The geometry of $Out(F_n)$ from Thurston to today and beyond

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Consider automorphisms of free groups, e.g.

$$f(a) = aaB, f(b) = bA$$

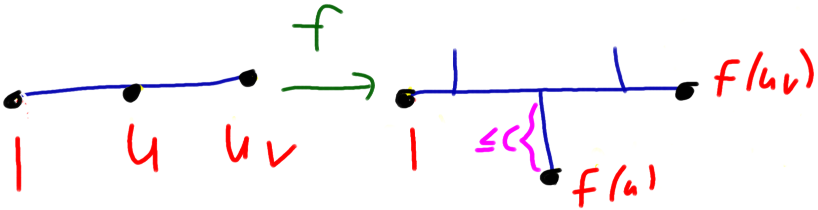
Note that a, b, ab are reduced words, but $f(a) \cdot f(b) = aaB \cdot bA$ is not, a word of length 2 cancels.

Notation: $[x]$ is the reduced word equivalent to x, e.g. $[aaBbA] = a$.

Bounded Cancellation Lemma

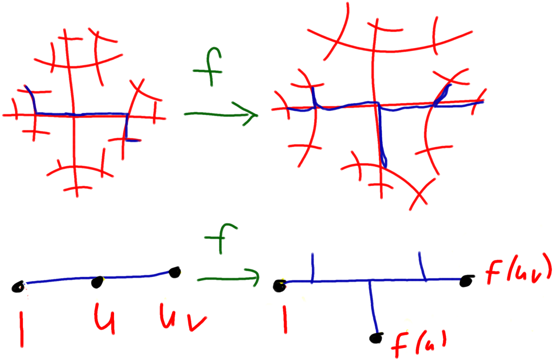
Theorem (Thurston's Bounded Cancellation Lemma, 1987)

For every automorphism $f : F_n \to F_n$ there is a constant $C = C(f)$ such that: whenever u, v, uv are reduced words the amount of cancellation in $[f(u)][f(v)]$ is at most C letters.



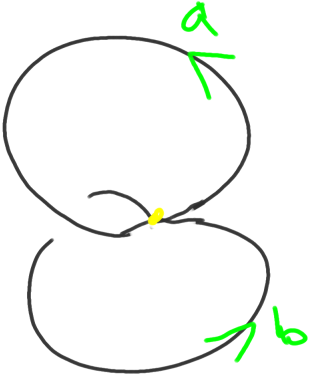
Proof:

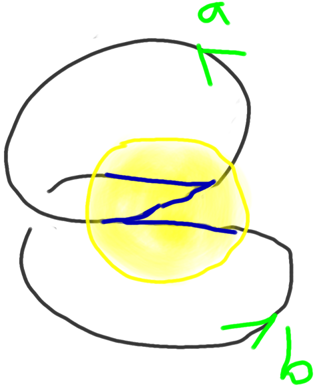
1. $f : F_n \to F_n$ is a quasi-isometry with respect to the word metric (it is even bilipschitz).
2. Quasi-isometries map geodesics to quasi-geodesics.
3. (Morse stability) Quasi-geodesics in trees (or Gromov hyperbolic spaces) are contained in Hausdorff neighborhoods of geodesics.



Train tracks

A train track structure on a graph Γ is a collection of 2-element subsets of the link of each vertex, called the set of legal turns.





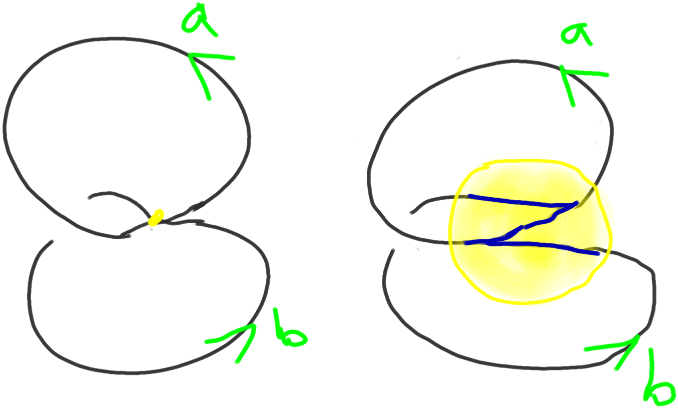
Bill Thurston:

The mental image is that of a railroad switch, or more generally a switchyard, where for each incoming direction there is a set of possible outgoing directions where trains can be diverted without reversing course.

Drawing by Conan Wu

A path on Γ is legal if it is a local embedding, and at each vertex it takes a legal turn.
Let $g : \Gamma \to \Gamma$ be a cellular map on a finite graph Γ. g is a train track map if it satisfies the following equivalent conditions:

1. For every $k > 0$ and every edge e, the path $f^k(e)$ has no backtracking (i.e. it is locally an embedding).
2. There is a train track structure preserved by g: legal paths are mapped to legal paths. Equivalently, edges are mapped to legal paths and legal turns are mapped to legal turns.



The map $a \mapsto aaB$, $b \mapsto bA$ is a train track map.

Theorem (B.-Handel, 1992)

Every fully irreducible automorphism can be represented by a train track map.

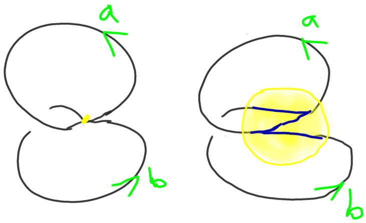
fully irreducible: no proper free factor is periodic up to conjugation.

Benefits of train track maps $g : \Gamma \to \Gamma$. Assume g is irreducible, i.e. no homotopically proper g-invariant subgraphs.

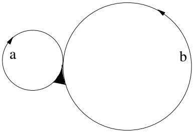
- Γ can be assigned a metric so that g stretches legal paths by a fixed factor λ, the dilatation.
- λ and the metric can be computed from the transition matrix.
- λ is the growth rate of the automorphism.
- λ is a weak Perron number.

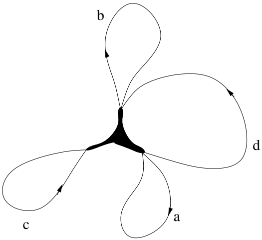
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$a \mapsto aaB$, $b \mapsto bA$.



Train track maps





$$a \mapsto ab$$

$$b \mapsto bab$$

$$|a| = 1, |b| = \lambda - 1$$
$$\lambda^2 - 3\lambda + 1 = 0$$

$$a \mapsto b$$

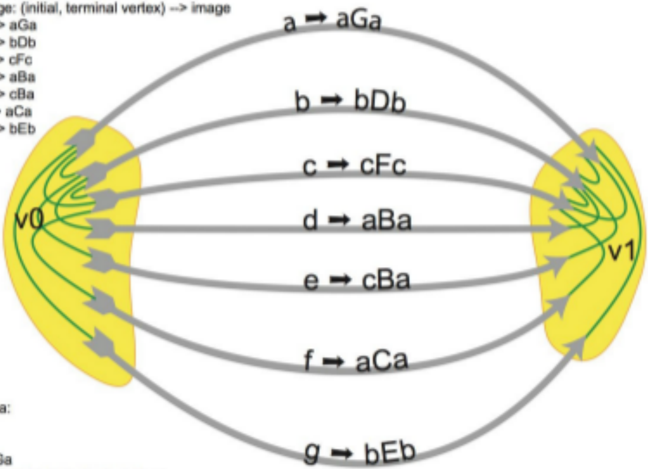
$$b \mapsto c$$

$$c \mapsto dA$$

$$d \mapsto DC$$

$$|a| = 1, |b| = \lambda$$
$$|c| = \lambda^2, |d| = \lambda^3 - 1$$
$$\lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

7 // number of edges
2 // number of vertices
// format: edge: (initial, terminal vertex) --> image
a: (v0, v1) --> aGa
b: (v0, v1) --> bDb
c: (v0, v1) --> cFc
d: (v0, v1) --> aBa
e: (v0, v1) --> cBa
f: (v0, v1) --> aCa
g: (v0, v1) --> bEb



Iterates of a:
a
aGa
aGaBeBaGa
aGaBeBaGaBdBcBaBdBaGaBeBaGa
aGaBeBaGaBdBcBaBdBaGaBeBaGaBdBaBaBdBcFcBdBaGaBdBaBaBdBaGaBeBaGaBdBcBaB\dBaGaBeBaGa

An automorphism of F(6) with λ = 3.

Theorem (Thurston, 2011)

For every weak Perron number λ *there is an irreducible train track map with dilatation* λ*. (No rank restriction.)*

Questions (Thurston)

- Characterize pseudo-Anosov dilatations, no bound on genus. Fried's conjecture.
- $\lambda(f^{-1})$ is typically different from $\lambda(f)$ for automorphisms of free groups. Characterize the pairs $(\lambda(f), \lambda(f^{-1}))$.

Mapping tori and 3-manifolds

If $g : \Gamma \to \Gamma$ is a homotopy equivalence representing an automorphism $f : F_n \to F_n$, the mapping torus

$$M_g = \Gamma \times [0,1]/(x,1) \sim (g(x),0)$$

has fundamental group

$$F_n \rtimes_f \mathbb{Z}$$

also called the mapping torus of f.
Principle: These are similar to 3-manifolds.

A group is coherent if each of its finitely generated subgroups is finitely presented.

Theorem (Scott, 1973)

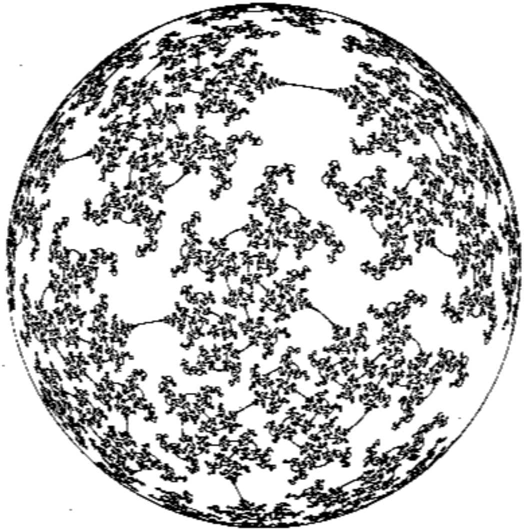
Every finitely generated 3-manifold group is coherent.

Theorem (Feighn-Handel, 1999)

Mapping tori of free group automorphisms are coherent.

Theorem (Thurston)

If $f : S \to S$ is a homeomorphism of a surface that does not have periodic isotopy classes of essential scc's, the mapping torus M_f is a hyperbolic 3-manifold.



Theorem (B-Feighn, Brinkmann)

If $f : F_n \to F_n$ does not have any nontrivial periodic conjugacy classes, then $F_n \rtimes_f \mathbb{Z}$ is a Gromov hyperbolic group.

Theorem (Hagen-Wise, 2014)

If $F_n \rtimes_f \mathbb{Z}$ is hyperbolic, then it can be cubulated. So by [Agol, Wise] it is linear.

Theorem (Bridson-Groves)

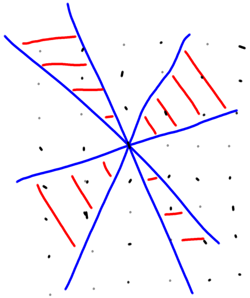
For any automorphism $f : F_n \to F_n$ the mapping torus $F_n \rtimes_f \mathbb{Z}$ satisfies quadratic isoperimetric inequality.

Theorem (Thurston)

If M is a hyperbolic 3-manifold, the set of classes in $H^1(M;\mathbb{Z})$ corresponding to fibrations is the intersection

$$\mathcal{C} \cap H^1(M;\mathbb{Z})$$

for a finite collection of polyhedral open cones $\mathcal{C} \subset H^1(M;\mathbb{R})$.



Theorem (Fried, 1982)

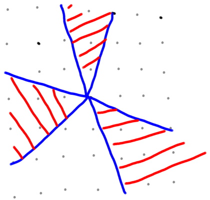
There is a continuous, homogeneous function of degree -1 *defined on* $\mathcal{C}$ *that on points of* $H^1(M;\mathbb{Z})$ *evaluates to* $\log(\lambda)$*, where* λ *is dilatation of the monodromy.*

Theorem (McMullen, 2000)

There is a (Teichmüller) polynomial $\Theta \in \mathbb{Z}[H_1(M)]$ *so that for every* $\alpha \in \mathcal{C} \cap H^1(M;\mathbb{Z})$*, the house of the specialization* $\Theta^\alpha \in \mathbb{Z}[\mathbb{Z}]$ *is the dilatation of the monodromy.*

Theorem (Dowdall-I.Kapovich-Leininger, Algom-Kfir-Hironaka-Rafi, 2013-14)

- *Let $G = F_n \rtimes_f \mathbb{Z}$ be hyperbolic. The set of classes in $H^1(G;\mathbb{Z})$ corresponding to fibrations $G = F_N \rtimes_F \mathbb{Z}$ with expanding train track monodromy is the intersection $\mathcal{C} \cap H^1(G;\mathbb{Z})$ for a collection of open polyhedral cones $\mathcal{C} \subset H^1(G;\mathbb{R})$.*
- *There is a continuous, homogeneous function of degree -1 that on integral points evaluates to $\log(\lambda)$, λ is the dilatation of the monodromy.*
- *There is a polynomial $\Theta \in \mathbb{Z}[H_1(G)/tor]$ so that for every $\alpha \in \mathcal{C} \cap H^1(G;\mathbb{Z})$, the house of the specialization $\Theta^\alpha \in \mathbb{Z}[\mathbb{Z}]$ is the dilatation of the monodromy.*

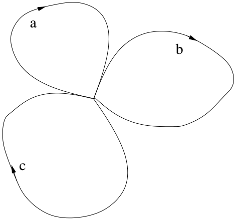


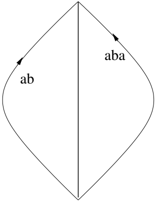
Cf. Bieri-Neumann-Strebel

Outer space

Definition

- graph: finite 1-dimensional cell complex Γ, all vertices have valence ≥ 3.
- rose $R = R_n$: wedge of n circles.



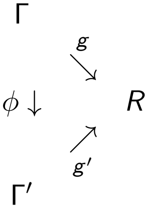


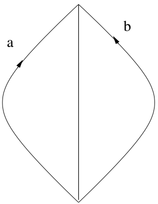
- marking: homotopy equivalence $g : \Gamma \to R$.
- metric on Γ: assignment of positive lengths to the edges of Γ so that the sum is 1.

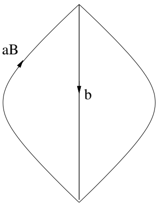
Outer space

Definition (Culler-Vogtmann, 1986)

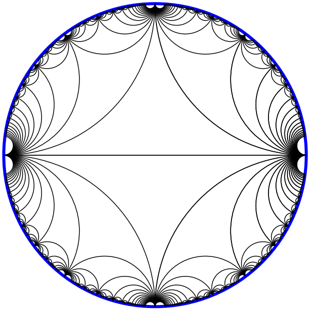
Outer space CV_n is the space of equivalence classes of marked metric graphs (g, Γ) where $(g, \Gamma) \sim (g', \Gamma')$ if there is an isometry $\phi : \Gamma \to \Gamma'$ so that $g'\phi \simeq g$.

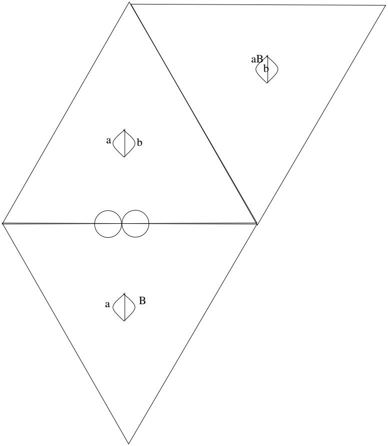


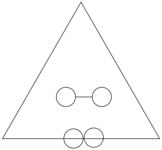




Outer space in rank 2







Triangles have to be added to edges along the base.

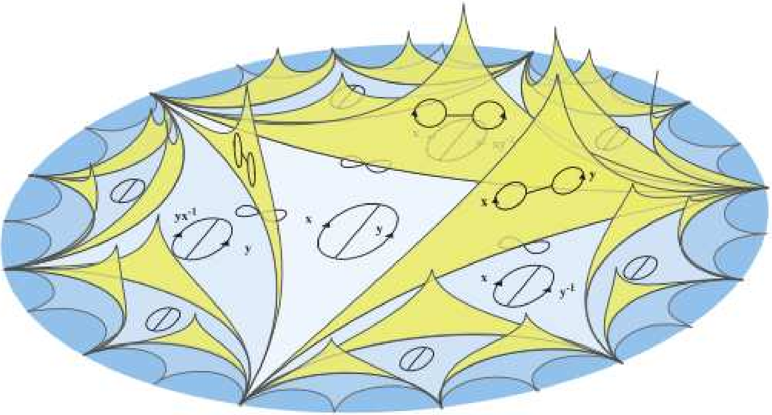


Figure 2: Outer space in rank 2

Picture of rank 2 Outer space by Karen Vogtmann

contractibility

Theorem (Culler-Vogtmann 1986)

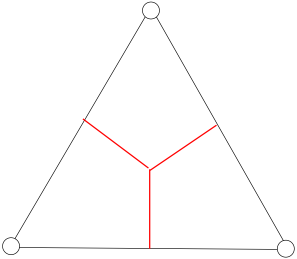
CV_n is contractible.

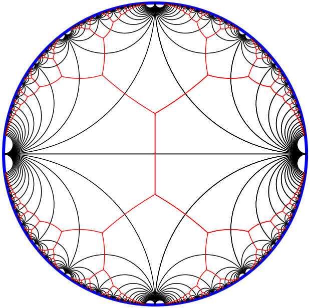
Action

If $\phi \in Out(F_n)$ let $f : R \to R$ be a h.e. with $\pi_1(f) = \phi$ and define

$$\phi(g, \Gamma) = (fg, \Gamma) \qquad\qquad \Gamma \xrightarrow{g} R_n \xrightarrow{f} R_n$$

- action is simplicial,
- point stabilizers are finite.
- there are finitely many orbits of simplices (but the quotient is not compact).
- the action is cocompact on the spine $SCV_n \subset CV_n$.





Topological properties

Finiteness properties:

- Virtually finite $K(G,1)$ (Culler-Vogtmann 1986).
- $vcd(Out(F_n)) = 2n - 3$ ($n \geq 2$) (Culler-Vogtmann 1986).
- every finite subgroup fixes a point of CV_n.

Other properties:

- every solvable subgroup is finitely generated and virtually abelian (Alibegović 2002)
- Tits alternative: every subgroup $H \subset Out(F_n)$ either contains a free group or is virtually abelian (B-Feighn-Handel, 2000, 2005)
- Bieri-Eckmann duality (B-Feighn 2000)

$$H^i(G; M) \cong H_{d-i}(G; M \otimes D)$$

- Homological stability (Hatcher-Vogtmann 2004)

$$H_i(Aut(F_n)) \cong H_i(Aut(F_{n+1})) \text{ for } n >> i$$

- Computation of stable homology (Galatius, 2011)

Lipschitz metric on Outer space

Motivated by Thurston's metric on Teichmüller space (1998).
If $(g,\Gamma),(g',\Gamma') \in CV_n$ consider maps $f:\Gamma\to\Gamma'$ so that $g'f \simeq g$ (such f is the difference of markings).

$$\begin{array}{ccc} \Gamma & & \\ & \overset{g}{\searrow} & \\ f\downarrow & & R \\ & \underset{g'}{\nearrow} & \\ \Gamma' & & \end{array}$$

Consider only f's that are linear on edges.
Arzela-Ascoli $\Rightarrow \exists f$ that minimizes the largest slope, call it $\sigma(\Gamma,\Gamma')$.

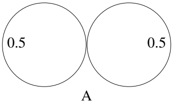
Lipschitz metric on Outer space

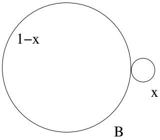
Definition

$d(\Gamma, \Gamma') = \log \sigma(\Gamma, \Gamma')$

- $d(\Gamma, \Gamma'') \leq d(\Gamma, \Gamma') + d(\Gamma', \Gamma'')$,
- $d(\Gamma, \Gamma') = 0 \iff \Gamma = \Gamma'$.
- in general, $d(\Gamma, \Gamma') \neq d(\Gamma', \Gamma)$.
- Geodesic metric.

Example





$$d(A, B) = \log \frac{1 - x}{0.5} \to \log 2$$

$$d(B, A) = \log \frac{0.5}{x} \to \infty$$

But [Handel-Mosher] The restriction of d to the spine is quasi-symmetric, i.e. $d(\Gamma, \Gamma')/d(\Gamma', \Gamma)$ is uniformly bounded.

Lipschitz metric on Outer space

Theorem (Thurston)

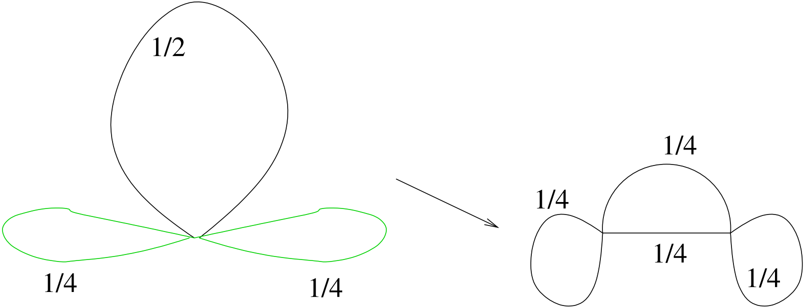
Let $f : S \to S'$ be a homotopy equivalence between two closed hyperbolic surfaces that minimizes the Lipschitz constant in its homotopy class. Then there is a geodesic lamination $\Lambda \subset S$ so that f is linear along the leaves of Λ with slope equal to the maximum. Moreover, f can be perturbed so that in the complement of Λ the Lipschitz constant is smaller than maximal.

For the optimal map, lines of tension form a geodesic lamination.

Lipschitz metric on Outer space

Theorem

Let $f : \Gamma \to \Gamma'$ be a homotopy equivalence between two points of CV_n that minimizes the Lipschitz constant in its homotopy class. Then there is a subgraph $\Gamma_0 \subset \Gamma$ so that f is linear along the edges of Γ_0 with slope equal to the maximum and Γ_0 has a train track structure so that legal paths are stretched maximally. Moreover, f can be perturbed so that in the complement of Γ_0 the Lipschitz constant is smaller than maximal.



For the optimal map, lines of tension form a train track.

Proof of existence of train track maps

Proof.

(Sketch) Parallel to Bers' proof of Nielsen-Thurston classification. Consider

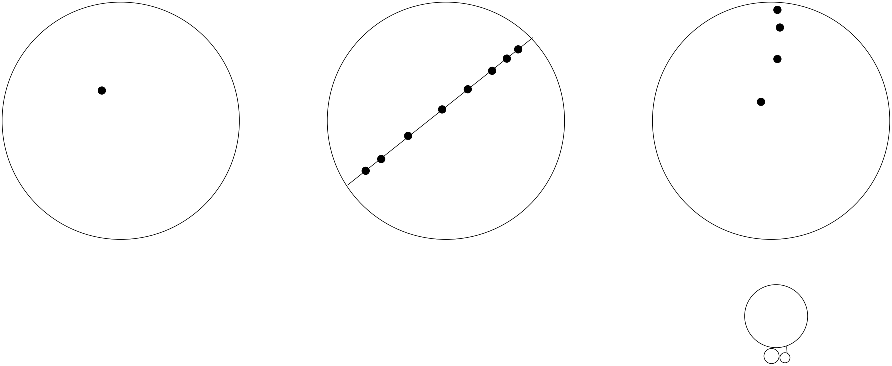
$$\Phi : CV_n \to [0, \infty)$$

$$\Phi(\Gamma) = d(\Gamma, \phi(\Gamma))$$

There are 3 cases:

- $\inf \Phi = 0$ and is realized. Then there is Γ with $\phi(\Gamma) = \Gamma$ so ϕ has finite order.
- $\inf \Phi > 0$ and is realized, say at Γ. Apply above Theorem to $\phi : \Gamma \to \phi(\Gamma)$. Argue that $\Gamma_0 = \Gamma$ or else ϕ is reducible. Train-track structure on Γ_0 can be promoted to give the theorem.
- $d = \inf \Phi$ is not realized. Let $\Gamma_i \in CV_n$ have $d(\Gamma_i, \phi(\Gamma_i)) \to d$. Argue that projections to $CV_n/Out(F_n)$ leave every compact set. Thus Γ_i has "thin part" which must be invariant, so ϕ is reducible.

Proof of existence of train track maps

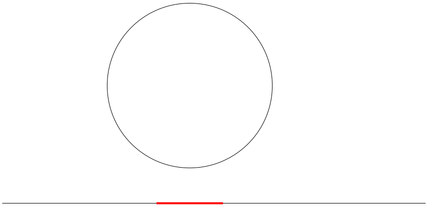


Axes

Irreducible ϕ has an axis with translation length $\log \lambda$, where λ is the expansion rate of ϕ.

Theorem (Yael Algom-Kfir, 2008)

Axes of fully irreducible elements are strongly contracting, i.e. the projection of any ball disjoint from the axis to the axis has uniformly bounded size.



The analogous theorem in Teichmüller space was proved by Minsky (1996).

Corollary (Yael Algom-Kfir)

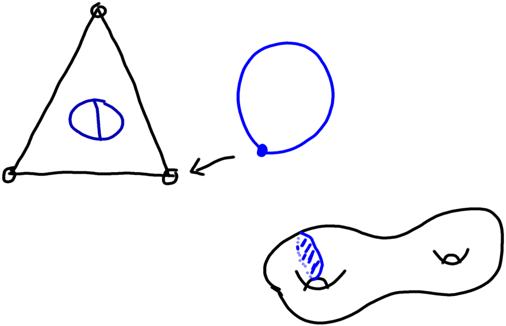
Axes of fully irreducible elements are Morse.

Very recent developments

- Hyperbolicity of associated complexes
- Boundary
- Subfactor projections and estimating distances
- Poisson boundary of $Out(F_n)$

Complex of free splittings $\mathcal{S}_n$

Add missing faces to CV_n. This simplicial complex is $\mathcal{S}_n$.
An ideal point represents a graph of groups decomposition of F_n with trivial edge groups.
Alternate description: complex of spheres in $M_n = \#_1^n S^1 \times S^2$.



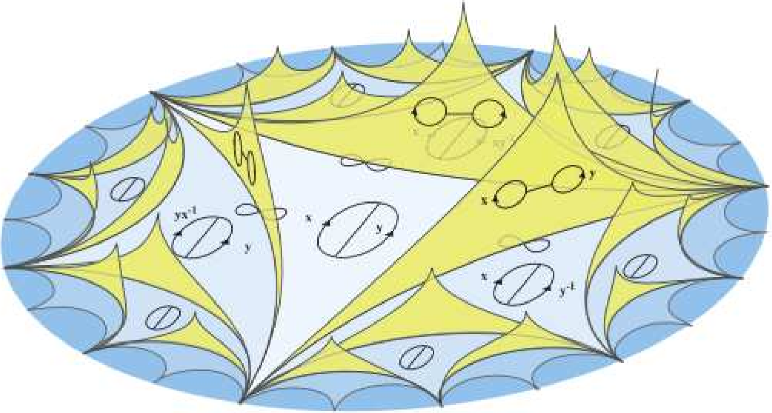
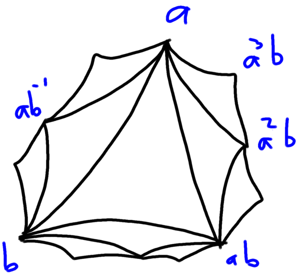


Figure 2: Outer space in rank 2

Complex of free factors $\mathcal{F}_n$

Analogous to the Bruhat-Tits building for $GL_n(\mathbb{Z})$.

- Vertex: conjugacy class of proper free factors
- Simplex: Flag, i.e. collection of vertices that become nested after appropriately conjugating.



Theorem (2011)

Both $\mathcal{F}_n$ [B-Feighn] and $\mathcal{S}_n$ [Handel-Mosher] are δ-hyperbolic. An automorphism acts hyperbolically on $\mathcal{F}_n$ iff it is fully irreducible.

Hyperbolicity criteria

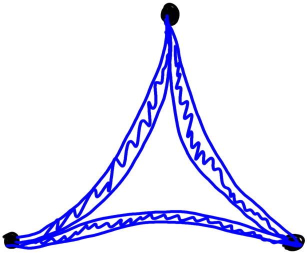
Masur-Minsky,...,Bowditch

Theorem (Masur-Schleimer, Bowditch, 2012)

Let X be a connected graph, $h \geq 0$, and for all $x, y \in X^{(0)}$ there is a connected subgraph $L(x, y) \ni x, y$ so that:

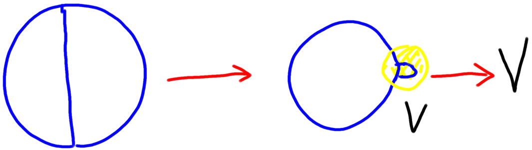
- *(thin triangles) for all x, y, z $L(x, y) \subseteq N_h(L(x, z) \cup L(z, y))$,*
- *$d(x, y) \leq 1$ implies $diam(L(x, y)) \leq h$.*

Then X is hyperbolic.



There are coarse maps:

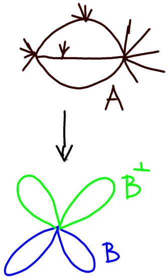
$$CV_n \to \mathcal{S}_n \to \mathcal{F}_n$$



Can take $L(\cdot, \cdot)$ to be images of folding paths [Stallings], or Hatcher's surgery paths (Horbez-Hilion).

Large scale geometry of $Out(F_n)$

Modeled on the Masur-Minsky theory of subsurface projections.
Goal: Construct many actions of $Out(F_n)$ (or a finite index subgroup) on δ-hyperbolic spaces.
Here we use splitting complexes – action is freer.



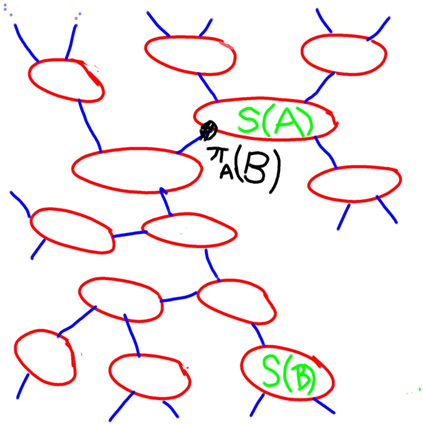
Theorem (B-Feighn)

If A, B are free factors "in general position" then there is a coarsely well defined projection $\pi_A(B) \in \mathcal{S}(A)$.

Taylor: Version for $\mathcal{F}(A)$, sharp notion of "general position".

Projection complexes

There is a "projection complex" [B-Bromberg-Fujiwara] that organizes subsurface and subfactor projections into individual hyperbolic spaces.



Theorem (B-Bromberg-Fujiwara)

The mapping class group $Mod(S)$ acts on a product $Y_1 \times \cdots \times Y_k$ of hyperbolic spaces so that an orbit map is a QI embedding.

Theorem (B-Feighn)

$Out(F_n)$ acts on a product $Y_1 \times \cdots \times Y_k$ of hyperbolic spaces so that every exponentially growing automorphism has positive translation length.

Question: Can a finite index subgroup of $Out(F_n)$ act on a δ-hyperbolic space so that $a \mapsto ab, b \mapsto b, \cdots$ has positive translation length?

Boundary of $\mathcal{F}_n$

- Outer space has a natural compactification $\overline{CV}_n$ where ideal points are represented by $F_n - \mathbb{R}$-trees (Culler-Morgan).
- Structure of individual $F_n - \mathbb{R}$-trees (Coulbois, Hilion, Reynolds)
- Notion of arational trees – these correspond to filling laminations in PML. Cf. Klarreich.

Definition

A tree $T \in \partial CV_n$ is arational if every proper factor $A < F_n$ acts on T discretely and freely.

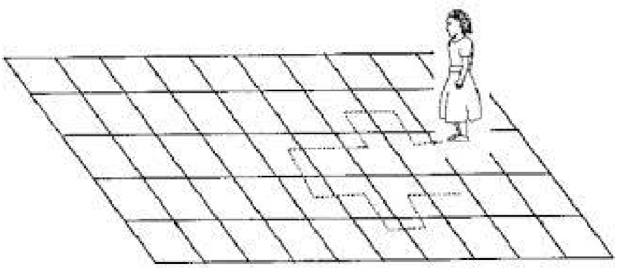
Theorem (B-Reynolds, Hamenstädt)

The Gromov boundary of $\mathcal{F}_n$ can be identified with the subquotient of $\partial CV_n = \overline{CV}_n - CV_n$, namely

$$\{arationaltrees\}/\sim$$

where the equivalence is equivariant homeomorphism. Equivalence classes are simplices.

Poisson boundary



Consider a random walk on $Out(F_n)$, with measure of finite support generating the whole group.

Theorem (Horbez, 2014)

The hitting measure is supported on the set of arational trees.

Theorem (Horbez, 2014)

$\partial\mathcal{F}_n$ serves as a model of the Poisson boundary of $Out(F_n)$.

Questions about the geometry of $Out(F_n)$

- Asymptotic dimension $asdim(Out(F_n)) < \infty$? $asdim(\mathcal{S}) < \infty$? $asdim(\mathcal{F}) < \infty$?
- Compute $rank(Out(F_n))$ (= largest N so that there is a qi embedding $R^N \to Out(F_n)$).
- Asymptotic cone of $Out(F_n)$. Is it tree graded? Dimension?
- QI rigidity?