation to the Dirichlet’s divisor problem mentioned in §1.1, we have the Wilton estimate which says that

$$\sum_{n \leq x} \lambda_f(n)e\left(\frac{an}{c}\right) \leq C(k, N, \varepsilon)x^{1/2+\varepsilon}$$

for any $x > 1$. This estimate corresponds to a long $n$-sum and a varying $c$. The case of short $n$-sums is studied in the work of Jutila [45], see [24] for the latest results. The case of long $n$-sums and varying $f$ are also studied, see the recent [29] and the references there. Finally the condition for (1.9) that $(N_1, N_2) = 1$ is subtle, and if it is dropped then it can happen [62] that the left-hand side of (1.12) be as large as $N^{1/2}x^{1/2}$.

### 1.6. Structure of the article.

Section 2 is a review of automorphic representations on $\text{GL}(2)$ and Section 3 is a proof of a Voronoï summation using Whittaker periods. Section 4 is a detailed study of the Bessel transforms, both at archimedean and non-archimedean places. Section 5 specializes the Voronoï formula to new vectors and Section 6 is an analysis of Atkin–Lehner operators in the context of representation theory. The final Section 7 is a review of classical automorphic forms and the relation to the Voronoï formula.

### 1.7. Notation.

In this paper we mostly work in the context of automorphic representations of $\text{GL}(2, \mathbb{A})$. Over the adeles there is little obstacle in working over general number fields $F$ compared to working over $\mathbb{Q}$, so we shall proceed with number fields. Denote by $\mathbb{A} = A_F$ the ring of adeles. For all places $v$ of $F$ we denote by $F_v$ the associated local
field and view $F$ as embedded in $F_v$. When $F_v$ is non-archimedean we denote by $\mathfrak{o}_v$ its ring of integers, by $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{o}_v$, and by $\varpi_v$ a uniformizer. Let $\psi = \otimes_v \psi_v$ be a non-trivial additive character on $F \backslash \mathbb{A}$.

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2. **Automorphic representations on GL(2)**

The integral representation of $L$-functions for GL(2) goes back to Hecke. It has been developed in the book by Jacquet–Langlands [41] in the adelic and representation theoretic framework. We shall use this framework in our study of the Voronoï formula.

2.1. **Definition.** Let $G = \text{GL}_2$ and denote by $\mathcal{A}_{\text{cusp}}$ the space of smooth automorphic cusp forms on $G(\mathbb{A})$. Let $\pi = \otimes_v \pi_v \subset \mathcal{A}_{\text{cusp}}$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. We denote by $\omega: F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}$ the central character of $\pi$.

We have an involution $g_\iota = t^{-1}g^{-1}$ on $G$ and then on $\mathcal{A}_{\text{cusp}}$ acting by $\iota \phi(g) = \phi(g_\iota)$. If $\phi \in \pi$, then $\iota \phi \in \tilde{\pi}$ where $\tilde{\pi} \subset \mathcal{A}_{\text{cusp}}$ is the contragredient representation of $\pi$.

We have $\tilde{\pi} \simeq \pi \otimes \omega^{-1}$, which follows from the fact that we have a well-defined invariant pairing

\[
\int_{G(F)Z(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} \phi(g) \tilde{\phi}(g) dg, \quad \phi \in \pi, \quad \tilde{\phi} \in \pi \otimes \omega^{-1}.
\]

Let $T$ be the maximal torus of $G$ consisting of diagonal matrices, let $N$ be the maximal unipotent subgroup consisting of upper-triangular matrices and $N^-$ the opposite unipotent subgroup of lower-triangular matrices. We denote by the same letter $\psi$ the character on $N(\mathbb{A})$ trivial on $N(F)$ given by $\psi(n) = \psi(x)$ if $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

2.2. **Automorphisms.** We denote by $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the long Weyl element and let $w_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = w \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. We have

\[
w_1 g w_1^{-1} = (\det g) g^t, \quad g \in G.
\]

This identity is specific to the group $G = \text{GL}(2)$ and there is no analogue for $\text{GL}(n)$ with $n \geq 3$. Since composition with the inner automorphism $g \mapsto w_1 g w_1^{-1}$ of $G(\mathbb{A})$ preserves the isomorphism class of a representation $\pi$, we derive from (2.2) that $\pi \circ \iota \simeq \pi \otimes \omega^{-1}$.

The same identity (2.2) is valid locally thus we have $\pi_v \circ \iota \simeq \pi_v \otimes \omega_v^{-1}$ for all places $v$. On the other hand we have that $\tilde{\pi}_v \simeq \pi_v \circ \iota$ ([28, Thm 2]). Thus $\tilde{\pi}_v \simeq \pi_v \otimes \omega_v^{-1}$ which also follows from the global isomorphism recorded in the previous §2.1.
2.3. **Global Whittaker functions.** For \( \phi \in A_{\text{cusp}} \) let \( W_\phi \) be the \( \psi \)-Whittaker function given by

\[
W_\phi(g) = \int_{N(F) \backslash N(A)} \phi(ng) \overline{\psi(n)} \, dn, \quad g \in G(A).
\]

Here \( dn \) is the Tamagawa measure on \( N(A) \). The \( \psi^{-1} \)-Whittaker function of \( \tilde{\phi} \) is given by \( W_{\tilde{\phi}}(g) = W_\phi(wg^\prime) \). We have the Fourier expansion

\[
\phi(g) = \sum_{\gamma \in F^x} \psi \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right) g, \quad g \in G(A).
\]

2.4. **Bruhat decomposition.** For \( q \in Z \geq 1 \) and \( h \in Z \) with \( hq \equiv 1 \mod q \), one has:

\[
\begin{pmatrix} q^{-1} & h \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -h/q & 1 \end{pmatrix} \in \text{SL}_2(Z)
\]

This is useful for the Voronoï formula for \( \text{SL}_2(Z) \) because it enables to directly relate a lower-triangular matrix and an upper-triangular matrix.

In the present work this identity (2.5) appears in disguised form as the following Bruhat decomposition in the big Bruhat cell:

\[
\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} = \begin{pmatrix} 1 & \zeta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \zeta^{-1} \end{pmatrix}.
\]

Here \( \zeta \in A^\times \) or \( \zeta \in F_v^\times \).

2.5. **Whittaker models.** Since \( \pi \) is generic (i.e. the global Whittaker integral (2.3) is nonzero for some \( \phi \in \pi \)), \( \pi_v \) is generic for all places \( v \) of \( F \). We denote by \( \mathcal{W}(\pi_v, \psi_v) \) the Whittaker model of \( \pi_v \) with respect to \( \psi_v \). All functions \( W \in \mathcal{W}(\pi_v, \psi_v) \) satisfy \( W(n) = \psi_v(n) W(g) \) for all \( n \in N(F_v) \) and \( g \in G(F_v) \). The Whittaker model is unique, which is a result due to Gelfand–Kazhdan if \( v \) is archimedean and to Shalika if \( v \) is non-archimedean.

It is important in the context of this paper to make transparent the relation between the Whittaker models associated to \( \pi_v \) and \( \tilde{\pi}_v \) and to the characters \( \psi_v \) and \( \tilde{\psi}_v \). This is the purpose of the remainder of this subsection.

2.5.1. **Contragredient.** If \( W \in \mathcal{W}(\pi_v, \psi_v) \), then

\[
\tilde{W}(g) := W(wg^\prime), \quad g \in G(F_v)
\]

is a Whittaker function in \( \mathcal{W}(\tilde{\pi}_v, \psi_v^{-1}) \). This is the local counterpart of the observation following (2.3).

The above claim follows from the fact that the automorphism \( g \mapsto wg^\prime w \) preserves \( N \) and sends \( \psi_v \) to \( \psi_v^{-1} \), and the fact that \( \tilde{\pi}_v \) is isomorphic to \( \pi_v \circ \iota \) recalled in §2.2.
2.5.2. Character twists. Let $\chi_v$ be a quasi-character of $F_v^\times$. If $W \in \mathcal{W}(\pi_v, \psi_v)$ then
\begin{equation}
(2.8) \quad g \mapsto W(g)\chi_v(\det g),
\end{equation}
is a Whittaker function in $\mathcal{W}(\pi_v \otimes \chi_v, \psi_v)$. We note the particular case $\chi_v = \omega_v^{-1}$ (inverse of the central character). Since $\pi_v \simeq \pi_v \otimes \omega_v^{-1}$, $W(\pi_v)\chi_v(\det g)^{-1}$ is a Whittaker function in $\mathcal{W}(\pi_v, \psi_v)$.

2.5.3. Change of additive character. Let $a \in F_v^\times$ and $W \in \mathcal{W}(\pi_v, \psi_v)$. Then
\begin{equation}
(2.9) \quad g \mapsto W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right),
\end{equation}
is a Whittaker function in $\mathcal{W}(\pi_v, \psi'_v)$ with $\psi'_v(x) = \psi_v(ax)$. This realizes a $G(F_v)$-isomorphism between $\mathcal{W}(\pi_v, \psi_v)$ and $\mathcal{W}(\pi_v, \psi'_v)$ (which are both isomorphic to $\pi_v$).

2.5.4. Complex conjugate. Let $W \in \mathcal{W}(\pi_v, \psi_v)$. Then the complex conjugate $g \mapsto \overline{W(g)}$ belongs to $\mathcal{W}(\pi_v, \psi_v^{-1})$. If $\pi_v$ is unitary then $\pi_v \simeq \tilde{\pi}_v$.

2.6. Kirillov model. We denote by $\mathcal{K}(\pi_v, \psi_v)$ the Kirillov model of $\pi_v$ with respect to $\psi_v$. Recall that the action of $G(F_v)$ on $\mathcal{K}(\pi_v, \psi_v)$ is such that $w' = \rho((a \ b \ 
abla d))w$ satisfies
\begin{equation}
(2.10) \quad w'(y) = \omega_v(d)\psi_v(by/d)w(ay/d), \quad y \in F_v^\times.
\end{equation}
We recall the isomorphism between the Kirillov model and the Whittaker model given by $W \mapsto w$,
\begin{equation}
(2.11) \quad w(y) = W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right), \quad y \in F_v^\times,
\end{equation}
where $w \in \mathcal{K}(\pi_v, \psi_v)$ and $W \in \mathcal{W}(\pi_v, \psi_v)$.

If $v$ is non-archimedean (resp. archimedean), $C^\infty_c(F_v^\times)$ denotes the space of locally constant (resp. smooth) functions of compact support on $F_v^\times$.

**Lemma 2.1.** $C^\infty_c(F_v^\times)$ is a subspace of $\mathcal{K}(\pi_v, \psi_v)$. If $v$ is non-archimedean, it is of codimension 0, 1 or 2.

For details, see [11, Lecture 4], [15, §2] and [41, Chapter 2].

2.7. Local functional equation. Let $v$ be a place of $F$ and let $\chi_v$ be a quasi-character of $F_v^\times$. Let $W \in \mathcal{W}(\pi_v, \psi_v)$ be a $\psi_v$-Whittaker function. Consider the zeta integral, convergent for $\Re s$ large enough,
\begin{equation}
(2.12) \quad \Psi(s, W, \chi_v) = \int_{F_v^\times} W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\chi_v(y) |y|^{s-1/2} dy.
\end{equation}
The data of $s$ and the character $\chi_v$ are somewhat redundant since $|y|^{s-1/2}$ could be absorbed into $\chi_v(y)$. It is however traditional and practical to keep this redundancy.

Jacquet–Langlands [41] proved that the integral $\Psi_v(s, W, \chi_v)$ admits a meromorphic continuation and the following functional equation [42–44]:
\begin{equation}
(2.13) \quad \Psi(1 - s, \tilde{W}, \chi_v^{-1}) = \chi_v(-1)\gamma(s, \pi_v \times \chi_v, \psi_v)\Psi(s, W, \chi_v).
\end{equation}
Here $\gamma(s, \pi_v \times \chi_v, \psi_v)$ is the ratio $\epsilon(s, \pi_v \times \chi_v, \psi_v) \cdot \frac{L(1 - s, \pi_v \times \chi_v^{-1})}{L(s, \pi_v \times \chi_v)}$.

2.8. Epsilon factors. From the local functional equation it follows that $\epsilon(s, \pi_v, \psi_v)\epsilon(1 - s, \pi_v, \psi_v) = \omega_v(-1)$. Thus if $\pi_v$ is unitary then $|\epsilon(1/2, \pi_v, \psi_v)| = 1$.

If $\pi_v$ is self-dual, then $\epsilon(1/2, \pi_v, \psi_v) = \pm 1$ and is independent of $\psi_v$. It is called root number of $\pi_v$ and denoted $\epsilon(1/2, \pi_v)$.

2.9. Global functional equation. Let $\chi$ be a quasi-character on $F^\times \backslash \mathbb{A}^\times$. The product over all places

$$\epsilon(s, \pi \times \chi) = \prod_v \epsilon(s, \pi_v \times \chi_v, \psi_v)$$

is independent of the additive character $\psi$. The completed $L$-function

$$\Lambda(s, \pi \times \chi) = \prod_v L(s, \pi_v \times \chi_v)$$

satisfies the functional equation

$$\Lambda(s, \pi \times \chi) = \epsilon(s, \pi \times \chi)\Lambda(1 - s, \pi_v \times \chi^{-1}).$$

2.10. Hecke operators. Let $K_v = G(\mathfrak{o}_v)$ and $T_v$ be the characteristic function of the double coset $K_v \left( \begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix} \right) K_v$. We have a disjoint decomposition

$$K_v \left( \begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix} \right) K_v = K_v \left( \begin{smallmatrix} 0 & \varpi_v \\ 1 & 0 \end{smallmatrix} \right) \bigsqcup_{a \in \mathfrak{o}_v/p_v} K_v \left( \begin{smallmatrix} \varpi_v & a \\ 0 & 1 \end{smallmatrix} \right).$$

This can be used to defined the Hecke operator at an unramified place. More generally the Hecke algebra is $C_c^\infty(G(\mathfrak{A}))$.

2.11. Unramified representations. Let $v$ be a non-archimedean place such that $\pi_v$ is unramified, namely has conductor $f(\pi_v) = 0$. Let $\varphi_{\mathfrak{o}_v} \in \pi_v$ be a nonzero vector invariant by $G(\mathfrak{o}_v)$. References for this subsection are for example [15, §2.3], [11, Lecture 7] and [56]. It is known that $\pi_v$ is a principal series representation fully induced from a character of $B(F_v)$. Thus (with the unitary normalization):

$$\pi_v = \text{Ind}_{B(F_v)}^{G(F_v)}(\mu_1 \otimes \mu_2),$$

where $\mu_1, \mu_2$ are two unramified quasi-characters of $F_v^\times$ with $\mu_1\mu_2^{-1} \neq |\pm 1$. The Satake parameters are $\alpha_i(\pi_v) = \mu_i(\varpi_v)$, and the Satake matrix is the conjugacy class of $A(\pi_v) = \left( \begin{smallmatrix} \alpha_1(\pi_v) & 0 \\ 0 & \alpha_2(\pi_v) \end{smallmatrix} \right)$ in $\text{GL}(2, \mathbb{C})$.

The Hecke algebra $\mathcal{H}(G(F_v)/G(\mathfrak{o}_v))$ is generated by the Hecke operator $T_v$ associated to the double class $G(\mathfrak{o}) \left( \begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix} \right) G(\mathfrak{o})$. It acts on $\mathbb{C}\varphi_{\mathfrak{o}_v}$ through a character. The center acts via the central character which is determined by $\omega_v(\varpi) = \alpha_1(\pi_v)\alpha_2(\pi_v)$. The Hecke operator $T_v$ acts by multiplication by $\lambda(\pi_v) = \alpha_1(\pi_v) + \alpha_2(\pi_v)$ (normalized eigenvalue).
The $L$-function associated to $\pi_v$ reads
\begin{equation}
L(s, \pi_v) = \det(1 - A(\pi_v)q_v^{-s})^{-1} = (1 - \lambda(\pi_v)q_v^{-s} + \omega_v(q_v)^{-2s})^{-1}.
\end{equation}
Suppose that $\psi_v$ is unramified, i.e. $n(\psi_v) = 0$ or equivalently $\psi_v$ is trivial on $\mathfrak{o}_v$ but nontrivial on $\omega_v^{-1}\mathfrak{o}_v$. Then $\epsilon(s, \pi_v, \psi_v) \equiv 1$.

Finally we have the following formula for the unramified Whittaker function $W_\circ \in \mathcal{W}(\pi_v, \psi_v)$
\begin{equation}
W_\circ \left( \begin{array}{cc} \omega^a & 0 \\ 0 & \omega^b \end{array} \right) = q_v^{-(a-b)/2} \sum_{k+l=a+b, \ k, l \geq b} \alpha_1(\pi_v)^k \alpha_2(\pi_v)^l.
\end{equation}
Note that this is zero unless $a \geq b$. One can verify that this is compatible with (5.3).

We note that $\tilde{W}_\circ(g) = \omega_v(\det g)^{-1}W_\circ(g)$ is the newvector of $\pi_v$. In particular $\tilde{w}_\circ(y) = \omega_v(y)^{-1}w_\circ(y)$ is the newvector of $\tilde{\pi}_v$ in the Kirillov model.

2.12. **Real place.** We have a non-split exact sequence
\begin{equation}
1 \to C^\times \to W_\mathbb{R} \to \mathbb{Z}/2\mathbb{Z} \to 1.
\end{equation}
Let $W_\mathbb{C} = C^\times$. Then $W_\mathbb{R}^{ab} \simeq \mathbb{R}^\times$ and we have the following compatibilities. The composition $W_\mathbb{C}^ab \to W_\mathbb{R}^ab$ is the norm map $C^\times \to \mathbb{R}^\times$. The transfer $W_\mathbb{R}^ab \to W_\mathbb{C}^ab$ is the natural inclusion $\mathbb{R}^\times \to C^\times$. References for this subsection are for example the survey article [59, 60], and [46,47], [11, Lecture 8].

Concretely one can write $W_\mathbb{R} = C^\times \cup jC^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$. We recall the classification of irreducible unitary representations $\pi$ of $\text{GL}(2, \mathbb{R})$. Such representation $\pi$ either factors through the determinant or is infinite dimensional. It can be realized as a component of the induced representation from a character $\chi_1, \chi_2$ on $B$. Let $\chi_i(x) = \text{sgn}(x)^{s_i} |x|^{s_i}$ with $s_i \in \mathbb{C}$. The classification is as follows:
\begin{itemize}
    \item If $\pi$ is square-integrable then it is a discrete series representation of weight $k \geq 2$.
    Also $\epsilon_1 + \epsilon_2 \equiv k(2)$, $k = 1 + s_1 - s_2$ and $s_1 + s_2 \in i\mathbb{R}$. The central character is $\text{sgn}(x)^k |x|^{s_1+s_2}$.
    The quadruple $(\epsilon_1, s_1, \epsilon_2, s_2)$ is unique up to a shift $(\epsilon_1+1, s_1, \epsilon_2+1, s_2)$. In other words we have that $\pi \simeq \pi \otimes \text{sgn}$. This implies that $\pi$ is self-dual if and only if $s_1 + s_2 = 0$.

    Under the local Langlands correspondence, the representation of $W_\mathbb{R}$ is given by
\begin{equation}
    z \mapsto \left( \begin{array}{cc} z^{s_1} \bar{z}^{s_2} & 0 \\ 0 & \bar{z}^{s_1} z^{s_2} \end{array} \right), \quad j \mapsto \left( \begin{array}{cc} 0 & (-1)^k \\ 1 & 0 \end{array} \right).
\end{equation}
\end{itemize}
A small computation [59, 3.3.1] shows that this is the induced representation from $W_\mathbb{C}$ to $W_\mathbb{R}$ of the quasi-character $\chi(z) = z^{s_1} \bar{z}^{s_2}$ (it is also isomorphic to the induced representation of $\tilde{\chi}$). The character is well-defined because $s_1 - s_2 \in \mathbb{Z}$, note that $\chi(z) = (zz)^{s_1+s_2} (\bar{z}/z)^{k+1}$. The representation is irreducible because $\chi$ is not invariant by conjugation which is equivalent to $s_1 \neq s_2$ or $k \neq 1$.

\begin{itemize}
    \item If $\pi$ is not square-integrable then it is a Langlands quotient $\chi_1 \boxplus \chi_2$. The central character is $\chi_1 \chi_2$. Under the local Langlands correspondence, the representation of $W_\mathbb{R}$ is reducible, it factors through $W_\mathbb{R}^{ab}$. It is identified with the direct sum $\chi_1 \oplus \chi_2$. There
are three cases to distinguish: principal and complementary series and a limit of discrete series.

Limit of discrete series. Then $s_1 = s_2 \in i\mathbb{R}$ (weight 1) and the central character is non-trivial. The quadruple $(\epsilon_1, s_1, \epsilon_2, s_2)$ is unique up to a permutation $1 \leftrightarrow 2$ and up to a shift $(\epsilon_1 + 1, s_1, \epsilon_2 + 1, s_2)$.

- Otherwise the weight is zero, in which case it is a spherical representation in the sense that there are $SO_2(\mathbb{R})$-invariant vectors. The complementary series arise when $s_1 - s_2 \in \mathbb{R}$ and $s_1 + s_2 \in i\mathbb{R}$ with $0 < |s_1 - s_2| < 1$ and $\epsilon_1 = \epsilon_2$. The quadruple $(\epsilon_1, s_1, \epsilon_2, s_2)$ is unique up to a permutation $1 \leftrightarrow 2$.

- The principal series arise for all $s_1, s_2 \in i\mathbb{R}$. The quadruple $(\epsilon_1, s_1, \epsilon_2, s_2)$ is unique up to a permutation $1 \leftrightarrow 2$.

2.12.1. $L$-factors. The $L$-factors are then easy to determine. If $\pi$ is a Langlands quotient $\chi_1 \boxplus \chi_2$ with $\chi_i(x) = \sgn(x)^{\epsilon_i} |x|^{s_i}$ and $\epsilon_i = 0, 1$, then $[59, (3.1)]$:

\begin{equation}
L(s, \pi) = \Gamma_{\mathbb{R}}(s + s_1 + \epsilon_1) \Gamma_{\mathbb{R}}(s + s_2 + \epsilon_2).
\end{equation}

If $\pi$ is a square-integrable, then $\sigma(\pi)$ is induced from the character $\chi(z) = z^{s_1} \bar{z}^{s_2}$.

\begin{equation}
L(s, \pi) = L(s, \chi) = \Gamma_{\mathbb{C}}(s + s_1).
\end{equation}

This follows from the fact that $\chi(z) = \bar{z}^{1-k}(z \bar{z})^{s_1}$ and $1 - k < 0$. Writing $s_1 + s_2 = 2it$ with $t \in \mathbb{R}$, the central character is $\sgn(x)^k |x|^{2it}$ and the $L$-function may be written

\begin{equation}
L(s, \pi) = \Gamma_{\mathbb{R}}(s + \frac{k-1}{2} + it) \Gamma_{\mathbb{R}}(s + \frac{k+1}{2} + it).
\end{equation}

Note that the formula is different from (2.23), even though $\pi$ can be realized as a component of the induced representation by $(\chi_1, \chi_2)$.

2.12.2. Epsilon factors. We recall that $\psi$ denotes the standard additive character on $\mathbb{R}$ (resp. $\mathbb{C}$) and $dx$ is the self-dual Haar measure. The epsilon factor attached to a multiplicative character is described in $[59, 3.2]$. A unitary character of $\mathbb{R}^\times$ can be written $||^\kappa \sgn^\kappa$ with $t \in \mathbb{R}$ and $\kappa = 0, 1$; then $\epsilon(s, \sgn^\kappa, \psi) = i^\kappa$. A unitary character of $\mathbb{C}^\times$ can be written $||^\kappa (z/|z|)^\kappa$ with $t \in \mathbb{R}$ and $\kappa \in \mathbb{Z}$; if $\kappa \leq 0$ then $\epsilon(s, (z/|z|)^\kappa, \psi) = i^{-\kappa}$; if $\kappa \geq 0$ we can use the contragredient to find that $\epsilon(s, (z/|z|)^\kappa, \psi) = i^\kappa$. Thus in all cases:

\begin{equation}
\epsilon(s, (z/|z|)^\kappa, \psi) = i^{\kappa}.
\end{equation}

As an exercise let us compute the epsilon factor of discrete series $\pi$ of weight $k \geq 2$. By definition the epsilon factors are inductive in degree zero. This implies their uniqueness and, at least in principle, a way to compute them. The induced representation of $1_\mathbb{C}$ is $1_\mathbb{R} \boxplus \sgn$. Thus we compute that

\begin{equation}
\epsilon(\pi, \psi) = \epsilon(1_\mathbb{R}, \psi) \epsilon(\sgn, \psi) \epsilon(\chi, \psi) \epsilon(1_\mathbb{C}, \psi)^{-1} = i^k,
\end{equation}

since $\chi(z) = (z\bar{z})^t(z/|z|)^k$ for some $t \in \mathbb{R}$.

More generally if $\pi$ (resp. $\pi'$) is a discrete series representation of weight $k$ (resp. $\kappa$), then

\begin{equation}
L(s, \pi \otimes \pi') = \Gamma_{\mathbb{C}}(s + \frac{k + \kappa}{2} - 1) \Gamma_{\mathbb{C}}(s + \frac{|k - \kappa|}{2}),
\end{equation}

where $\kappa$ is a non-negative integer.
and \( \epsilon(\pi \otimes \pi', \psi) = (-1)^{\max(k, \kappa)} \). This may be seen by decomposing the Weil representation \( \sigma(\pi) \otimes \sigma(\pi') \) in the direct sum of two representations of weight \( k + \kappa - 1 \) and \( \max(k - \kappa, \kappa - k) + 1 \), respectively.

3. Voronoï summation

Let \( S \) be a finite set of places of \( F \) including all the places where \( \psi \) ramifies and all the archimedean places. We also fix \( T \) a finite set of places disjoint from \( S \). Let \( \pi = \otimes_v \pi_v \subset \mathcal{A}_{\text{cusp}} \) be an irreducible automorphic cuspidal representation of \( \text{GL}_2(\mathbb{A}) \). We assume that \( \pi \) is unramified outside \( S \cup T \).

Denote by \( w^{ST}_0 = \prod_{v \notin S \cup T} w_v \) the unramified Kirillov function of \( \pi^{ST} = \otimes_v \pi_v \). Then \( \tilde{w}^{ST}_0(y) = \omega^{ST}(y)^{-1} w^{ST}_0(y) \) is the unramified Kirillov function of \( \tilde{\pi}^{ST} \) (see §2.11). The following may be viewed as a \textit{primitive version} of the Voronoï summation formula.

**Theorem 3.1.** For all \( v \in S \cup T \), let \( w_v \in C_c^\infty(F_v^\times) \) and let \( w := \prod_{v \in S \cup T} w_v \cdot w^{ST}_0 \) which is a global Kirillov function in \( K(\pi, \psi) \). Let \( \zeta \in \mathbb{A}_v^\times \) be such that \( \zeta_v = 1 \) for all \( v \in S \) and \( v(\zeta) \leq 0 \) for all \( v \notin S \). Then

\[
(3.1) \quad \sum_{\gamma \in F_v^\times} \psi^S(\gamma \zeta) w(\gamma) = \omega(\zeta) \sum_{\gamma \in F_v^\times} \psi^S(\gamma \zeta^{-1}) (\rho(k)w)(\gamma \zeta^{-1}),
\]

where \( k \in G(\mathbb{A}) \) is such that \( k_v = w \) for all \( v \in S \), \( k_v = \begin{pmatrix} -\zeta^{-1} & 0 \\ \zeta^{-1} & 1 \end{pmatrix} \) for all \( v \in T \) and \( k_v = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \) for all \( v \notin S \cup T \).

**Remarks.**

(1) For almost all places \( v \), \( \pi_v, \psi_v \), and \( w_v = w_{ov} \) are unramified, while \( \zeta_v \in \mathfrak{o}_v^\times \) is a unit. Without loss of generality we may further assume that \( \zeta_v = 1 \) for all such unramified places \( v \). Indeed the value of the left and right hand side of the identity remains the same for all \( \zeta_v \in \mathfrak{o}_v^\times \).

(2) We investigate the places \( v \in S \) in the following Section 4. Above the places \( v \in T \), it could happen that \( \pi_v \) is ramified and \( v(\zeta) < 0 \); the analysis of this situation is taken up in Section 5.

The remaining of this section is devoted to the proof of Theorem 3.1. In the subsequent sections we would like to state a formula which depends as little as possible on the choice of vector \( \phi \in \pi \) (although the proof may involve such a choice), and which would rather depend directly on the invariants attached to \( \pi \).

The starting point of the proof is the following identity.

**Lemma 3.2.** Let \( \zeta \in \mathbb{A}_v^S \subset \mathbb{A} \). For all \( \phi \in \mathcal{A}_{\text{cusp}}, \)

\[
(3.2) \quad \sum_{\gamma \in F_v^\times} \psi^v(\gamma \zeta) W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\right) = \sum_{\gamma \in F_v^\times} \tilde{W}_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}\right)\right).
\]

**Proof.** Define \( A := \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \) in \( G(\mathbb{A}_v^S) \subset G(\mathbb{A}) \). It is easily seen that the left-hand side is equal to \( \tilde{\phi}(A) \). Similarly the right-hand side is equal to \( \tilde{\phi}(A') = \phi(A) \). \( \square \)
3.1. **Local vectors.** The next step is to choose a suitable vector $\phi \in \pi$. The vector $\phi = \otimes_v \phi_v$ will be a pure tensor where the local components are chosen as follows.

(i) Above unramified places $v \notin S \cup T$, let $\phi_v = \otimes_v \phi_{ov}$ be a non-zero newvector.

(ii) For all places $v \in S \cup T$, Lemma 2.1 implies that there exists a (unique) Whittaker function $W_v \in W(\pi_v, \psi_v)$ such that

$$w_v(y) = W_v\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right), \quad y \in F_v^\times.$$  

Let $\phi_v$ be the corresponding vector.

We shall abbreviate $W$ for $W_{\phi}$, $W_{\infty}$ for $W_{\phi_{\infty}}$, $W_v$ for $W_{\phi_v}$, $W_{\phi_v}$ for $W_{\phi_{ov}}$. We adopt similar conventions for the dual Whittaker functions, and write $\tilde{W}$, $\tilde{W}_{\infty}$, $\tilde{W}_v$ and $\tilde{W}_{\phi_v}$.

The left-hand side of (3.2) coincides with the left-hand side of Theorem 3.1. Thus it remains to compute

$$\tilde{W}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} A^\gamma\right) = W_S\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} w\right) \cdot \prod_{v \notin S} W_v\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} w\left(\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}\right)\right),$$

which is achieved place by place in the following subsections and will conclude the proof.

3.2. **Places** $v \in S$. We write $W_S\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} w\right) = (\rho(w)w_S)(\gamma)$ with the action of $w$ of the Kirillov model. This is a generalized Bessel transform as will be detailed in the following Section 4.

3.3. **Places** $v \in T$. We have

$$w\left(\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \zeta^{-1} \\ 0 & \zeta \end{pmatrix} \begin{pmatrix} -\zeta^{-1} & 0 \\ \zeta^{-1} & 1 \end{pmatrix},$$

Thus we can write $W_v\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} w\left(\begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}\right)\right)$ as

$$\omega_v(\zeta)\psi(\gamma\zeta^{-1})W_v\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\zeta^{-1} & 0 \\ \zeta^{-1} & 1 \end{pmatrix}\right).$$

3.4. **Unramified places.** Suppose now that $v \notin S \cup T$. Then $v(\zeta) \leq 0$ and $w_v$ is unramified. Thus the previous display simplifies to $\omega_v(\zeta)\psi(\gamma\zeta^{-1})w_{ov}(\gamma\zeta^{-2})$.

4. **Local transforms**

This section provides some background on the local transforms that appear in the Voronoï formula. In Theorem 3.1 the transform appears as the action on the Kirillov model of $w \in G$ (and of more general elements). We shall make clear the relationship with the classical Hankel transform at the archimedean places. At nonarchimedean places one can establish similar properties with the $p$-adic Bessel function $j_{\pi}$. We also
explain the equivalent formulation in terms of epsilon factor appearing in the functional equation.

4.1. Hankel transform. For a complex \( \nu \in \mathbb{C} \), the Hankel transform or Bessel transform is defined on functions on \((0, \infty)\) as

\[
(4.1) \quad \mathcal{H}_\nu f(y) := \int_0^\infty f(x) J_\nu(xy) xd\nu.
\]

The Hankel transform may be viewed as a generalization of the Fourier transform since \( J_{1/2}(y) = \left( \frac{2}{\pi y} \right)^{1/2} \sin(y) \). Therefore it is also sometimes referred to as Fourier–Bessel transform. It appears in different problems in physics and engineering, see e.g. [14, §15].

The representation theoretic interpretation of the Hankel transform is the action of \( \rho_\pi(w) \) on the Kirillov model [13, Theorem 4.1]. Let \( \pi \) be a generic irreducible representation of \( G(\mathbb{R}) \) and \( \psi(x) = e^{2\pi ix} \) be the standard additive character. For all \( w \in \mathcal{C}_c^\infty(\mathbb{R}^\times) \subset K(\pi, \psi) \),

\[
(4.2) \quad \rho_\pi(w) y(\pi) = \int_{\mathbb{R}^\times} \omega_\pi(x)^{-1} j_\pi(xy) w(x) \frac{dx}{x}.
\]

Here \( j_\pi \) is the Bessel function attached to \( \pi \), which may be expressed in terms of the classical \( I, J, K \)-Bessel functions as follows.

**Proposition 4.1.** For all characters \( \chi \) of \( \mathbb{R}^\times \), \( j_{\chi \otimes \pi}(y) = \chi(y) j_\pi(y) \). We assume below that \( \pi \) has central character \( \omega_\pi = 1 \) or \( \omega_\pi = \text{sgn} \).

If \( \pi \) is a discrete character representation of weight \( k \geq 2 \), then the central character is \( \text{sgn}^k \) and

\[
(4.3) \quad j_\pi(y) = \begin{cases} 
2\pi i^k \sqrt{y} J_{k-1}(4\pi \sqrt{y}), & y > 0, \\
0, & y < 0.
\end{cases}
\]

If \( \pi \) is a principal series representation with trivial central character \( \omega_\pi = 1 \) and Langlands parameters \( \{-ir, ir\} \), then \( \epsilon = \epsilon(s, \pi, \psi) = \pm 1 \) and

\[
(4.4) \quad j_\pi(y) = \begin{cases} 
\frac{i\pi \sqrt{y}}{\sinh(\pi r)} \left( J_{2ir}(4\pi \sqrt{y}) - J_{-2ir}(4\pi \sqrt{y}) \right), & y > 0, \\
4\epsilon \cosh(\pi r) \sqrt{|y|} K_{2ir}(4\pi \sqrt{|y|}), & y < 0.
\end{cases}
\]

If \( r = 0 \), then \( j_\pi(y) = -2\pi \sqrt{y} Y_0(4\pi \sqrt{y}) \) for \( y > 0 \) and \( j_\pi(y) = 4\epsilon \sqrt{|y|} K_0(4\pi y) \) for \( y < 0 \).

If \( \pi \) is a principal series representation with non-trivial central character \( \omega_\pi = \text{sgn} \) then \( \epsilon(s, \pi, \psi) = i \). If \( \pi = \text{sgn} \ | |^i r \oplus | |^{-ir} \), then

\[
(4.5) \quad j_\pi(y) = \begin{cases} 
\frac{i\pi \sqrt{y}}{\cosh(\pi r)} \left( J_{2ir}(4\pi \sqrt{y}) + J_{-2ir}(4\pi \sqrt{y}) \right), & y > 0, \\
-4\sinh(\pi r) \sqrt{|y|} K_{2ir}(4\pi \sqrt{|y|}), & y < 0.
\end{cases}
\]

If \( \pi = | |^i r \oplus \text{sgn} \ | |^{-ir} \) then we only need to replace \( r \) by \( -r \) in the above formula or twist by \( \otimes \text{sgn} \). The effect is to change \( j_\pi(y) \) into \( -j_\pi(y) \) if \( y < 0 \).
If π is a complementary series representation with Langlands parameters \{-r, r\} with \(-\frac{1}{2} < r < \frac{1}{2}\), then \(\omega_{\pi} = 1\) and

\[
(4.6) \quad j_{\pi}(y) = \begin{cases} 
\frac{-\pi \sqrt{y}}{\sin(\pi r)} \left( J_{2r}(4\pi \sqrt{y}) - J_{-2r}(4\pi \sqrt{y}) \right), & y > 0, \\
4 \cos(\pi r) \sqrt{|y|} K_{2r}(4\pi \sqrt{|y|}), & y < 0.
\end{cases}
\]

**Remarks.**

(i) If \(n\) is an integer, then \(J_n = (-1)^n J_{-n}\).

(ii) Recall the classification of irreducible representations from §2.12. We may unify the above formulas in the case of trivial central character as follows. Let \(\nu = k - 1\) (resp. \(\nu = 2i\nu\), \(\nu = 2\nu\)) if \(\pi\) is a discrete series representation of even weight \(k \geq 2\) (resp. principal series \(\|\nu\| \|\nu\|^{-1}\), or complementary series \(\|\nu\| \|\nu\|^{-1}\)). Then for all \(y > 0\),

\[
j_{\pi}(y) = \frac{-\pi \sqrt{\nu}}{\sin(\frac{\pi \nu}{2})} \left( J_{\nu}(4\pi \sqrt{\nu}) - J_{-\nu}(4\pi \sqrt{\nu}) \right).
\]

If \(y < 0\), the same formula holds with the \(J\)-Bessel function needs to be replaced by the \(I\)-Bessel function \(I_{\nu}\) and a multiplicative factor \(\epsilon(\pi)\).

(iii) Similarly if \(\pi\) has non-trivial central character, then in all cases

\[
j_{\pi}(y) = \frac{i\pi \sqrt{\nu}}{\sin(\frac{\pi \nu}{2})} \left( J_{\nu}(4\pi \sqrt{\nu}) + J_{-\nu}(4\pi \sqrt{\nu}) \right).
\]

**Proof.** See Cogdell–Piatetskii-Shapiro [13, Proposition 6.1]. We only need to outline the relationship between our notation and convention [13]. We use the Weyl element \(w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) in the definition (4.2) instead the element \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) used in [13].

For principal series representations the formula in [13] is in terms of the \(I\)-Bessel function. To express \(j_{\pi}\) in term of the \(K\)-Bessel function we use the following identity [39, (B.34)]

\[
K_{\nu}(z) = \frac{-\pi}{2 \sin(\pi \nu)} \left( J_{\nu}(z) - I_{-\nu}(z) \right).
\]

The calculation is analogous for complementary series.

There are differences for notation for discrete series representation, namely \(\sigma(d)\) in [13] corresponds to discrete series of weight \(2d\) (and trivial central character). Our proposition above also covers the case of discrete series of odd weight.

For principal series representations with non-trivial central character \(\omega_{\pi} = \text{sgn}\) the formula is in Harcos–Michel [34] in terms of the \(Y\)-Bessel function. To express \(j_{\pi}\) in terms of the \(J\)-Bessel function we use the following identity:

\[
Y_{\nu}(z) = \frac{1}{\sin(\pi \nu)} \left( J_{\nu}(z) \cos(\pi \nu) - J_{-\nu}(z) \right).
\]
We note that the action of $G(\mathbb{R})$ on its Kirillov model is completely determined by its central character $\omega_\pi$ and the action $\rho_\pi(w)$ of the Weyl element. Hence the representation $\pi$ is uniquely determined by $\omega_\pi$ and $j_\pi$.

**Example 4.2** (Parseval identity for the Hankel transform). Let $f$ and $g$ be Schwartz functions on $(0, \infty)$, let $v$ be such that

$$v(y) = \begin{cases} f(y) & \text{if } y > 0, \\ 0 & \text{if } y < 0 \end{cases}$$

and similarly for $w$ in terms of $g$. Consider the principal series representations $\pi_1 := ||^r \boxplus ||^{-r}$ and $\pi_2 := \text{sgn} ||^r \boxplus ||^{-r}$. We shall view $v, w$ as vectors in the respective Kirillov models $\mathcal{K}(\pi_1, \psi)$ and $\mathcal{K}(\pi_2, \psi)$. We form the sum

$$\sinh(\pi r) \langle \rho_{\pi_1}(w)v, w \rangle + \cosh(\pi r) \langle \rho_{\pi_2}(w)v, w \rangle$$

which is proportional to $\int_0^\infty \mathcal{H}_{2ir} v \overline{w}$. Since the respective actions by $\rho_{\pi_1}(w)$ and $\rho_{\pi_2}(w)$ are self-adjoint, we deduce the equality with $\int_0^\infty v \mathcal{H}_{-2ir} \overline{w}$. We have recovered the following Parseval identity for the Hankel transform (4.1):

$$\int_0^\infty f(y) \overline{g(y)} y dy = \int_0^\infty (\mathcal{H}_\nu f)(y) \overline{(\mathcal{H}_\nu g)(y)} y dy.$$

**Example 4.3** (Inversion of the Hankel transform). Similarly since $w^2 = e$, we can also recover the inversion formula $\mathcal{H}_\nu(\mathcal{H}_\nu f) = f$ for the Hankel transform.

An alternative approach is in the work of Goodman–Wallach [32] where the Jacquet-Whittaker functional appears as a linear combination of certain transforms of conical functionals and the Hankel transform appears in the line model of the representation.

As pointed out by the referee, the analogous formulas for the complex place are due to Bruggemann-Motohashi in their study of the Kuznetsov formula. We refer to [6] and [13] for details.

### 4.2. $p$-adic analogues.

There is an analogous theory at the non-archimedean places. To an irreducible generic representation $\pi$ of $G(F_v)$ one can again attach a Bessel function $j_\pi$ on $F_v^\times$. The definition and properties of $j_\pi$ may be found in the articles by Soudry [58] and Baruch [3]. Here we only need to recall [58, Lemma 4.2] that for any Kirillov function $w \in \mathcal{C}^\infty_c(F_v^\times)$,

$$\rho_\pi(w)(y) = \int_{F_v^\times} \omega_\pi(t)^{-1} j_\pi(ty) w(t) d^\times t.$$  \hspace{1cm} (4.7)

Thus the Hankel transform over a non-archimedean field again represents the action of $\rho_\pi(w)$ on the Kirillov model. Over non-archimedean places and for ramified vectors the situation is even richer.

**Lemma 4.4.** Let $f \in \mathcal{C}^\infty_c(F_v^\times) \subset \mathcal{K}(\pi, \psi)$. The following holds for all $y, \zeta \in F^\times$,

$$\rho \left( \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \right) f(y) = \psi(y) \cdot \int_{F_v^\times} \omega_\pi^{-1}(t) j_\pi(-ty) \psi(t) f(\zeta t) d^\times t.$$  \hspace{1cm} (4.8)
Proof. We have the following Bruhat decomposition, see also (2.6):

$$
\begin{pmatrix}
1 & 0 \\
\zeta & 1 
\end{pmatrix} = \begin{pmatrix}
\zeta^{-1} & 1 \\
0 & \zeta 
\end{pmatrix} w \begin{pmatrix}
1 & \zeta^{-1} \\
0 & -1 
\end{pmatrix}.
$$

Thus, the strategy is that $\rho\left(\begin{pmatrix}
\zeta & 0 \\
\zeta & 1 
\end{pmatrix}\right)f$ can be obtained in terms of the above action of $\rho(w)$ in the Kirillov model.

By construction, the action of the Borel on the Kirillov model yields

$$
\rho\left(\begin{pmatrix}
1 & \zeta^{-1} \\
0 & -1 
\end{pmatrix}\right)f(y) = \omega_{\pi}(-1)\psi(-\zeta^{-1}y)f(-y).
$$

We then deduce from (4.7) that $\rho\left(\begin{pmatrix}
1 & \zeta^{-1} \\
0 & -1 
\end{pmatrix}\right)f(y)$ is equal to

$$
\omega_{\pi}(-1) \int_{F^\times} \omega_{\pi}^{-1}(t)j_{\pi}(ty)\psi(-\zeta^{-1}t)f(-t)d^\times t
$$

which we rearrange as follows after the change of variable $t \mapsto -\zeta t$:

$$
\omega_{\pi}(\zeta)^{-1} \int_{F^\times} \omega_{\pi}^{-1}(t)j_{\pi}(-\zeta yt)\psi(t)f(\zeta t)d^\times t.
$$

We then treat similarly the action of $\begin{pmatrix}
\zeta^{-1} & 1 \\
0 & \zeta 
\end{pmatrix}$ and combining with the decomposition (4.9) we obtain overall:

$$
\rho\left(\begin{pmatrix}
1 & 0 \\
\zeta & 1 
\end{pmatrix}\right)f(y) = \psi(\zeta^{-1}y) \int_{F^\times} \omega_{\pi}^{-1}(t)j_{\pi}(-\zeta^{-1}yt)\psi(t)w_o(\zeta t)d^\times t.
$$

The proposition now follows by replacing $y$ by $\zeta y$. \hfill \Box

If we assume that $f$ is $\sigma^\times$-invariant, then $\rho\left(\begin{pmatrix}
\zeta & 0 \\
\zeta & 1 
\end{pmatrix}\right)f$ depends only on $v(\zeta)$. It is then convenient to introduce the matrices $k_i := \begin{pmatrix}
\varpi & 0 \\
\varpi_i & 1 
\end{pmatrix}$ for all $i \geq 0$, and then we only need to consider $\rho(k_i)f$. See [61,62] for examples of computations of $\rho(k_i)f$ when $f$ is the newvector.

If $f' = \rho(k_i)f$, then we have

$$
\int_{F^\times} |f'(y)|^2 d^\times y = \int_{F^\times} |f(y)|^2 d^\times y,
$$

which follows from the fact that $\pi$ is unitary and preserves the above invariant scalar product on the Kirillov model. In fact more generally the map

$$
w \mapsto \int_{F^\times} \omega_{\pi}(t)^{-1}j_{\pi}(ty)w(t)d^\times t
$$

is always an isometry of $L^2(F^\times, d^\times y)$. 

4.3. Epsilon factors and functional equation. Finally we work out the relationship of the Hankel transform with epsilon factors. It can be verified that the Mellin transform of the Bessel function \( j_\pi \) coincides with the twisted gamma factors. In higher rank this plays an important role in the work of Cogdell–Shahidi on the stabilization of \( \gamma \)-factors with application to functoriality.

**Lemma 4.5.** For all multiplicative characters \( \chi \) and \( s \in \mathbb{C} \) with \( \Re s \) large enough,

\[
\int_{F^\times} (\omega_\pi \chi)^{-1}(y) |y|^\frac{1}{2}-s \ j_\pi(y) d^\times y = \chi(-1) \gamma(s, \pi \times \chi, \psi).
\]

**Proof.** This follows from the Jacquet–Langlands functional equation, see e.g. [12, 61]. □

The Voronoï formula is related to the global functional equation of \( L \)-functions twisted by multiplicative characters. The local transform relates to the local functional equation as follows.

**Lemma 4.6.** For all \( w \in \mathcal{C}_c^\infty(F^\times) \), multiplicative character \( \chi \) and \( s \in \mathbb{C} \) with \( \Re s \) large enough,

\[
\int_{F^\times} (\rho_\pi(w)(y)(\omega_\pi \chi)^{-1}(y) |y|^s \ j_\pi(y) d^\times y = \chi(-1) \gamma(1-s, \pi \times \chi, \psi) \int_{F^\times} w(y) \chi(y) |y|^\frac{1}{2}-s d^\times y.
\]

**Proof.** This follows from the previous Lemma 4.5. Indeed we can relate directly the identity of the lemma with the Hankel transforms (4.2) and (4.7). See also [38, Lemma 5.2].

**Example 4.7.** For the sake of consistency we illustrate the identity in Lemma 4.5 in the case of principal series representations. We begin with a trivial central character in which case we may assume that \( \pi = ||^r \boxplus ||^{-r} \). The left-hand side in Lemma 4.5 splits into two terms:

\[
\int_0^\infty j_\pi(y) y^{\frac{1}{2}-s} d^\times y + \chi(-1) \int_{-\infty}^0 j_\pi(y) |y|^\frac{1}{2}-s d^\times y.
\]

According to [33, §6.56],

\[
\int_0^\infty J_\nu(y) y^{s} d^\times y = 2^{s-1} \frac{\Gamma\left(\frac{\nu+s}{2}\right)}{\Gamma\left(1 + \frac{\nu-s}{2}\right)}
\]

\[
\int_0^\infty K_\nu(y) y^{s} d^\times y = 2^{s-2} \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right).
\]

Thus one can compute using (4.4) that (4.11) is equal to \((2\pi)^{2s-2}\) times

\[
\frac{i\pi}{\sinh(\pi r)} \left( \frac{\Gamma(1-s+ir)}{\Gamma(s+ir)} - \frac{\Gamma(1-s-ir)}{\Gamma(s-ir)} \right) + 2\chi(-1) \cosh(\pi r) \Gamma\left(\frac{1}{2} - s + ir\right) \Gamma\left(\frac{1}{2} - s - ir\right).
\]

\(^1\)Note that in the notation of [38], \( \tilde{w}(y) = \omega_\pi(y)^{-1} \rho_\pi(w(y)). \)

\(^2\)There is a mistake in some older editions of [33] in the power of 2 of the Mellin transform of the \( K \)-Bessel function. See the newest edition of [33] and [http://dlmf.nist.gov/10.43](http://dlmf.nist.gov/10.43) for the correct version.
A trigonometric calculation gives that it is equal to the right-hand side in Lemma 4.5, that is:

\[ \chi(-1)\gamma(s, \pi \times \chi, \psi) = \frac{L(1 - s, \pi \times \chi^{-1})}{L(s, \pi \times \chi)}. \]

Note that \( \epsilon(s, \pi \times \chi, \psi) = \chi(-1). \) For instance if \( \chi(-1) = -1 \) then

\[ \frac{L(1 - s, \pi \times \chi^{-1})}{L(s, \pi \times \chi)} = \frac{\Gamma_R(2 - s + ir)\Gamma_R(2 - s - ir)}{\Gamma_R(1 + s + ir)\Gamma_R(1 + s - ir)} = \pi^{2s - 1} C^-(s), \]

where \( C^-(s) \) is the notation from [48, Appendix A].

Now consider the case of \( \pi = \text{sgn} \| i \| \). Then the left-hand side of Lemma 4.5 is

\[ \int_{-\infty}^{\infty} j_\pi(y)y^{\frac{1}{2} - s}d^\times y - \chi(-1) \int_{-\infty}^{0} j_\pi(y)|y|^{\frac{1}{2} - s}d^\times y. \]

One can compute that (4.13) is equal to \((2\pi)^{2s - 2}\) times

\[ \frac{i\pi}{\cosh(\pi r)} \left( \frac{\Gamma(1 - s + ir)}{\Gamma(s + ir)} + \frac{\Gamma(1 - s - ir)}{\Gamma(s - ir)} \right) + 2\chi(-1) \sinh(\pi r)\Gamma(\frac{1}{2} - s + ir)\Gamma(\frac{1}{2} - s - ir). \]

it is equal to the right-hand side in Lemma 4.5, that is:

\[ \chi(-1)\gamma(s, \pi \times \chi, \psi) = i \left\{ \begin{array}{ll}
\frac{\Gamma_R(1 - s + ir)\Gamma_R(2 - s - ir)}{\Gamma_R(1 + s - ir)\Gamma_R(1 + s + ir)} & \text{if } \chi(-1) = 1, \\
\frac{\Gamma_R(1 + s + ir)\Gamma_R(2 - s - ir)}{\Gamma_R(1 - s - ir)\Gamma_R(1 - s + ir)} & \text{if } \chi(-1) = -1.
\end{array} \right. \]

5. Formula with newvectors

In this section we assume further that for all \( v \in T, w_v = w_{ov} \) is the newvector.

5.1. Definitions. Let \( v \) be a non-archimedean place. For all \( i \geq 1 \) we let

\[ K[p_v^i] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{o}_v), \ c \in p_v^i, \ d \in 1 + p_v^i \right\}. \]

For \( i = 0 \) the convention is that \( K[\mathfrak{o}] = GL_2(\mathfrak{o}). \)

There exists a least nonnegative integer \( c = c(\pi_v) \) such \( \pi_v \) has a non-trivial vector invariant by \( K[p_v^c] \). The space of \( K[p_v^c] \)-invariant vectors is one-dimensional and by definition consists of the newvectors of \( \pi_v \). The integer \( c(\pi_v) \) is the conductor of \( \pi_v \). We have \( c(\pi_v) = c(\pi_v) \) and \( c(\pi_v) = 0 \) if and only if \( \pi_v \) is unramified. The dimension of the subspace of vectors invariant by \( K[p_v^c] \) is equal to \( i + 1 - c(\pi_v) \) for all \( i \geq c(\pi_v) \).

Automorphic forms are viewed in the classical framework as functions on the upper-half plane. The theory of newforms is developed in [1]. A representation theoretic interpretation is given in [9, 15]. There is a bijective correspondence between newforms and automorphic representations. A newform generates an automorphic representations and conversely an automorphic representations contain an unique line of newforms.

An equivalent definition of the conductor is given by the epsilon factor. Namely we have

\[ \epsilon(s, \pi_v, \psi_v) = \epsilon\left(\frac{1}{2}, \pi_v, \psi_v\right)q_v^{(2n(\psi_v)+c(\pi_v))(\frac{1}{2} - s)}. \]
Here $n(\psi_v)$ is the conductor of the additive character $\psi_v$, namely the smallest integer $n \in \mathbb{Z}$ such that $\psi_v$ is trivial on $p_v^n$.

We denote by $W_o \in \mathcal{W}(\pi_v, \psi_v)$ the unique newvector which is normalized by the condition $W_o(e) = 1$. The following is proven in Jacquet–Langlands [41], see also Atkin–Lehner [1, §4] and [27, (6.37)]. By Mellin inversion this is equivalent to the Casselman–Shintani formula for $W_o \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right)$.

**Proposition 5.1.** If $\psi_v$ and $\chi_v$ are unramified, then

$$L(s, \pi_v \times \chi_v) = \Psi(s, W_o, \chi_v).$$

5.2. **Central character.** In this subsection we make a few observations concerning the central character $\omega_v$ of $\pi_v$ and how it relates to newvectors. Consider the Hecke congruence subgroups

$$K_0[p_v^i] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathfrak{o}_v), \ c \in p_v^i \right\}$$

which contains the subgroup $K[p_v^i]$ defined in (5.1). In the theory of newvectors there is a slight difference between working with $K[p_v^i]$ and with $K_0[p_v^i]$ which is clarified by the following.

**Lemma 5.2.** Let $i \geq 1$ and $\phi$ be a vector in $\pi_v$ invariant by $K[p_v^i]$. Then

$$\rho(k_0)\phi = \omega_v(d)\phi, \quad \text{for all } k_0 = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in K_0[p_v^i].$$

**Proof.** This is [9, Eq.(1.3)]. (there is a typo there where “a”’ should read “d”’.) The idea of the proof is that since $i \geq 1$, we can write $k_0 = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} k$ with $k \in K[p_v^i]$, from which the statement follows.

This applies in particular to $i = c(\pi_v)$ and $\phi$ the newvector. Choosing furthermore $k_0 \in K[p_v^{c(\pi_v)}]$, we deduce that $\omega_v(d) = 1$ for all $d \in 1 + p_v^{c(\pi_v)}$. In other words we have recovered the fact that $c(\omega_v) \leq c(\pi_v)$.

5.3. **A rewriting.** Notation being the same as in the beginning of Section 3, we assume that $w_v = w_{ov}$ is the newvector for all $v \not\in S$, and we rewrite the Voronoï formula in the following way.

**Proposition 5.3.** For all $v \in S$ let $w_v \in C_c^\infty(F_v^\times)$ and $w := \prod_{v \in S} w_v \cdot w_o^S$ which is a global Kirillov function in $\mathcal{K}(\pi, \psi)$. Let $\beta \in F_v^\times$ be such that $\nu(\beta) \leq 0$ for all $v \not\in S$ and let $u \in \mathfrak{d}_v^S := \prod_{v \not\in S} \mathfrak{o}_v^\times$. Then

$$\sum_{\gamma \in F_v^\times} \psi^S(u\beta\gamma)w(\gamma) = \omega_{\pi}(u) \sum_{\gamma \in F_v^\times} \psi^S(u^{-1}\beta\gamma)(\rho(k)w)(\gamma).$$

where $k \in G(\mathbb{A})$ is such that $k_v = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} w$ for all $v \in S$, $k_v = \begin{pmatrix} -u^{-1} & 0 \\ \beta^{-1} & 1 \end{pmatrix}$ for all $v \not\in S$. 


Proof. We write \( \zeta := t\beta u \) with \( t \in F_S^\times \) chosen in such a way that \( \zeta_v = 1 \) for all \( v \notin S \). We can apply Theorem 3.1 in which the LHS coincides with \( \sum_{\gamma \in F^\times} \psi^S(\gamma)t\beta u_w(\gamma) \). We make the change of variable \( \gamma \mapsto \gamma \beta^2 \) in the RHS from which the identity follows after some calculations. Note that \( \omega_\pi(\beta) = 1 \) and \( \omega_\pi(\zeta) = \omega_\pi(u)^{-1}\omega_\pi(\beta_0)^{-1} \).

If \( S \) is large enough such that \( o_S \) is principal, then we can recover conversely the statement of Theorem 3.1. Indeed \( A^\times = F_S^\times F^\times \widehat{\mathfrak{o}}^S \times \) and thus any \( \zeta \in A^\times \) may be written in the form \( \zeta = t\beta u \) for some \( t \in F_S^\times \), \( \beta \in F^\times \) and \( u \in \widehat{\mathfrak{o}}^S \). The decomposition is unique up to units \( \mathfrak{o}_S \).

The advantage of rewriting the formula as above is that we can read off the significance of each term clearly. The parameter \( \beta \in F^\times \) may be thought of as the conductor of the additive twist. The Kirillov function \( w_S = \prod_{v \in S} w_v \) play the role of test functions. The term \( \rho(k)w \) encodes a local transform above places \( v \in S \). And the parameter \( u \in \widehat{\mathfrak{o}}^S \) affects mildly the size of each term.

**Corollary 5.4.** Assuming furthermore that \( \beta \in F^\times \) is such that \( v(\beta) + c(\pi_v) \leq 0 \) for all \( v \notin S \), the identity (5.6) holds with \( k \in G(A) \) such that \( k_v = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} w \) for all \( v \in S \), and \( k_v = e \) for all \( v \notin S \).

The condition is for example satisfied if \( \pi_v \) is unramified for all \( v \notin S \).

### 6. Atkin–Lehner operators

We now take into account the Atkin–Lehner operators. The theory of Atkin–Lehner operators is further developed in [2, 49] and a representation theoretic interpretation over \( \mathbb{Q} \) has been given in [25]. In this section we want to clarify the local and global aspects of the construction of the operators. We take the opportunity to develop these facts in details for future reference and since it may be of independent interest.

**6.1. Local Atkin–Lehner operators.** A significant part of Atkin–Lehner theory is local in nature. Let \( v \) be a non-archimedean place and \( c = c(\pi_v) \) be the conductor of \( \pi_v \).

It will be convenient to use the notation \( a_v := \begin{pmatrix} \omega_v^c & 0 \\ 0 & 1 \end{pmatrix} \). The matrix \( wa_v \) normalizes \( K_0[p_v^\mathfrak{o}] \).

Let \( \phi_o \) be a newvector of \( \pi_v \). Then the vector \( \phi' = \rho(wa_v)\phi_o \) is stabilized by \( K_0[p_v^\mathfrak{o}] \). However it is not a newvector in general as it may not be invariant by \( K[p_v^\mathfrak{o}] \) which is not normalized by \( wa_v \). In fact it is not difficult to verify that

\[
\rho(k_0)\phi' = \omega_v(a)\phi', \quad \text{for all } k_0 = \begin{pmatrix} a & * \\ * & * \end{pmatrix} \in K_0[p_v^\mathfrak{o}],
\]

which is to be compared with Lemma 5.2 for the newvector \( \phi_o \).

The subgroup \( K[p_v^\mathfrak{o}] \) is stable under the anti-involution \( k \mapsto a_v^{-1}k^t a_v \). Because of the identity (2.2), \( K[p_v^\mathfrak{o}] \) is also stable under the involution \( k \mapsto \det(k)^{-1}a_v^{-1}w^twa_v \).
Thus the vector $\phi'$ is a newvector for the representation $\pi_v \otimes \omega_v^{-1}$ which is isomorphic to $\tilde{\pi}_v$. We can see this from (6.1) as well since we note that $\omega_v(\det k) = \omega_v(ad)$ for all $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K[\mathfrak{p}_v^e]$. 

To make a precise sense of this we need to choose a particular vector space for the representations $\pi_v$ and $\tilde{\pi}_v$. We fix an additive character $\psi_v$ and work with generic representations and their Whittaker models. The local Atkin–Lehner operators will depend on $\psi_v$.

The effects on the Whittaker model of twisting by a character and taking the contragredient has been reviewed in §2.5. This leads us to making the following definition.

**Definition 6.1.** The Atkin–Lehner operator $AL : \mathcal{W}(\pi_v, \psi_v) \to \mathcal{W}(\tilde{\pi}_v, \psi_v)$ maps a Whittaker function $W$ for $\pi_v$ to the Whittaker function $AL(W)$ for $\tilde{\pi}_v$ such that:

$$AL(W)(g) := \omega_v(\det(g))^{-1}W(g\omega_v), \quad g \in G(F_v).$$

At the level of the Kirillov model the operator $AL : \mathcal{K}(\pi_v, \psi_v) \to \mathcal{K}(\tilde{\pi}_v, \psi_v)$ maps a Kirillov function $w$ to:

$$AL(w)(y) := \omega_v(\omega_v^{-1}y)^{-1}(\rho(ww)(\omega_v^{-1}y), \quad y \in F_v^\times.$$

Note that $AL$ is not $G(F_v)$-equivariant. The main properties are summarized in the following which follows from the considerations above.

**Lemma 6.2.** If $W_0 \in \mathcal{W}(\pi_v, \psi_v)$ is a newvector then $AL(W_0) \in \mathcal{W}(\tilde{\pi}_v, \psi_v)$ is a newvector. The Atkin–Lehner operator is an involution in the sense that $AL \circ AL$ corresponds to multiplication by $\omega_v(-1)$.

If $\pi_v$ is self-dual we can view $AL$ as an involution of $\mathcal{W}(\pi_v, \psi_v)$. Furthermore it can be verified that in this case $AL$ is independent of the choice of $\psi_v$.

### 6.2. Local root numbers

We assume in this subsection that $\psi_v$ is unramified. Let $c = \epsilon(\pi_v)$ and let $W_0 \in \mathcal{W}(\pi_v, \psi_v)$ be the normalized newvector with $W_0(e) = 1$.

The newvector $AL(W_0) \in \mathcal{W}(\tilde{\pi}_v, \psi_v)$ might not be normalized in general which is related to the local functional equation.

**Lemma 6.3.** Suppose $\psi_v$ is unramified and $W_0$ (resp. $w_0$) is the normalized Whittaker (resp. Kirillov) newvector for $\pi_v$. Then the following holds:

$$AL(W_0)(e) = AL(w_0)(1) = \epsilon_{1/2}(\pi_v, \psi_v).$$

**Remarks.**

(i) We have $|\epsilon(1/2, \pi_v, \psi_v)| = 1$ and $\epsilon(1/2, \pi_v, \psi_v)\epsilon(1/2, \tilde{\pi}_v, \psi_v) = \omega_v(-1)$ which is consistent with Lemma 6.2.

(ii) If $\pi_v$ is self-dual then $\epsilon(1/2, \pi_v, \psi_v) = \epsilon(1/2, \pi_v)$ is independent of $\psi_v$. We obtain that $AL(W_0) = \epsilon(1/2, \pi_v)W_0$. This is consistent with the observation above that $AL$ could be defined on $\pi_v$ independently of the choice of $\psi_v$, in which case we can say that $AL(\phi_v) = \epsilon(1/2, \pi_v)\phi_v$ for any newvector $\phi_v \in \pi_v$.

(iii) If $\omega_v$ is trivial the lemma is shown in [56, Theorem 3.2.2].
Proof. For all \( g \in G(F_v) \) we have

\[
AL(W_o)(g) = \omega_v(\det(g))^{-1}W_o(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g \mathfrak{a}_v).
\]

By a change of variable we deduce that for all \( s \in \mathbb{C} \) and \( \chi_v \) unramified,

\[
\Psi(1 - s, AL(W_o), \chi_v^{-1}) = \chi_v(\varpi_v)^{-\epsilon} q_v^{(s-\frac{1}{2})} \Psi(1 - s, W_o, \chi_v^{-1}).
\]

We apply the local functional equation and then the Proposition 5.1 to obtain that

\[
\Psi(1 - s, \tilde{W}_o, \chi_v^{-1}) = \gamma(s, \pi_v \times \chi_v, \psi_v) \Psi(s, W_o, \chi_v)
\]

\[
= \epsilon(s, \pi_v \times \chi_v, \psi_v)L(1 - s, \tilde{\pi}_v \times \chi_v^{-1}).
\]

Hence we have the identity

\[(6.3) \quad \Psi(1 - s, AL(W_o), \chi_v^{-1}) = \epsilon(\frac{1}{2}, \pi_v, \psi_v)L(1 - s, \tilde{\pi}_v \times \chi_v^{-1}).\]

We can again apply Proposition 5.1 to the left hand side because \( AL(W_o) \) is proportional to the newvector of \( \tilde{\pi}_v \) to obtain:

\[
\Psi(1 - s, AL(W_o), \chi_v^{-1}) = AL(W_o)(e)L(1 - s, \tilde{\pi}_v \times \chi_v^{-1}).
\]

By comparison with (6.3) this concludes the proof of the lemma. (the final result is independent of \( s \) and \( \chi_v \) as should be). \( \square \)

6.3. Global Atkin–Lehner operators. There is a difficulty that applying the Atkin– Lehner operators at a few places is a local operation that might not be compatible with the global invariance by \( G(F) \). This is because we are twisting by the local central characters \( \omega_v \), at some finite places. There are different ways to go around this problem and define global operators. Here we choose a flexible solution inspired from [57]. The solution is to define Atkin–Lehner operators depending on the choice of a global Hecke character with some prescribed ramification.

Therefore we begin by recalling some background and notation on Hecke Grössencharacters. A Hecke character is a continuous character \( \eta : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C} \). Attached to \( \eta \) are its archimedean part \( \eta_\infty \) which is a character of \( \mathbb{R}^\times \), its non-archimedean part \( \eta_f \) which is a character of \( (\mathfrak{o}_F/\mathfrak{m})^\times \) and its conductor \( \mathfrak{m} \) which is an integral ideal of \( \mathfrak{o}_F \). The character \( \eta_f \) is sometimes referred to as a numerical character. For all archimedean places \( v \), we have a sign \( \eta_v(-1) = \pm 1 \) which could be recorded in the form of the modulus at infinity.

Let \( I_\mathfrak{m} \) denote the group of fractional ideals of \( \mathfrak{o}_F \) prime to \( \mathfrak{m} \). The classical realization of \( \eta \) (which we denote with the same letter) is a character \( \eta : I_\mathfrak{m} \to S^1 \). For all prime ideals \( \mathfrak{p} \) prime to \( \mathfrak{m} \), the classical \( \eta(\mathfrak{p}) \) is equal to the adelic \( \eta(\varpi_\mathfrak{p}) \). It satisfies the relation

\[
\eta((\alpha)) = \eta_\infty(\alpha)^{-1}\eta_f(\alpha), \quad \forall \alpha \in \mathfrak{o}_F \text{ such that } (\alpha, \mathfrak{m}) = 1.
\]

We necessarily have \( \eta_\infty(\epsilon) = \eta_f(\epsilon) \) for all \( \epsilon \in \mathfrak{o}_F^\times \) and this is a restriction on the characters \( \eta_\infty \) that may occur as infinity types.

Conversely if \( \eta_f, \eta_\infty \) are such that \( \eta_\infty(\epsilon) = \eta_f(\epsilon) \) for all \( \epsilon \in \mathfrak{o}_F^\times \), then there exist Hecke characters with archimedean part \( \eta_\infty \) and non-archimedean part \( \eta_f \), as follows from the strong approximation theorem. Given a numerical character \( \eta_f \) on \( (\mathfrak{o}_F/\mathfrak{m})^\times \) there exist
characters \( \eta_\infty \) satisfying the above condition, and therefore there exist Hecke characters extending \( \eta_f \), in the sense that the non-archimedean part is \( \eta_f \).

The above constructions extend to quasi-characters by replacing \( S^1 \) with \( \mathbb{C}^\times \).

Consider a finite set of non-archimedean places \( Q \). Choose \( \eta \) an Hecke character which extends the numerical character \( \omega_Q = \prod_{v \in Q} \omega_v \) on \( \hat{o} \times Q \).

\[
(6.4) \quad AL_Q(\phi)(g) := \eta(\det g)^{-1} \phi(gb)
\]

where \( b_v \in G(\mathbb{A}) \) is such that \( b_v = e \) for all \( v \not\in Q \) and \( b_v = w \begin{pmatrix} \omega_v(\pi_v) & 0 \\ 0 & 1 \end{pmatrix} \) for all \( v \in Q \).

This is an automorphic form in \( \pi' := \pi \otimes \eta^{-1} \). We have that \( \pi'_v \) is an unramified twist of \( \pi_v \) (resp. \( \tilde{\pi}_v \)) for all non-archimedean places \( v \not\in Q \) (resp. places \( v \in Q \)).

One can make a canonical choice under some simplifying assumptions such as if the ground field is \( F = \mathbb{Q} \) in which case we recover the familiar construction reviewed in §7 below or if the central character is trivial. If the central character \( \omega \) is trivial then the global and local constructions simplify. The representation \( \pi \) is self-dual. The same applies locally to \( \omega_v \) and \( \pi_v \). Because of the above discussion, it is more traditional to work out the theory of newvectors with the subgroups \( K_0[p_i^1] \) if the central character is trivial. Also the epsilon factor \( \epsilon(\frac{1}{2}, \pi_v, \psi_v) = \epsilon(\frac{1}{2}, \pi_v) \) is independent of \( \psi_v \) and equals \( \pm 1 \).

### 7. Classical automorphic forms

We review the classical theory of automorphic forms on \( GL(2) \) over \( F = \mathbb{Q} \) and recall some essential facts. We adopt notation that are consistent with some recent papers in the subject that uses the Voronoï formula, e.g. [17, 34, 48]. We shall explain whenever possible the correspondence with the representation theory framework. We try to emphasize on the aspects of the theory which may be confusing such as ramification, infinity type and Atkin–Lehner theory; these aspects sometimes can be understood more directly from a representation theoretic perspective.

Details can be found in several places in the literature, e.g. [7, 8, 11, 15, 26, 27]. We don’t provide any proof in this section.

Let \( N \geq 1 \) be a positive integer and \( \chi \) be a Dirichlet character of modulus \( N \). For an integer \( k \geq 2 \) we denote by \( S_k(N, \chi) \) the complex vector space of weight \( k \) holomorphic cusp forms with level \( N \) and nebentypus \( \chi \). For a positive real number \( \lambda \in \mathbb{R}_{>0} \) let \( S_\lambda(N, \chi) \) be the complex vector space of Maass cusp forms with level \( N \), nebentypus \( \chi \) and eigenvalue \( \lambda \). In particular we have the transformation law

\[
(7.1) \quad f_\gamma(z) := j_\gamma(z)^{-k} f(\gamma z) = \chi(d) f(z), \quad z \in \mathbb{H},
\]

for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \). Here the multiplier is \( j_\gamma(z) = \frac{cz + d}{|cz + d|} \) and the convention is that \( k = 0, 1 \) for Maass forms.
We have an expansion in Fourier series
\[ f(z) = y^{\frac{k}{2}} \sum_{n=1}^{\infty} \psi_f(n)n^{\frac{k-1}{2}} e(nz) \]
for holomorphic forms \( f \in S_k(N, \chi) \) and for Maass cusp forms of weight 0 the expansion reads
\[ f(z) = 2y^{\frac{1}{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \psi_f(n) e(nx) K_{ir}(2\pi |n| y), \quad z = x + iy, \]
where \( \frac{1}{4} + r^2 = \lambda \). Up to some multiplicative constants, the above expansions can be unified in a single formula as follows:
\[ (7.2) \quad f(z) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\psi_f(n)}{\sqrt{n}} W_{ir}^{sgn(n)}(4\pi |n| y)e(nx). \]
The reason is that for holomorphic forms, \( ir = \frac{k-1}{2} \) and we have the formula \( W_{a,a-\frac{1}{2}}(y) = y^a e^{-y/2} \); for Maass forms of weight 0, we have \( W_{0,ir}(4\pi y) = 2y^{\frac{1}{2}} K_{ir}(2\pi y) \); for Maass forms of weight 1, there is no formula for \( W_{\pm\frac{1}{2},ir}(y) \). The representation theoretic interpretation is that these functions are the lowest weight vector in the Kirillov model \( \mathcal{K}(\pi_{\infty}, \psi_{\infty}) \). For simplicity we ignore anti-holomorphic forms which may be handled similarly.

For holomorphic forms of weight \( k \) we have necessarily \( \chi(-1) = (-1)^k \), while for Maass forms of weight 0 (resp. weight 1) we have \( \chi(-1) = 1 \) (resp. \( \chi(-1) = -1 \)). The central character at infinity is 1 or \( sgn \) depending on the sign of \( \chi(-1) \).

Maass forms of weight 0 are eigenfunctions of the reflection operator \( f(z) \mapsto f(-\bar{z}) \) acting on \( S_\lambda(N, \chi) \) (this is because it commutes with Hecke operators). We call \( f \) even or odd depending of the eigenvalue being \( \pm 1 \). In terms of Fourier coefficients, we have equivalently \( \psi_f(-n) = \pm \psi_f(n) \) depending of \( f \) being even or odd.

We discuss the isomorphism class of the representation at infinity. For holomorphic forms of weight \( k \), the component at infinity is the discrete series representation of weight \( k \geq 2 \). For Maass forms \( f \) of weight 0 the component at infinity is \( ||^{ir} \boxplus ||^{-ir} \) or \( sgn ||^{ir} \boxplus sgn ||^{-ir} \) depending on \( f \) begin even or odd. For Maass forms of weight 1 the component at infinity is either \( sgn ||^{ir} \boxplus ||^{-ir} \) or \( ||^{ir} \boxplus sgn ||^{ir} \).

We explain that there is a slight imprecision in [48] concerning the type of the representation at infinity for weight zero Maass forms. For a weight zero Maass form it is necessary that the central character be trivial at infinity, i.e. \( \chi(-1) = 1 \).

There are Hecke operators \( T_n \) with \( (n, N) = 1 \) acting on \( S_\bullet(N, \chi) \). The Hecke operators are normal with respect to the Petersson inner product and more precisely \( T_n^* = \chi(n)T_n \). There is an orthogonal basis of \( S_\bullet(N, \chi) \) consisting of eigenvectors of all the Hecke operators. This is identical to the Hecke operators discussed in \( \S 2.10 \).

The subspace of newforms is the orthogonal complement of the subspace generated by old forms of type \( f(dz) \) with \( f \) of level strictly dividing \( N \). The set of primitive forms \( S_\bullet^*(N, \chi) \) is an orthogonal basis of the subspace of newforms. By definition a primitive
form \( f \) is a newform which is an eigenfunction of all \( T_n \) with \( (n, N) = 1 \) and such that \( \psi_f(1) = 1 \). This notion of newform is identical to the one discussed in §5.

A primitive form is in fact an eigenfunction of all Hecke operators, and \( \lambda_f(n) = \psi_f(n) \) is the normalized eigenvalue for all \( n \geq 1 \).\(^3\) We have the Hecke relation

\[
\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \chi(d)\lambda_f\left(\frac{mn}{d^2}\right), \quad m, n \geq 1.
\]

(7.3)

Note also that we have the relation \( \lambda_f(n) = \chi(n)\lambda_f(n) \) for all \((n, N) = 1\).

If the nebentypus is trivial, then \( \lambda_f(p^2) = p \) for all \( p \mid N \). If furthermore \( N \) is square-free then \( \mu(N)\lambda_f(N)N^{-1/2} \) is equal to \( \eta_f(N) \) which is defined below.

We review the theory of Atkin–Lehner operators for general nebentypus \( \chi \), see [1,2,49] and the summary in [48, § A.1]. Given a factorization \( N = N_1N_2 \) with \((N_1, N_2) = 1 \), we have a unique factorization \( \chi = \chi_1\chi_2 \) where \( \chi_i \) is a Dirichlet character of modulus \( N_i \). The classical way to define the Atkin–Lehner operators is to extend the action (7.1) to \( GL_2(\mathbb{R})^+ \). We introduce the Atkin–Lehner matrices \( W_{N_i} = \begin{pmatrix} xN_1 & y \\ zN & wN_1 \end{pmatrix} \) with the conditions \( x \equiv 1(N_2) \), \( y \equiv 1(N_1) \) and \( N_1^2xw - Nyz = N_1 \). Then we have a linear map

\[
W_{N_1} : S_\bullet(N, \chi) \rightarrow S_\bullet(N, \chi_1\chi_2)
\]

that is independent of the choice of the integers \( w, x, y, z \). We have the relation \( W_{N_1} \circ W_{N_1} = \chi_1(-1)\chi_2(N_1) \). If \( N_1 = N \), then one can choose \( W_N = \begin{pmatrix} 0 & 1 \\ -D & 0 \end{pmatrix} \). More generally if \( N_1N_2 \mid N \) with \((N_1N_2, N/N_1N_2) = 1 \), then \( W_{N_2} \circ W_{N_1} = \chi_2(N_1)W_{N_1N_2} \).

Adelicly the matrix \( W_{N_1} \in GL_2(\mathbb{Q}) \) has the property that it belongs to \( GL_2(\mathbb{Z}_p) \) for all \( p \mid N \), to \( K[\nu N] \) for all \( p \mid N_1 \). Because the class number is one, the Dirichlet character \( \chi_1 \) has a unique extension to a finite order Hecke character \( \eta \) on \( \mathbb{A}^\times_Q \). This produces a canonical choice for the definition of Atkin–Lehner operators in Section 6. Then it is a direct verification to see that the adelic definition coincides with the construction above.

The Atkin–Lehner operator sends newforms to newforms. For a primitive form \( f \in S^\bullet_\chi(N, \chi) \) there is a primitive form \( f_1 \in S^\bullet_{\chi_1\chi_2}(N) \) and a complex number \( \eta_f(N_1) \) with \( |\eta_f(N_1)| = 1 \) such that

\[
W_{N_1}(f) = \eta_f(N_1)f_1.
\]

(7.5)

The constant \( \eta_f(N_1) \) is called the “pseudo-eigenvalue” of \( W_{N_1} \). If \( \chi_1 \) is trivial, then \( f_1 = f \) and \( \eta_f(N_1) \) is a true eigenvalue.

\(^3\)This identity \( \lambda_f = \psi_f \) corresponds to the statement seen in §5.1 that the local L-factor is the Mellin transform of the Kirillov newvector.
This again has an interpretation in the context of representation theory, in terms of epsilon factors:

\[(7.6) \quad \eta_f(N_1) = \prod_{p \mid N_1} \epsilon\left(\frac{1}{2}, \pi_p, \psi_p\right)\]

In the classical setting the dependence on the additive character isn’t made explicit, although it may be seen to enter in the notion of primitive forms, see also [2, p.224] and [25]. Adelically primitive forms are normalized in such a way that \(W_f(e) = 1\). Thus the relation (7.6) follows from Lemma 6.3 and Section 6.

The eigenvalues of \(f_1\) are

\[(7.7) \quad \lambda_{f_1}(n) = \begin{cases} \chi_1(n)\lambda_f(n), & \text{if } (n, N_1) = 1, \\ \chi_2(n)\lambda_f(n), & \text{if } n \mid N_1^\infty. \end{cases}\]

If \(N_1 = N\) then \(\lambda_{f_1}(n) = \lambda_f(n)\) for all \(n \geq 1\).

If \(\lambda(N_1) \neq 0\), then

\[(7.8) \quad \eta_f(N_1) = \frac{G(\chi_1)}{\lambda_f(N_1)\sqrt{N_1}},\]

where \(G(\chi_1) = \sum_{x \in (\mathbb{Z}/N_1\mathbb{Z})^\times} \chi_1(x)e(x/N_1)\) is a Gauss sum. This can be used to explain the value of \(\eta\) in (1.9).

Let \(a, c \geq 1\). To establish the formula (1.9) we let \(\zeta_v = 1\) if \(v = \infty\) or if \(v(c) = 0\), and \(\zeta_v = a/c\) if \(v(c) \geq 1\). We apply Proposition 5.3 where \(\beta = 1/c \in \mathbb{Q}^\times\) and

\[u_v = \begin{cases} a & \text{if } v(c) \geq 1, \\ 1/c & \text{otherwise}. \end{cases}\]

Thus \(u \in \hat{\mathbb{Z}}^\times\), we set \(S = \{\infty\}\), and make a change of variable \(\zeta \sim N_2\zeta\).

The formula for the divisor function \(\tau(n)\) and \(c = 1\) is due to Voronoï [63] whose motivation was to derive new estimates on the Dirichlet’s divisor problem as explained in the introduction. For a general modulus \(c \geq 1\) the formula is

\[(7.9) \quad \sum_{n \geq 1} \tau(n)e\left(\frac{an}{c}\right)w(n) = 2 \int_0^\infty \left(\log \frac{\sqrt{x}}{c} \gamma\right) w(x)dx + \sum_{n \geq 1} \tau(n)e\left(-\frac{an}{c}\right)\tilde{w}_1(n) + \sum_{n \geq 1} \tau(n)e\left(\frac{an}{c}\right)\tilde{w}_2(n).\]

Here \(\gamma\) is Euler constant and for all \(y > 0\),

\[(7.10) \quad \tilde{w}_1(y) = -2\pi \int_0^\infty Y_0\left(\frac{4\pi \sqrt{xy}}{c}\right)w(x)dx,\]

\[\tilde{w}_2(y) = 4\int_0^\infty K_0\left(\frac{4\pi \sqrt{xy}}{c}\right)w(x)dx.\]

The precise expression for the transforms appears in Jutila [45], see also [16] for a version with a smoothed divisor function and [40, §4.5].
References


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